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On the class of *b*-weakly compact operators

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Abstract

We study the b-weakly compact operators using the Banach-Saks sets. More precisely, we will establish that an operator T from a Banach lattice E into a Banach space Y is b-weakly compact if and only if T carries b-order bounded subsets of E onto Banach-Saks subsets of Y. Next we give a sequential characterization of these operators without requiring the sequences to be disjoint. Also, we describe the relationships between b-weakly compact, and b-L-weakly compact operators.

Keywords Positive operator · Banach-Saks set · Banach lattice · b-weakly compact operator

Mathematics Subject Classification Primary 47B65; Secondary 46B42

1 Introduction

The class of *b-weakly compact* operators were introduced by S. Alpay, B. Altin and C. Tonyali in [3]. Since then, this concept has been studied by many authors; see, for instance, [2, 4, 6]. Recall that an operator T from a Banach lattice E to a Banach space X is said *b-weakly compact* whenever T carries each b-order bounded subset of E into a relatively weakly compact subset of X. A subset B of E is said *b-order bounded* if it is order bounded in $E^{''}$ (the topological bidual of E). It is not difficult to check that an order bounded subset of E is b-order bounded. However, the unit ball of c_0 is b-order bounded but not order bounded. Note that each weakly compact operator T is b-weakly compact, but the converse is not always true. In fact, the identity operator $Id_{L_1[0,1]} : L_1[0, 1] \longrightarrow L_1[0, 1]$ is b-weakly compact, but not weakly compact (see

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Example 2.6 (a) in [3]). Some characterization of b-weakly compact operators are given by Alpay et al ([3], Proposition 2.8) and B. Altin ([4], Proposition 1). More precisely, if T is a bounded operator from a Banach lattice E into a Banach space X, the following assertions are equivalent:

- T is b-weakly compact.
- $\lim_n \|Tx_n\| = 0$ for every b-order bounded disjoint sequence $(x_n)_{n\in\mathbb{N}}$ of E
- $(Tx_n)_{n\in\mathbb{N}}$ is norm convergent for every positive increasing sequence $(x_n)_{n\in\mathbb{N}}$ of the closed unit ball B_E of E.

The main aim of the present paper is studying b-weakly compact operators using the Banach-Saks sets. In Sect. 2 we introduce some basic definitions and facts concerning Banach-Saks and b-order bounded sets. In particular, we prove that the notions of an L-weakly compact and a Banach-Saks set coincide for intervals. In Sect. 3 we present some characterizations of the b-weakly compact operators. Mainly, we prove that an operator T from a Banach lattice E into a Banach space Y is b-weakly compact if and only if T carries b-order bounded subsets of E onto Banach-Saks subsets of Y if and only if $\lim_n \|Tx_n\| = 0$ for every $b-$order bounded sequence $(x_n)_{n\in\mathbb{N}}$ of E_+ satisfying that the sequence of arithmetic means $\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_n$ converge in norm to zero (Theorems 3.3 and 3.6). In Sect. 4, we establish some relationships between the notions of a b-weakly compact, and a b-L-weakly compact operator. In particular, we prove that these two notions coincide for positive operators between two Banach lattices E and F such that F has an order continuous norm.

We refer the reader to [1, 16] for any unexplained terms from the theory of Banach lattices and operators.

2 Banach-Saks and *b*-order bounded sets

A *Banach lattice* is a Banach space $(E, ||.||)$ such that E is a Riesz space and its norm satisfies the following property: For each $x, y \in E$ such that $|x| \leq |y|$, we have $||x|| \leq ||y||$. Note that the topological dual E', endowed with the dual norm and the dual order is a Banach lattice. The maximum, respectively the minimum of the set $\{x_i, 1 \leq i \leq n\}$ is denoted by $\vee_{i=1}^{n} x_i$, respectively $\wedge_{i=1}^{n} x_i$. A net (u_α) in a Banach lattice is said to be *disjoint* whenever $|u_\alpha| \wedge |u_\beta| = 0$ holds for $\alpha \neq \beta$.

