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Pseudo metric subregularity and its stability in Asplund spaces

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Abstract
As a variant of metric subregularity, pseudo metric subregularity is studied via general limit critical sets using the techniques of variational analysis. In terms of limit critical sets, we provide some sufficient conditions for the validity of pseudo/Hölder metric subregularity. Usually, the property of pseudo metric subregularity is not stable under small smooth perturbation. We provide a characterization for pseudo metric subregularity to be stable under small $C^{1,p}$ smooth perturbation. In particular, some existing results on metric subregularity are extended to pseudo metric subregularity. Finally, we consider the pseudo weak sharp minimizer of a proper lower semicontinuous function and its relation with pseudo metric subregularity of the corresponding subdifferential mapping.

Keywords Pseudo/Hölder metric subregularity · Regular normal cone · Multifunctions · Stability

Mathematics Subject Classification 49J53 · 90C31 · 90C48

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1 Introduction

It is well known that metric (sub)regularity plays an important role in many fields of nonlinear analysis as well as its applications, i.e., optimization, constraint qualification conditions and stability analysis etc. (for more details see [1,5–7,14,21,22,27] and references therein). Let X, Y be Banach spaces, $F : X \rightrightarrows Y$ be a multifunction and $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$. Recall that F is metrically regular at $(\bar{x}, \bar{y})$ if there exist $\tau, \delta \in (0, +\infty)$ such that

$$\tau d(x, F^{-1}(y)) \leq d(y, F(x)) \ \ \forall (x, y) \in B(\bar{x}, \delta) \times B(\bar{y}, \delta). \tag{1.1}$$

The supremum of τ over all such combinations of τ and δ is called the regularity modulus for F at $(\bar{x}, \bar{y})$ and denoted by $\mathrm{reg}(F, \bar{x}, \bar{y})$ (cf. [10,11,16]). It is well known that metric regularity is persistent with respect to small Lipschitz perturbation (cf. [2, Theorem 3.3]), i.e., if $F : X \rightrightarrows Y$ is a closed multifunction and is metrically regular at $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$, then for any $f : X \to Y$, which is locally Lipschtiz continuous around $\bar{x}$ with $\mathrm{lip}(f, \bar{x}) < \mathrm{reg}(F, \bar{x}, \bar{y})$, $F + f$ is metrically regular at $(\bar{x}, \bar{y} + f(\bar{x}))$.

Fixing $y = \bar{y}$ in (1.1), we obtain the following weaker version of metric regularity of F at $(\bar{x}, \bar{y})$, i.e.,

$$\tau d(x, F^{-1}(\bar{y})) \leq d(\bar{y}, F(x)) \ \ \forall x \in B(\bar{x}, \delta) \tag{1.2}$$

for some $\tau, \delta \in (0, +\infty)$, which is called metric subregularity and is closely related to properties such as calmness, weak sharp minima and error bound. The latter has important applications in sensitivity and convergence analysis of mathematical programming (cf. [9,12,26]). Unlike metric regularity, the property of metric subregularity is usually unstable even under small smooth perturbation (see [5, Example 1.2]). For this, Gfrerer introduced the so-called limit critical set of a multifunction F, in terms of which a point-based characterization is obtained for stability of metric subregularity of F under small C^1 perturbations in Asplund spaces (cf. [5,8]).

However, the metric subregularity is quite restrictive in some applications. A useful variant of metric subregularity is the following Hölder metric subregularity: F is said to be Hölder metrically subregular of order p (with $p \in [1, +\infty)$) at $(\bar{x}, \bar{y})$, if there exists $\tau, \delta \in (0, +\infty)$ such that

$$\tau d(x, F^{-1}(\bar{y}))^p \leq d(\bar{y}, F(x)) \ \ \forall x \in B(\bar{x}, \delta) \tag{1.3}$$

