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Positivity

Singular value inequalities and applications

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Abstract
It is shown among other inequalities that if A, B and X are $n \times n$ complex matrices such that A and B are positive semidefinite, then $s_j(AX - XB) \leq s_j\left(\left(\frac{1}{2}A + \frac{1}{2}A^{1/2}\left|X^*\right|^2 A^{1/2}\right) \oplus \left(\frac{1}{2}B + \frac{1}{2}B^{1/2}\left|X\right|^2 B^{1/2}\right)\right)$ for $j = 1, 2, \ldots, 2n$. Several related singular value inequalities and norm inequalities are also given.

Keywords Concave function · Positive semidefinite matrix · Singular value · Unitarily invariant norm · Inequality

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1 Introduction

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices. For $A \in \mathbb{M}_n$, we denote the eigenvalues of $|A| = (A^*A)^{1/2}$ by $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$, they are called the singular values of A. Note that $s_j(A) = s_j(A^*) = s_j(|A|)$ for $j = 1, 2, \ldots, n$. Note that the spectral (usual operator) norm $\|.\|$ is the largest singular value, i.e. $\|A\| = s_1(A)$, and the Schatten p-norms $\|.\|_p$ are defined interms of the singular values, where $\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{1/p}$ for $1 \leq p \leq \infty$. Apart from the spectral (usual operator) norm and the Schatten p-norms, we have the wider class of unitarily invariant norms $|||.|||$. Unitarily invariant norms are characterized by the invariance property which states that $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n$ and for all unitary matrices U and V. Unitarily invariant norms are increasing functions of singular values (see, e.g., [4] or [9]).

For $A, B, X \in \mathbb{M}_n$, a matrix of the form $AX - XA$ is called a commutator, a matrix of the form $AX - XB$ is called a generalized commutator, a matrix of the form $AX + XA$ is called anticommutator, and a matrix of the form $AX + XB$ is called a

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generalized anticommutator. In this paper, we present singular value inequalities for these types of matrices.

Kittaneh in [11] has proved that if $A, B \in \mathbb{M}_n$ are positive semidefinite, then

$$s_j(A+B) \le s_j\left(\left(A+\left|B^{1/2}A^{1/2}\right|\right) \oplus \left(B+\left|A^{1/2}B^{1/2}\right|\right)\right) \tag{1.1}$$

for $j = 1, 2, \dots, 2n$. Inequality (1.1) can be extended to unitarily invariant norms. For $j = 1$, this inequality is the spectral norm inequality,

$$||A+B|| \le \max\left\{\left\|A+\left|B^{1/2}A^{1/2}\right|\right\|, \left\|B+\left|A^{1/2}B^{1/2}\right|\right\|\right\}. \tag{1.2}$$

Specifying inequality (1.1) to the Schatten p-norms, we get

$$\|A+B\|_p \le \left(\left\|A+\left|B^{1/2}A^{1/2}\right|\right\|_p^p + \left\|B+\left|A^{1/2}B^{1/2}\right|\right\|_p^p\right)^{1/p} \tag{1.3}$$

for $1 \le p \le \infty$. Kittaneh in [10] has been proved that if $A, B \in \mathbb{M}_n$ are positive semidefinite, then

$$|||A+B||| \le |||A \oplus B||| + \left|\left|\left|A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2}\right|\right|\right|. \tag{1.4}$$

It should be mentioned here that inequality (1.4) is trivial consequence of inequality (1.1) by application of triangular inequality. Specifying inequality (1.4) to the spectral norm $\|.\|$, leads to

$$||A+B|| \le \max\{\|A\|, \|B\|\} + \left\|A^{1/2}B^{1/2}\right\|. \tag{1.5}$$

Davidson and Power in [8] has been shown a weaker version of inequality (1.5). Bourin

in [7] provides an equivalent formulation of inequality (1.5). Specifying inequality (1.4) to the Schatten p-norms, we have

$$\|A+B\|_p \le \left(\|A\|_p^p + \|B\|_p^p\right)^{1/p} + 2^{1/p}\left\|A^{1/2}B^{1/2}\right\|_p. \tag{1.6}$$

For recent studies and details for generalizations of singular value inequalities, we refer to [1,2] and [3]. In this paper, we give a remarkable generalizations of the inequalities (1.1), (1.2), and (1.3). Several applications are also given.

2 Main results

To reach our findings, we need the following lemmas. The first lemma has been shown by Bhatia and Kittaneh in [5]. The second lemma has been proved by Bourin in [6]. The third lemma has been given by Bhatia in [4].