Recall from [15] that a subset A of a Banach space X is called *Banach-Saks* if each bounded sequence $(x_n)_{n\in\mathbb{N}}$ of A has a subsequence $(y_n)_{n\in\mathbb{N}}$ whose arithmetic means converge in norm. That is, there exists $y \in E$ such that:

$$\lim_{n\to\infty} \|\frac{1}{n}\sum_{k=1}^{n} y_k - y\| = 0.$$

In relying upon Proposition 2.3 in [15], a Banach-Saks set is weakly relatively compact. The converse statement is in general not true [7]. We have the following result.

Lemma 2.1 *Let $(h_n)_{n\in\mathbb{N}}$ be a b-order bounded disjoint sequence of a Banach lattice E. Then* $\lim_n \|\frac{1}{n}\sum_{k=1}^{n} h_k\| = 0$.

Proof Let $(h_n)_{n\in\mathbb{N}}$ be a disjoint sequence of E such that $0 \leq h_n \leq x''$ holds for all $n \in \mathbb{N}$ and for some $x'' \in E''$. Observe that $0 \leq \vee_{i=1}^{m} h_i = \sum_{i=1}^{m} h_i \leq x''$ for all $m \in \mathbb{N}$. Therefore, $0 \leq \frac{1}{n}\sum_{i=1}^{n} h_i \leq \frac{x''}{n}$ for all $n \in \mathbb{N}$, which implies that

$$\|\frac{1}{n}\sum_{i=1}^{n} h_i\| \leq \frac{\|x''\|}{n} \to 0.$$

□

Recall that a Banach lattice E is said to be *order continuous* if $\lim_\alpha \|x_\alpha\| = 0$ for every decreasing net $(x_\alpha)_\alpha$ in E such that $\inf(x_\alpha) = 0$. Let E be an order continuous Banach lattice, an element $e \in E$ is said to be a weak unit if for $h \in E$, $e \wedge h = 0$ implies $h = 0$. The set of all positive vectors of E is denoted by E_+. The ideal generated by a vector $x \in E$ is denoted by E_x and is given by

$$E_x = \{y \in E;\ \exists\, \lambda > 0 \quad \text{with} \quad |y| \leq \lambda|x|\}.$$

Recall from ([16], Definition 3.6.1) that A bounded subset S of E is said to be *L-weakly compact* , if $\|x_n\| \longrightarrow 0$ for every disjoint sequence $(x_n)_n$ in the solid hull of S. The solid hull of S is given by

$$\mathrm{Sol}(S) = \{x \in E : |x| \leq |a| \text{ for some } a \in S\}.$$

The maximal closed ideal in E on which the induced norm is order continuous is denoted by E^a. A Grothendieck type characterization of L-weakly compact sets is expressed as follows.

Theorem 2.2 (Proposition 3.6.2 in [16]) *Let A be a non-empty bounded subset of E. The following assertions are equivalent:*

(1) *A is L-weakly compact.*
(2) *For each $\epsilon > 0$ there exists some $u \in (E^a)_+$ such that $A \subset [-u, u] + \epsilon B_E$, where B_E is the closed unit ball of E.*

The notions of L-weakly compact and Banach-Saks sets coincide for intervals. The details follow.

Theorem 2.3 *Let E be a Banach lattice, and let $b \in E_+$. Then $[-b, b]$ is L-weakly compact if and only if it is Banach-Saks.*

Proof Let $b \in E_+$ be such that $[-b, b]$ is L-weakly compact. Since $[-b, b] \subseteq E^a$ and E^a is a Banach lattice with order continuos norm (where E^a is the maximal closed ideal in E on which the induced norm is order continuous), we conclude from Lemma 2.3 in [12] that $[-b, b]$ is a Banach-Saks in E^a. So, $[-b, b]$ is a Banach-Saks set in E.

Conversely, if $[-b, b]$ is Banach-Saks, it follows from Proposition 2.3 in [15], that $[-b, b]$ is weakly compact, so by Corollary 5.54 in [1], $[-b, b]$ is $L-$weakly compact.
□

Note that Theorem 2.3 is not true for arbitrary order bounded subsets. Indeed, if (e_n) denotes the sequence of the basic unit vectors of l_∞, then (e_n) is an order bounded Banach-Saks set of l_∞ but not L-weakly compact.