(cf. [4,13,14,25,30] and references therein). In terms of coderivative, Li and Mordukhovich [14] provided sufficient conditions for Hölder metric subregularity, while Kruger [13] uses slope to study it. In [30], Zheng and Zhu studied a more general concept called generalized metric subregularity. Using $\bar{x}$ instead of $F^{-1}(\bar{y})$ in (1.3),we obtain the Hölder strong metric subregularity, i.e. F is said to be Hölder strongly metrically subregular of order p (with $p \in [1, +\infty)$) at $(\bar{x}, \bar{y})$, if there exists $\tau, \delta \in (0, +\infty)$ such that

$$\tau \|x - \bar{x}\|^p \leq d(\bar{y}, F(x)) \ \ \forall x \in B(\bar{x}, \delta). \tag{1.4}$$

It is clear that the validity of (1.4) is equivalent to the fact that (1.3) holds with $\bar{x}$ being an isolated point of $F^{-1}(\bar{y})$.

In [7], Gfrerer introduced a new concept called pseudo metric subregularity with directions and Ngai et al. [23] studied a more general case of pseudo metric subregularity with directions. For convenience, we consider the next notion (without directions): F is said to be pseudo metrically subregular of order p (with $p \in [1, +\infty)$) at $(\bar{x}, \bar{y})$, if there exists $\tau, \delta \in (0, +\infty)$ such that

$$\tau \|x - \bar{x}\|^{p-1} d(x, F^{-1}(\bar{y})) \leq d(\bar{y}, F(x)) \ \ \forall x \in B(\bar{x}, \delta). \tag{1.5}$$

It is clear that Höder/pseudo metric subregularity of order one goes back to metric subregularity. To some extent, pseudo metric subregularity can be understood as metric subregularity with modulus τ behaving like $O(\|x-\bar{x}\|^{p-1})$ as $x \to \bar{x}$ (for more details see [7]). Note that $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$, it is clear that pseudo metric subregularity implies (usually strictly stronger, see [25, Example, page 1975]) Hölder metric subregularity and is weaker (usually strictly weaker) than Hölder strong metric subregularity. The following example helps to illustrate this fact in detail:

Example 1.1 Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $(\bar{x}, \bar{y}) = ((0, 0), 0)$ and $F(s, t) = [s^2 - t^2, +\infty)$ for all $(s, t) \in \mathbb{R}^2$. Then F is pseudo metrically subregular of order 2 at $(\bar{x}, \bar{y})$, but is not Hölder strongly metrically subregular of order 2 at the referred point. Indeed, $F^{-1}(0) = \{(s, t) \in \mathbb{R}^2 : s^2 \leq t^2\}$ and $(0, 0)$ is not the isolated point of $F^{-1}(0)$, so F is not Hölder strongly metrically subregular of order 2 at $((0, 0), 0)$. For any $(s, t) \notin F^{-1}(0)$, one has $|s| > |t|$. If $s > t \geq 0$, we have

$$\begin{aligned} \|(s, t) - (0, 0)\| d((s, t), F^{-1}(0)) &= \frac{\sqrt{2}}{2}\sqrt{s^2 + t^2}|s - t| \leq \frac{\sqrt{2}}{2}|s + t||s - t| \\ &= \frac{\sqrt{2}}{2} d(0, F(s, t)). \end{aligned} \tag{1.6}$$

If $s > -t \geq 0$ or $-s > t \geq 0$ or $s < t \leq 0$, it is also easy to calculate that (1.6) holds. This implies that F is pseudo metrically subregular of order 2 at $((0, 0), 0)$.