Lemma 2.1 *Let $A, B \in \mathbb{M}_n$. Then*

$$s_j(AB^*) \le \frac{1}{2} s_j(A^*A + B^*B)$$

for $j = 1, 2, \ldots, n$

Lemma 2.2 *Let $A, B \in \mathbb{M}_n$ be normal and let f be a nonnegative concave function on $[0, \infty)$. Then*

$$|||f(|A + B|)||| \le |||f(|A|) + f(|B|)|||$$

for every unitarily invariant norm.

Lemma 2.3 *Let $A, B \in \mathbb{M}_n$ such that AB is Hermitian. Then*

$$|||AB||| \le |||Re(BA)|||\,.$$

From now until the end of the paper, we will assume that all functions considered are continuous and all matrices denoted by the symbol A or B are positive semidefinite. Our first result is the following singular value inequality for generalized commutator.

Theorem 2.4 *Let $A, B, X \in \mathbb{M}_n$. Then*

$$s_j(AX - XB) \le s_j\,(K \oplus L) \tag{2.1}$$

for $j = 1, 2, \ldots, 2n$, where

$$K = \frac{1}{2}A + \frac{1}{2}A^{1/2}\left|X^*\right|^2 A^{1/2}$$

and

$$L = \frac{1}{2}B + \frac{1}{2}B^{1/2}\left|X\right|^2 B^{1/2}.$$

Proof Let

$$S = \begin{bmatrix} A^{1/2} & XB^{1/2} \\ 0 & 0 \end{bmatrix},$$

$$R^* = \begin{bmatrix} A^{1/2}X & 0 \\ -B^{1/2} & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} A & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & B^{1/2}\left|X\right|^2 B^{1/2} \end{bmatrix},$$

and

$$N = \begin{bmatrix} A^{1/2}\left|X^*\right|^2 A^{1/2} & -A^{1/2}XB^{1/2} \\ -B^{1/2}X^*A^{1/2} & B \end{bmatrix}.$$

Then for $j = 1, 2, \ldots, 2n$, we have

$$\begin{aligned} s_j(AX - XB) &= s_j(SR^*) \\ &\leq \frac{1}{2} s_j(S^*S + R^*R) (\text{by Lemma 2.1}) \\ &= s_j\left(\frac{1}{2}M + \frac{1}{2}N\right) \\ &= s_j\left(\begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix}\right) \\ &= s_j\left(K \oplus L\right). \end{aligned}$$

Our inequality has thus been proved. □

Remark 2.5 Letting $X = I$ in inequality (2.1), we give

$$s_j(A - B) \leq s_j(A \oplus B) \tag{2.2}$$

for $j = 1, 2, \ldots, 2n$. Inequality (2.2) has been proved by Zhan in [12].

Remark 2.6 Letting $B = A$ in inequality (2.1), we give the following singular value inequality for commutator.

$$s_j(AX - XA) \leq s_j\left(Y \oplus Z\right)$$

for $j = 1, 2, \ldots, 2n$, where

$$Y = \frac{1}{2}A + \frac{1}{2}A^{1/2}\left|X^*\right|^2 A^{1/2}$$

and

$$Z = \frac{1}{2}A + \frac{1}{2}A^{1/2}\left|X\right|^2 A^{1/2}.$$

We present the following generalization of inequality (1.1), which is singular value inequality for generalized anticommutator.

Theorem 2.7 *Let $A, B, X \in \mathbb{M}_n$. Then*

$$s_j(AX + XB) \leq s_j(C \oplus D) \tag{2.3}$$

for $j = 1, 2, \ldots, 2n$, where

$$\begin{aligned} C &= K + \left|B^{1/2}X^*A^{1/2}\right|, \\ D &= L + \left|A^{1/2}XB^{1/2}\right|, \\ K &= \frac{1}{2}A + \frac{1}{2}A^{1/2}\left|X^*\right|^2 A^{1/2}, \end{aligned}$$

and

$$L = \frac{1}{2}B + \frac{1}{2}B^{1/2}|X|^2 B^{1/2}.$$

Proof Let

$$S = \begin{bmatrix} A^{1/2} & XB^{1/2} \\ 0 & 0 \end{bmatrix},$$
$$T = \begin{bmatrix} X^*A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix},$$
$$U = \begin{bmatrix} A & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & B^{1/2}|X|^2 B^{1/2} \end{bmatrix},$$

and

$$V = \begin{bmatrix} A^{1/2}|X^*|^2 A^{1/2} & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & B \end{bmatrix}.$$