A Banach lattice E is said to be a *Kantorovich-Banach space* (or briefly a *KB-space*) whenever every increasing norm bounded sequence of E_+ is norm convergent ([1], Definition 4.58). For instance, each reflexive Banach lattice is a KB-space ([1], Theorem 4.70). Also, for $1 < p < \infty$ the space $L^p[0, 1]$ is an example of a KB-space ([5], Proposition 2.1).

Recall from ([1], p. 52) that an ideal I of E is called a σ*-ideal* whenever for every sequence $(x_n)_{n\in\mathbb{N}}$ of I, if $\sup(x_n) = x$ in E, then $x \in I$.

Theorem 2.4 *Let E be a Banach lattice, then the following statements are equivalent:*

(1) *E is a σ-ideal of E''.*
(2) *E is KB-space.*
(3) *Every b-order bounded subset A of E has the Banach-Saks property.*
(4) *Every b-order bounded subset A of E is relatively weakly compact.*

Proof (1) $\Longrightarrow$ (2) Let (x_n) be a norm bounded sequence in E satisfying $0 \leq x_n \uparrow$. Then $0 \leq x_n \uparrow x''$ holds in E'' for some x'' (see page 232 in [1]). By hypothesis, $x'' \in E$. Since E is an ideal in E'', it has an order continuous norm (see Theorem 4.9 in [1]). So, by Theorem 2.4.2 iii) of [16], (x_n) is convergent. Thus E is KB-space.

(2) $\Longrightarrow$ (3) Let A be a b-order bounded subset of E. By Proposition 2.1 in [3], A is order bounded, so there exists $b \in E^+$ such that $A \subseteq [-b, b]$. The rest of the proof follows from Theorem 2.4.2 in [16] and Theorem 2.3.

(3) $\Longrightarrow$ (4) Follows immediately from Proposition 2.3 in [15].

(4) $\Longrightarrow$ (1) Relying on our hypothesis we have $I : E \to E$ is b-weakly compact. Hence, we deduce from Proposition 2.10 in [3] that E is a KB-space, which implies that E is a band of E'' ([1], Theorem 4.60). In particular E is a σ-ideal of E''. □

Notice that the equivalence (2) , (4) of Theorem 2.4 is exactly Proposition 2.10 of [3].

3 Some characterizations of *b*-weakly compact operators

The main objective of this section is to characterize the b-weakly compact operators. For this, we need to fix some notations and recall some definitions.

Recall from [11] that an operator T between a Banach lattice E and a Banach space Y is said to be *order weakly compact* if $T([-x, x])$ is relatively weakly compact for every positive element $x \in E$. Order weakly compact operators are characterized as follows.

Theorem 3.1 ([16], Theorem 3.4.6) *Suppose that T is a bounded operator from a Banach lattice E into a Banach space Y. Then there exist a Banach lattice G, a lattice homomorphism $\phi : E \to G$, and an operator $R : G \to Y$ with $T = R\phi$ such that G has order continuous norm if and only if T is order weakly compact.*

A Grothendieck type characterization of the Banach-Saks sets is the next.

Lemma 3.2 *A subset B of a Banach space X is Banach-Saks if and only if for each $\epsilon > 0$ there exists a Banach-Saks subset S of X such that*

$$B \subset S + \epsilon B_X.$$

Proof If B is Banach-Saks, and $\epsilon > 0$, then $B \subset B + \epsilon B_X$.
Conversely, let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of B and let $\epsilon > 0$. From our hypothesis, there exists a Banach-Saks set S such that $\{x_n,\ n \in \mathbb{N}\} \subset S + \epsilon B_X$, and hence $x_n = y_n + \epsilon z_n$, where $(y_n) \subset S$ and $(z_n) \subset B_X$. Without loss of generality we can assume that the sequence $(\frac{1}{n}\sum_{k=1}^{n} y_k)_n$ converges in norm to some $y \in X$. Then, there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ we have

$$||\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right) - y|| \leq \epsilon.$$