To study pseudo metric subregularity, Gfrerer [7, Theorem 1-(2)] provided a sufficient condition through an approach similar to the limit critical set. Usually, the property of pseudo metric subregularity is also unstable under small perturbations. Let $C^1(X, Y, \bar{x})$ denote the set of all mappings from X to Y which are Fréchet continuously differentiable on some neighbourhood of $\bar{x}$. For a mapping $f : X \to Y$ and $\bar{x} \in X$, we say that f is $C^{1,p}$ around $\bar{x}$ if $f \in C^1(X, Y, \bar{x})$ and for any $\varepsilon \in (0, +\infty)$, there exists $\delta \in (0, +\infty)$ such that

$$\|\nabla f(x) - \nabla f(\bar{x})\| \leq \varepsilon \|x - \bar{x}\|^{p-1} \ \forall x \in B(\bar{x}, \delta).$$

Let $C^{1,p}(X, Y, \bar{x})$ denote the set of all mappings from X to Y which are $C^{1,p}$ around $\bar{x}$.

The following example demonstrates that, even for a convex multifunction in finite dimensional space, the property of pseudo metric subregularity is unstable under small $C^{1,p}$ smooth perturbation.

Example 1.2 Let $p \in [1, +\infty)$ be fixed. Consider the convex multifunction $F : \mathbb{R} \rightrightarrows \mathbb{R}$ with $F(x) := [0, +\infty)$ for $x \in (-\infty, 0)$ and $F(x) := [x^p, +\infty)$ for $x \in [0, +\infty)$. Let $\bar{x} = \bar{y} = 0$, then $F^{-1}(0) = (-\infty, 0]$ and $d(0, F(x)) = x^p$ for all $x \in (0, +\infty)$. It is easy to see that F is pseudo metrically subregular of order p at $(0, 0)$. Define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) := |x|^{1+p}$ for all $x \in \mathbb{R}$, then we have

$$f'(x) := \begin{cases} (1+p)x^p, & if\ x \in (0, +\infty), \\ 0, & if\ x = 0, \\ -(1+p)(-x)^p, & if\ x \in (-\infty, 0), \end{cases}$$

and $|f'(x) - f'(0)|/|x - 0|^{p-1} \leq (1 + p)|x| \to 0$ when $x \to 0$. This shows that f is $C^{1,p}$ around 0 with $f(0) = 0$ and $f'(0) = 0$. However, $F + f$ is no longer pseudo metrically subregular of order p at $(0, 0)$. Indeed, for $x_k = -\frac{1}{k}$, $k \in \mathbb{N}$, it is clear that $(F + f)^{-1}(0) = \{0\}$, $d(x_k, (F + f)^{-1}(0)) = |x_k| = \frac{1}{k}$ and $d(0, (F + f)(x_k)) = |x_k|^{1+p} = \frac{1}{k^{1+p}}$, hence $\frac{d(0,(F+f)(x_k))}{\|x_k - 0\|^{p-1} d(x_k,(F+f)^{-1}(0))} = \frac{1}{k} \to 0$.

Hence, it is natural and essential to investigate qualifications on multifunctions for which pseudo metric subregularity remains true under certain smooth perturbations. For convenience, we define two different kinds of stability for pseudo metric subregularity as below:

Definition 1.1 Let $F : X \rightrightarrows Y$ be a multifunction, $(\bar{x}, \bar{y}) \in \text{gph(F)}$ and $p \in [1, +\infty)$.

(i) We say that $F : X \rightrightarrows Y$ is pseudo metrically subregular of order p stable at $(\bar{x}, \bar{y})$ under $C^{1,p}$ perturbation, if for any $f \in C^{1,p}(X, Y, \bar{x})$ with $f(\bar{x}) = 0$ and $\nabla f(\bar{x}) = 0$, the mapping $F + f$ is pseudo metrically subregular of order p at $(\bar{x}, \bar{y})$;

(ii) We say that F is pseudo metrically subregular of order p stable at $(\bar{x}, \bar{y})$ under p-bounded smooth perturbation, if there exists $c \in (0, +\infty)$ such that, for any $f \in C^1(X, Y, \bar{x})$ with $f(\bar{x}) = 0$ and $\|\nabla f(x)\| \leq c\|x - \bar{x}\|^{p-1}$ for all x sufficiently close to $\bar{x}$, the mapping $F + f$ is pseudo metrically subregular of order p at $(\bar{x}, \bar{y})$.