Then for $j = 1, 2, \ldots, 2n$, we have

$$\begin{aligned}
s_j(AX + XB) &= s_j(ST^*) \\
&\le \frac{1}{2}s_j(S^*S + T^*T), \text{ (by Lemma2.1)} \\
&= \frac{1}{2}s_j(U + V) \\
&= s_j\left(\begin{bmatrix} K & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & L \end{bmatrix}\right) \\
&= s_j\left(\begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix} + \begin{bmatrix} 0 & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & 0 \end{bmatrix}\right) \\
&\le s_j\left(\begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix} + \begin{bmatrix} \left|B^{1/2}X^*A^{1/2}\right| & 0 \\ 0 & \left|A^{1/2}XB^{1/2}\right| \end{bmatrix}\right) \\
&= s_j\left(\begin{bmatrix} K + \left|B^{1/2}X^*A^{1/2}\right| & 0 \\ 0 & L + \left|A^{1/2}XB^{1/2}\right| \end{bmatrix}\right) \\
&= s_j\left(\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}\right).
\end{aligned}$$

Inequality (2.3) has thus been substantiated. □

Remark 2.8 Letting $X = I$ in inequality (2.3), we give inequality (1.1). In that sense inequality (2.3) is certainly a generalization of inequality (1.1).

We are now in a position to present our next norm inequality, which is a generalization of the generalized anticommutator.

Theorem 2.9 *Let $A, B, X \in \mathbb{M}_n$ and let f be a nonnegative increasing concave function on* $[0, \infty)$. *Then*

$$|||f(|(AX + XB) \oplus 0|)||| \leq |||I \oplus J||| \tag{2.4}$$

for every unitarily invariant norm, where

$$\begin{aligned} I &= \left(f(K) + f\left(\left|B^{1/2}X^*A^{1/2}\right|\right)\right), \\ J &= \left(f(L) + f\left(\left|A^{1/2}XB^{1/2}\right|\right)\right), \\ K &= \frac{1}{2}A + \frac{1}{2}A^{1/2}\left|X^*\right|^2 A^{1/2}, \end{aligned}$$

and

$$L = \frac{1}{2}B + \frac{1}{2}B^{1/2}|X|^2 B^{1/2}.$$

Proof Let

$$\begin{aligned} S &= \begin{bmatrix} A^{1/2} & XB^{1/2} \\ 0 & 0 \end{bmatrix}, \\ T &= \begin{bmatrix} X^*A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix}, \\ E &= \begin{bmatrix} A & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & B^{1/2}|X|^2 B^{1/2} \end{bmatrix}, \end{aligned}$$

and

$$F = \begin{bmatrix} A^{1/2}|X^*|^2 A^{1/2} & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & B \end{bmatrix}.$$

Then for $j = 1, 2, \ldots, 2n$, we have

$$\begin{aligned} s_j(f(|(AX + XB) \oplus 0|)) &= s_j(f(\left|ST^*\right|)) \\ &= f(s_j(ST^*)) \\ &\leq f\left(\frac{1}{2}s_j(S^*S + T^*T)\right) \text{ (by Lemma 2.1)} \\ &= f\left(\frac{1}{2}s_j(E + F)\right) \\ &= s_j\left(f\left(\left|\frac{1}{2}E + \frac{1}{2}F\right|\right)\right) \\ &= s_j\left(f\left(\left|\begin{bmatrix} K & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & L \end{bmatrix}\right|\right)\right). \end{aligned}$$

This implies that,

$$
\begin{aligned}
|||f(|(AX+XB)\oplus 0|)||| &\le \left|\left|\left| f\left(\left|\begin{bmatrix} K & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & L \end{bmatrix}\right|\right)\right|\right|\right| \\
&= \left|\left|\left| f\left(\left| \begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix} + \begin{bmatrix} 0 & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & 0 \end{bmatrix} \right|\right)\right|\right|\right| \\
&\le \left|\left|\left| f\left(\left|\begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix}\right|\right) + f\left(\left| \begin{matrix} 0 & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & 0 \end{matrix} \right|\right)\right|\right|\right|, \\
&\text{(by Lemma 2.2)}, \\
&\le \left|\left|\left| f\left(\begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix}\right) + f\left(\begin{bmatrix} \left|B^{1/2}X^*A^{1/2}\right| & 0 \\ 0 & \left|A^{1/2}XB^{1/2}\right| \end{bmatrix}\right)\right|\right|\right| \\
&= \left|\left|\left| \begin{bmatrix} f(K) & 0 \\ 0 & f(L) \end{bmatrix} + \begin{bmatrix} f\left(\left|B^{1/2}X^*A^{1/2}\right|\right) & 0 \\ 0 & f\left(\left|A^{1/2}XB^{1/2}\right|\right) \end{bmatrix}\right|\right|\right| \\
&= \left|\left|\left| \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}\right|\right|\right| \\
&= |||I \oplus J|||,
\end{aligned}
$$