Let $n \geq N_0$, then

$$\begin{aligned}
||\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) - y|| &= ||\left[\frac{1}{n}\sum_{k=1}^{n}(y_k + \epsilon z_k)\right] - y\ || \\
&\leq ||\ \left(\frac{1}{n}\sum_{k=1}^{n} y_k\right) - y\ || + \frac{\epsilon}{n}||\sum_{k=1}^{n} z_k\ || \\
&\leq ||\ \left(\frac{1}{n}\sum_{k=1}^{n} y_k\right) - y\ || + \frac{\epsilon}{n}\sum_{k=1}^{n}||z_k|| \\
&\leq \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

Consequently, the sequence $\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_{n\in\mathbb{N}}$ converges in norm to y. □

From Proposition 2.8 in [3], T is b-weakly compact if and only if $(Tx_n)_n$ is norm convergent to zero for every b-order bounded disjoint sequence $(x_n)_{n\in\mathbb{N}}$ of E_+. Our next theorem characterizes the b-weakly compact operators using the Banach-Saks sets.

Theorem 3.3 *Let E be a Banach lattice and Y a Banach space. If $T : E \to Y$ is a bounded operator, then the following assertions are equivalent:*

(1) *T is b-weakly compact.*
(2) *T carries b-order bounded subsets of E onto Banach-Saks subsets of Y.*

Proof (2) $\Longrightarrow$ (1) According to ([15], Proposition 2.3), every Banach-Saks set is relatively weakly-compact. This leads up to the result.

(1) $\Longrightarrow$ (2) Let B be a b-order bounded subset of E. Since $B^+ := \{x^+, x \in B\}$ and $B^- := \{x^-, x \in B\}$ are both b-order bounded subsets of E_+ and $B \subset B^+ - B^-$, it is enough to show that $T(A)$ is a Banach-Saks subset of Y for each b-order bounded subset A of E_+.

For this, let A be a b-order bounded subset of E^+. If (w_n) is a disjoint sequence in the solid hull of A, then $(w_n)_n$ is also b-order bounded, and therefore $\lim_n \|Tw_n\| = 0$ ([3], Proposition 2.8). Now, let $\epsilon > 0$ be fixed. By Theorem 4.36 in [1], there exists some $u_\epsilon \in E_+$ such that $\|T[(x - u_\epsilon)^+]\| < \epsilon$, for all $x \in A$. Using the equality $x = x \wedge u_\epsilon + (x - u_\epsilon)^+$, we see that $Tx \in T([-u_\epsilon, u_\epsilon]) + \epsilon B_Y$, and hence $T(A) \subseteq T([-u_\epsilon, u_\epsilon]) + \epsilon B_Y$. According to Lemma 3.2 it remains to show that $T[-u_\epsilon, u_\epsilon]$ is Banach-Saks.

Since T is b-weakly compact, in particular it is order weakly compact, it follows from Theorem 3.1 that there exist an order continuous Banach lattice G, a lattice homomorphism $\phi : E \longrightarrow G$ and a bounded operator $R : G \longrightarrow Y$, with $T = R\phi$. Clearly, $\phi[-u_\epsilon, u_\epsilon]$ is an order bounded subset of G. From the order continuity of G, it follows that $\phi[-u_\epsilon, u_\epsilon]$ is L- weakly compact ([1], Theorem 4.14). Therefore, by Lemma 2.3 in [12], $\phi[-u_\epsilon, u_\epsilon]$ is Banach-Saks. Since R is bounded, it is easy to see that $T[-u_\epsilon, u_\epsilon]$ is likewise Banach-Saks, and the proof is concluded. □

If E is an order continuous Banach lattice which has a weak unit, then there exist a probability space (Ω, Σ, μ), an order ideal I of $L_1(\Omega, \Sigma, \mu)$, a lattice norm $\| \,.\, \|_I$ on I and an order isometry j from E onto $(I, \| \,.\, \|_I)$ such that the canonical inclusion from I into $L_1(\Omega, \Sigma, \mu)$ is continuous with $\|f\|_1 \leq \|f\|_I$ (see Theorem 1.b.14 in [14]). This implies that j is continuous as an operator from E into $L_1(\Omega, \Sigma, \mu)$. Note that a separable subspace X of an order continuous Banach lattice E is included in some closed order ideal Y of E with a weak unit (see Proposition 1.a.9 in [14]). Thus, E_X (the closed ideal generated by X) has a weak unit. An operator $T : E \to X$ is $M-$weakly compact if for every bounded disjoint sequence (w_n) we have $\|Tw_n\| \to 0$ ([16]).