It is clear that the stability of pseudo metric subregularity under p-bounded smooth perturbation implies the one under $C^{1,p}$ perturbation. It is worth to note that the condition $\|\nabla f(x)\| \leq c\|x - \bar{x}\|^{p-1}$ for all x sufficiently close to $\bar{x}$ ensures $\|\nabla f(\bar{x})\| = 0$ for any $p \in (1, +\infty)$, but when $p = 1$, this condition reduces to the boundedness of $\nabla f(x)$ around $\bar{x}$.

The rest of the paper is organized as follows. Section 2 contains basic definitions and required preliminary results used in what follows. In Sect. 3, via new-defined general limit critical sets, we provide some sufficient conditions for pseudo metric subregularity. We show that two kinds of stability for pseudo metric subregularity in Definition 1.1 are equivalent and furthermore provide a characterization for the aforementioned property, which is also generalization of [8, Theorem 2.8 ($(i) \Leftrightarrow (ii)$)]. In Sect. 4, we consider the pseudo weak sharp minimizer of a proper lower semicontinuous function and its relation with pseudo metric subregularity of the corresponding subdifferential mapping.

2 Notations and preliminaries

In this section, we summarize some fundamental notations and tools in variational analysis, more details see [17,20,24]. Recall that a Banach space X is called an Asplund space if every continuous convex function on X is Fréchet differentiable at each point of a dense subset of X. It is well known that X is an Asplund space if and only if every separable subspace of X has a separable dual space. In particular, every reflexive Banach space is an Asplund space. Let X be an Asplund space with topological dual X^* and B_X and S_X denote the closed unit ball and unit sphere of X, respectively. We denote by $B(x, r)$ ($B[x, r]$) the open (closed) ball with center x and radius r and $d(x, A) := \inf_{a\in A} d(x, a)$ the point-to-set distance from x to A (in the usual convention, the infimum of the empty set equals $+\infty$). For a proper lower semi-continuous function $\phi : X \to \mathbb{R} \cup \{+\infty\}$, let $\mathrm{dom}(\phi)$ and $\mathrm{epi}(\phi)$ denote the domain and the epigraph of ϕ, respectively, that is,

$$\mathrm{dom}(\phi) := \{x \in X|\ \phi(x) < +\infty\} \text{ and } \mathrm{epi}(\phi) := \{(x, r) \in X \times \mathbb{R}|\phi(x) \le r\}.$$

For $x \in \mathrm{dom}(\phi)$, recall that regular subdifferential (Fréchet subdifferential) $\hat{\partial}\phi(\bar{x})$ of ϕ at x is defined as

$$\hat{\partial}\phi(x) := \left\{x^* \in X^*|\ \liminf_{x'\to x} \frac{\phi(x') - \phi(x) - \langle x^*, x' - x\rangle}{\|x' - x\|} \ge 0\right\}.$$

When f is convex, regular subdifferentials reduce to the one in the sense of convex analysis; that is,

$$\hat{\partial}\phi(x) = \partial\phi(x) := \{x^* \in X^*|\langle x^*, x' - x\rangle \le \phi(x') - \phi(x)\ \forall x' \in X\}\ \forall x \in \mathrm{dom}(\phi).$$

For a closed subset A of X and a point a in A, let $\hat{N}(A, a)$ denote the regular normal cone (Fréchet normal cone) of A to a, defined by

$$\hat{N}(A, a) := \left\{x^* \in X^*|\ \limsup_{x \xrightarrow{A} a} \frac{\langle x^*, x - a\rangle}{\|x - a\|} \le 0\right\}.$$

Let the indicator function δ_A be defined as

$$\delta_A(x) := \begin{cases} 0, & if\ x \in A, \\ +\infty, & otherwise. \end{cases}$$

It well known that $\hat{N}(A, a) = \hat{\partial}\delta_A(a)$ for all $a \in A$ and

$$\hat{\partial}\phi(x) = \{x^* \in X^*|(x^*, -1) \in \hat{N}(\mathrm{epi}(\phi), (x, \phi(x)))\}\ \forall x \in \mathrm{dom}(\phi).$$