which is precisely inequality (2.4). □

Remark 2.10 Letting $f(t)=t$ in inequality (2.4), we give norm inequality for generalized anticommutator. In that sense, inequality (2.4) is certainly a generalization of generalized anticommutator norm inequalities.

Specifying inequality (2.4) to the spectral norm and the Schatten p-norms, we give the following norm inequalities for generalized anticommutator which are generalizations of the inequalities (1.2) and (1.3), respectively.

Corollary 2.11 *Let $A, B, X \in \mathbb{M}_n$. Then*

$$\|AX+XB\| \le \max\left\{\left\|K+\left|B^{1/2}X^*A^{1/2}\right|\right\|, \left\|L+\left|A^{1/2}XB^{1/2}\right|\right\|\right\} \tag{2.5}$$

where

$$K=\frac{1}{2}A+\frac{1}{2}A^{1/2}\left|X^*\right|^2A^{1/2}$$

and

$$L = \frac{1}{2}B + \frac{1}{2}B^{1/2}\left|X\right|^2 B^{1/2}.$$

Proof Inequality (2.3) follows by substituting $f(t) = t$ and by considering the spectral norm in Theorem 2.9. □

Remark 2.12 Letting $X = I$ in Corollary 2.11, we give inequality (1.2).

Corollary 2.13 *Let $A, B, X \in \mathbb{M}_n$. Then for $1 \le p \le \infty$, we have*

$$\|AX + XB\|_p \le \left(\left\|K + \left|B^{1/2}X^*A^{1/2}\right|\right\|_p^p + \left\|L + \left|A^{1/2}XB^{1/2}\right|\right\|_p^p\right)^{1/p} \quad (2.6)$$

where

$$K = \frac{1}{2}A + \frac{1}{2}A^{1/2}\left|X^*\right|^2 A^{1/2}$$

and

$$L = \frac{1}{2}B + \frac{1}{2}B^{1/2}\left|X\right|^2 B^{1/2}.$$

Proof Inequality (2.6) follows by substituting $f(t) = t$ and by considering the Schatten p-norms in Theorem 2.9. □

Remark 2.14 Letting $X = I$ in Corollary 2.13, we give inequality (1.3).

The following two corollaries are applications of Theorem 2.9.

Corollary 2.15 *Let $A, B, X \in \mathbb{M}_n$. Then*

$$|||\log\left(|(AX + XB)| + I\right)||| \le |||M \oplus N|||$$

for every unitarily invariant norm, where

$$M = \left(\log(K + I) + \log\left(\left|B^{1/2}X^*A^{1/2}\right| + I\right)\right),$$
$$N = \left(\log(L + I) + \log\left(\left|A^{1/2}XB^{1/2}\right| + I\right)\right),$$
$$K = \frac{1}{2}A + \frac{1}{2}A^{1/2}\left|X^*\right|^2 A^{1/2},$$

and

$$L = \frac{1}{2}B + \frac{1}{2}B^{1/2}\left|X\right|^2 B^{1/2}.$$

Proof The inequality is an immediate consequence of Theorem 2.9 by letting $f(t) = \log(t+1)$. □

Corollary 2.16 *Let $A, B, X \in \mathbb{M}_n$. Then, for $r \in (0,1]$, we have*

$$\left|\left|\left| |(AX + XB)|^r \right|\right|\right| \le |||P \oplus Q|||$$

for every unitarily invariant norm, where

$$P = \left(K^r + \left|B^{1/2}X^*A^{1/2}\right|^r\right),$$
$$Q = \left(L^r + \left|A^{1/2}XB^{1/2}\right|^r\right),$$
$$K = \frac{1}{2}A + \frac{1}{2}A^{1/2}\left|X^*\right|^2 A^{1/2},$$

and

$$L = \frac{1}{2}B + \frac{1}{2}B^{1/2}\left|X\right|^2 B^{1/2}.$$

Proof The inequality is an immediate consequence of Theorem 2.9 by letting $f(t) = t^r$ and $r \in (0,1]$. □

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Conflict of interest The author declare that she/he have no conflict of interest.

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