At this state of analysis we need this following result.

Theorem 3.4 (Theorem 1.2.8 in [17]) *Let $(x_n)_n$ be a normalized sequence of a Banach lattice E with order continuous norm. Then,*

(1) *either $(\|x_n\|_{L_1})$ is bounded away from zero,*
(2) *or there exist a subsequence (x_{n_k}) and a disjoint sequence $(z_k) \subset E$ such that $\|z_k - x_{n_k}\| \longrightarrow 0$.*

To continue our discussion, we need the next Lemma:

Lemma 3.5 *Let Y be a Banach space and E be a Banach lattices such that E' has an order continuous norm. For every bounded linear operator $T : E \to Y$ the following assertions are equivalent.*

(1) *T is $M-$weakly compact.*
(2) *$\|Tx_n\| \to 0$ as $n \to +\infty$ for every bounded sequence (x_n) of E_+ satisfying $x_n \to 0$ in $\sigma(E, E')$ as $n \to +\infty$.*

Proof (1) $\Rightarrow$ (2) Suppose that T is $M-$weakly compact. Then there exist a reflexive Banach lattice G, an $M-$weakly compact lattice homomorphism $\phi : E \longrightarrow G$ and an $M-$weakly compact operator $R : G \longrightarrow Y$ with $T = R\phi$ (see Exercice 10 page 338 in [1]). Now let (x_n) be a bounded sequence of E_+ satisfying $x_n \to 0$ in $\sigma(E, E')$ as $n \to +\infty$, so that $\phi x_n \to 0$ in $\sigma(G, G')$ as $n \to +\infty$. Since $V := [\phi x_n]$, the closure of the subspace spanned by the vectors $(\phi x_n)_n$, is a separable subspace of G, it follows from Proposition 1.a.9 in [14] that E_V is an order ideal with a weak order unit. By applying ([14], Theorem 1.b.14), we infer that E_V can be represented as a dense order ideal of $L_1(\Omega, \Sigma, \mu)$ for some probability measure μ, such that the formal inclusion

$$i : E_V \hookrightarrow L_1(\Omega, \Sigma, \mu)$$

is continuous. It follows that $(i(\phi x_n))_n$ converges weakly to 0 in $L_1(\Omega, \Sigma, \mu)$. Since $L_1(\Omega, \Sigma, \mu)$ has the positive schur property, $\lim_n \|i(\phi x_n)\|_1 = 0$. Now, let (y_n) be an arbitrary subsequence of (x_n). Since $\lim_n \|i(\phi y_n)\|_1 = 0$, it follows from Theorem 3.4 that

(1) either $\|\phi y_n\|_1 \geq \gamma \|\phi y_n\|$ for some $\gamma > 0$,
(2) or there is a subsequence $(z_n)_{n\in\mathbb{N}}$ of (y_n) and a disjoint sequence (w_n) in the solid hull of (ϕz_n) such that $\|\phi z_n - w_n\| \longrightarrow 0$.

Assume first that (1) is satisfied, then $(\|\phi y_n\|)$ and hence $(\|T y_n\|)$ converges to 0. Next, suppose that (2) is satisfied. Since $\|\phi z_n - w_n\| \to 0$, so $\|T z_n - R w_n\| \to 0$. On the other hand, since the disjoint sequence (w_n) is bounded and R is $M-$weakly compact, then $\lim \|R w_n\| = 0$, which implies $\lim \|T z_n\| = 0$. Thus, we have shown that every subsequence of $(T x_n)$ has a subsequence that is norm convergent to zero. This leads up to $\lim \|T x_n\| = 0$, which concludes the proof.

(2) $\Rightarrow$ (1) This assertion follows from Theorem 2.4.14 in [16].