If A is convex, then

$$\hat{N}(A, a) = \{x^* \in X^*|\langle x^*, x - a\rangle \le 0\ \ \forall x \in A\}\ \forall a \in A.$$

Let $F : X \rightrightarrows Y$ be a multifunction and its graph and domain be defined as

$$\mathrm{gph}(F) := \{(x, y) \in X \times Y |\ y \in F(x)\ \ \forall x \in X\} \text{ and } \mathrm{dom}(F) := \{x \in X | F(x) \neq \emptyset\}.$$

We say that F is a closed multifunction if its graph $\mathrm{gph}(F)$ is closed in the product space $X \times Y$. We will use the following notion of the coderivative which were first constructed by Mordukhovich (cf. [15]). For $(x, y) \in \mathrm{gph}(F)$, the coderivative $\hat{D}^* F(x, y)$ is a multifunction between Y^* and X^* defined by

$$\hat{D}^* F(x, y)(y^*) := \{x^* \in X^* | (x^*, -y^*) \in \hat{N}(\mathrm{gph}(F), (x, y))\}\ \ \forall y^* \in Y^*.$$

The details of the coderivatives can be found in [17]. For a closed multifunction $F : X \rightrightarrows Y$ and a single-valued mapping $f : X \to Y$, if $(x, y) \in \mathrm{gph}(F + f)$ and f is Fréchet differentiable at x, then one has that (cf. [17, Theorem 1.62]):

$$(x^*, -y^*) \in \hat{N}(\mathrm{gph}(F + f), (x, y)) \Leftrightarrow (x^* - \nabla f(x)^* y^*, -y^*) \in \hat{N}(\mathrm{gph}(F), (x, y - f(x))). \tag{2.1}$$

Recall that the duality mapping $J : Y \rightrightarrows Y^*$ denotes the normal dual mapping of Y, that is,

$$J(y) := \{y^* \in S_{Y^*} | \langle y^*, y\rangle = \|y\|\}\ \ \forall y \in Y \backslash \{0\}.$$

It is clear that $J(y) = \partial \|\cdot\|(y)$ and, when Y is smooth, $J(y)$ is single-valued and $J(y) = \nabla \|\cdot\|(y)$ for all $y \in Y \backslash \{0\}$. For $\bar{x} \in X$ and $A \subset X$, let $\mathrm{P}_A(\bar{x})$ denote the projection from $\bar{x}$ to A, that is

$$\mathrm{P}_A(\bar{x}) := \{x \in A | \|x - \bar{x}\| = d(\bar{x}, A)\}.$$

For convenience, we use the following notations (cf. [27,30]). Let $\varepsilon \in (0, +\infty)$, the normalized ε-enlargement of the duality mapping is

$$J_\varepsilon(y) := \{y^* \in S_{Y^*} | d(y^*, J(y)) < \varepsilon\}\ \forall y \in Y \backslash \{0\}$$

and the ε-approximation of $\bar{x}$ to A is defined as

$$P_A^\varepsilon(\bar{x}) := \{x \in A | \|x - \bar{x}\| < \min\{(1 + \varepsilon) d(\bar{x}, A), d(\bar{x}, A) + \varepsilon\}.$$

Several kinds of subdifferential sum rules are employed in the main results of this paper. Below we provide these rules for completeness (cf. [17]).