□

Let E be a Banach lattice, $x'' \in E''$, and let $I_{x''}$ be the principal ideal generated by x'' in E''. By Theorem 4.21 in [1] the ideal $Y_{x''} = E \cap I_{x''}$ under the norm $\|.\|_\infty$ defined by

$$\|x\|_\infty = \inf\{\lambda > 0; \quad |x| \leq \lambda |x''|\}; \quad x \in Y_{x''},$$

is an AM-space.

The next result gives a sequential characterization of b-weakly compact operators in the spirit of ([3], Proposition 2.8) without requiring the sequences to be disjoint.

Theorem 3.6 *Let E be a Banach lattice and Y a Banach space. If $T : E \to Y$ is a bounded operator, then the following assertions are equivalent:*

(1) *T is b-weakly compact.*
(2) *$\|T x_n\| \to 0$ as $n \to +\infty$ for every b-order bounded sequence (x_n) of E_+ satisfying $0 \leq x_n \leq x''$ for all $n \in \mathbb{N}$ and $x_n \to 0$ in $\sigma(Y_{x''}, Y'_{x''})$ as $n \to +\infty$ for some $x'' \in E''$.*

Proof (1) $\Rightarrow$ (2) Let (x_n) be a bounded sequence of E_+ satisfying $0 \leq x_n \leq x''$ for all $n \in \mathbb{N}$ and $x_n \to 0$ in $\sigma(Y_{x''}, Y'_{x''})$ as $n \to +\infty$ for some $x'' \in E''$. Let $T_{x''}$ be the restriction of the operator T to $Y_{x''}$. Since T is b-weakly compact, then $T_{x''}$ is weakly compact. Thus, by Theorem 5.62 in [1], $T_{x''}$ is $M-$weakly compact. Since $Y'_{x''}$ has an order continuous norm, it follows from Lemma 3.5 that $\|Tx_n\| \to 0$.
(2) $\Rightarrow$ (1) Let $(w_n)_n$ be a disjoint sequence of E satisfying $0 \leq w_n \leq x''$ for all $n \in \mathbb{N}$ for some $x'' \in E''$. Since (w_n) is an order bounded sequence of $I_{x''}$ (the principal ideal generated by x'' in E'' under the norm $\|.\|_\infty$), then $w_n \to 0$ in $\sigma(I_{x''}, I'_{x''})$ as $n \to +\infty$ (see Lemma 2.1), an so $\|Tw_n\| \to 0$ as $n \to +\infty$. Consequently, by Proposition 2.8 in [3], T is b-weakly compact.

□

Theorem 3.7 *Let E be a Banach lattice and Y be a Banach space. If $T : E \to Y$ is a bounded operator, then the following assertions are equivalent:*

(1) *T is b-weakly compact.*
(2) *There is no b-order bounded disjoint sequence of unit vectors (w_n) in E such that the restriction of T to the subspace $[w_n]$ is an isomorphism.*

Proof (1) $\Longrightarrow$ (2) Let $(w_n)_n$ be a b-order bounded disjoint sequence of unit vectors in E. Suppose that $T_{|[w_n]}$ is an isomorphism. Since T is b-weakly compact, it follows from Proposition 2.8 in [3] that $\lim_n \|Tw_n\| = 0$, and so $\lim_n \|w_n\| = 0$. This clearly leads to a contradiction.
(2) $\Longrightarrow$ (1) Suppose that T is not b-weakly compact. Again by Proposition 2.8 in [3] there is a positive b-order bounded disjoint sequence (w_n) of unit vectors in E such that $\|Tw_n\| > 1$ for all $n \in \mathbb{N}$. Now, observe that there is some $x'' \in E''$, such that

$$0 \leq \sum_{i=1}^{n} w_i = \vee_{i=1}^{n} w_i \leq x'',$$

and therefore $\|\sum_{i=1}^{n} w_i\| \leq \|x''\|$. The rest of the proof follows from Proposition 2.3.13 in [16].

□

Recall that an operator T between a Banach lattice E and a Banach space Y is said to be *disjointly strictly singular* if, there is no disjoint sequence of non null vectors $(x_n)_n$ in E such that the restriction of T to the subspace $[x_n]$ spanned by the vectors $(x_n)_n$ is an isomorphism [13].

Corollary 3.8 *Let E be a Banach lattice and X a Banach space. Then every disjointly strictly singular operator $T : E \to X$ is b-weakly compact.*

4 Relationships with *b-L*-weakly compact operators

The class of *b-L-weakly compact* operators was introduced by D. Lhaimer et al in their paper [9]. An operator T between two Banach lattices E and F is called *b-L-weakly* *compact*if it maps b-order bounded subsets of E into L-weakly compact subsets of F. The notions of b-weakly compact and b-L-weakly compact operators may coincide. The next result provides a condition for this to happen.

Theorem 4.1 *Let E and F be Banach lattices such that F has an order continuous norm. If $T : E \to F$ is a positive operator, then the following assertions are equivalent:*

(1) *T is b-weakly compact.*
(2) *T carries b-order bounded subsets of E onto Banach-Saks subsets of F.*
(3) *T is b-L-weakly compact.*

Proof (1) $\Leftrightarrow$ (2) : See Theorem 3.3.
(3) $\Rightarrow$ (1) According to ([16], Proposition 3.6.5), every L-weakly compact subset of a Banach lattice is relatively weakly compact. This yields the result.
It remains to show that (1) $\Longrightarrow$ (3).

For this, let A be a b-order bounded subset of E, and let ϵ be given. Arguing as in the proof of Theorem 3.3, we see that

$$TA \subseteq T[-u, u] + \epsilon B_F,$$

for some $u \in E_+$. Since T is positive, $T[-u, u] \subseteq [-Tu, Tu]$. Consequently,

$$TA \subseteq [-Tu, Tu] + \epsilon B_F.$$

Now taking into account the facts that $Tu \in F = F^a$, we conclude that TA is $L-$weakly compact (by Theorem 2.2). Thus T is $b - L-$weakly compact. □

Next, we provide a Grothendieck type characterization of the L-weakly compact sets.

Lemma 4.2 *A subset B of a Banach lattice E is L-weakly compact if and only if for each $\epsilon > 0$ there exist an L-weakly compact subset L of E satisfying*

$$B \subset L + \epsilon B_E.$$

Proof If B is L-weakly compact, then $B \subset B + \epsilon B_X$ for all $\epsilon > 0$.

Conversely, let B be a subset of Banach lattice E such that for each $\epsilon > 0$, there exists an $L-$weakly compact subset L of E satisfying $B \subseteq L + \epsilon B_E$. By Theorem 2.2, we have $L \subseteq [-u, u] + \epsilon B_E$. for some $u \in (E^a)_+$. Consequently, $B \subseteq [-u, u] + 2\epsilon B_E$, and by applying Theorem 2.2 once more, we conclude that B is $L-$weakly compact. □

Recall from [10] that an operator T from a Banach lattice E into a Banach lattice F is called *order L-weakly compact* whenever $T[0, x]$ is an L-weakly compact subset of F for each $x \in E_+$.

Theorem 4.3 *Let E and F be two Banach lattices. If $T : E \to F$ is a bounded operator, then the following assertions are equivalent:*

(1) *T is b-L-weakly compact.*
(2) *T is both order L-weakly compact and b-weakly compact.*

Proof (1) $\Longrightarrow$ (2) Let T be a b-L-weakly compact operator. According to ([16], Proposition 3.6.5), every L-weakly compact subset of F is relatively weakly compact. Then T is b-weakly compact. On the other hand, since $[0, x]$ is b-order bounded for each $x \in E_+$, it follows that T is order L-weakly compact.
(2) $\Longrightarrow$ (1) Let A be a b-order bounded set of E, and let $\epsilon > 0$. Arguing as in the proof of Theorem 3.3 , we see that there exists some $u_\epsilon \in E_+$ such that

$$T(A) \subset T[-u_\epsilon, u_\epsilon] + \epsilon B_F.$$

Since T is order L-weakly compact, $T[-u_\epsilon, u_\epsilon]$ is L-weakly compact subset of F. The rest of the proof follows from Lemma 4.2.

□

Declaration

Conflict of interest The authors declare that they have no competing interests.

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