Lemma 2.1 *Suppose that X is an Asplund space and $\phi, \psi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous function. Let $\bar{x} \in \mathrm{dom}(\psi)$ such that ϕ is Lipschitz continuous at $\bar{x}$. Then the following statements hold:*
(i) *For any $x^* \in \hat{\partial}(\phi + \psi)(\bar{x})$ and any $\sigma \in (0, +\infty)$, there exist $x_1, x_2 \in B(\bar{x}, \sigma)$ such that $|\psi(x_2) - \psi(\bar{x})| < \sigma$ and*

$$x^* \in \hat{\partial}\phi(x_1) + \hat{\partial}\psi(x_2) + \sigma B_{X^*}.$$

(ii) *If ϕ is Fréchet differentiable at $\bar{x}$ with derivative $\nabla\phi(\bar{x})$, then*

$$\hat{\partial}(\phi+\psi)(\bar{x}) = \nabla\phi(\bar{x}) + \hat{\partial}\psi(\bar{x}).$$

We conclude this section with the the Ekeland variational principle (cf. [3,17]), which plays a key role in the proof of the main result.

Lemma 2.2 (Ekeland variational principle) *Suppose that X is a Banach space, and $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and bounded from below. Let $\varepsilon > 0$ and $\bar{x} \in X$ be given such that*

$$f(\bar{x}) < \inf_{x \in X} f(x) + \varepsilon.$$

Then for any $\lambda > 0$, there exists $x \in X$ satisfying

(i) $\|x - \bar{x}\| < \lambda$,
(ii) $f(x) \leq f(\bar{x})$,
(iii) $f(x) < f(u) + \frac{\varepsilon}{\lambda}\|u - x\|$ *for all* $u \in X \setminus \{x\}$.

3 The stability of pseudo metric subregularity under smooth disturbance

Without any other statement, throughout the remainder of this paper, we always assume that X and Y are Asplund spaces and $F : X \rightrightarrows Y$ is a closed multifunction.

For convenience, for $(\bar{x}, \bar{y}) \in X \times Y$ and $\delta, \varepsilon > 0$, we set

$$\mathcal{B}(F, \bar{x}, \bar{y}, \delta, \varepsilon) := \{(x, y) \in B(\bar{x}, \delta) \times B(\bar{y}, \delta) | x \notin F^{-1}(\bar{y}), y \in P^{\varepsilon}_{F(x)}(\bar{y})\},$$
$$\mathcal{C}(F, \bar{x}, \bar{y}, \beta, p) := \{(x, y) \in \mathrm{gph}(F) | \|y - \bar{y}\| < \beta\|x - \bar{x}\|^{p-1} d(x, F^{-1}(\bar{y}))\}$$

and

$$\mathcal{D}(F, \bar{x}, \bar{y}, \beta, p) := \{(x, y) \in \mathrm{gph}(F) | \|y - \bar{y}\| < \beta d(x, F^{-1}(\bar{y}))^p\}.$$

We first state the following lemma which provides a new sufficient condition for pseudo/Hölder metric subregularity of order p and estimates its modulus for a generalized multifunction.

Lemma 3.1 *Let $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$, $\alpha, \beta, \varepsilon, \delta \in (0, +\infty)$ and $p \in [1, +\infty)$.*
(i) *If*

$$d(0, \hat{D}^* F(x, y)(J_\varepsilon(y' - \bar{y}))) \geq \alpha\|x - \bar{x}\|^{p-1} \ \forall (x, y) \in \mathcal{B}(F, \bar{x}, \bar{y}, \delta, \varepsilon) \cap \mathcal{C}(F, \bar{x}, \bar{y}, \beta, p), \quad y' \in B(y, \min\{\varepsilon, \|y - \bar{y}\|^2\}) \backslash \{\bar{y}\}. \tag{3.1}$$

Then F is pseudo metrically subregular of order p at $(\bar{x}, \bar{y})$, explicitly,

$$\kappa\|x - \bar{x}\|^{p-1} d(x, F^{-1}(\bar{y})) \leq d(\bar{y}, F(x)) \ \forall x \in B(\bar{x}, \delta'), \tag{3.2}$$

where

$$\kappa := \min\left\{\frac{\alpha}{3\cdot 2^{p-1}}, \frac{\beta}{2^p}\right\} \text{ and } \delta' := \min\left\{\frac{2\delta}{3}, \left(\frac{\delta}{\kappa}\right)^{\frac{1}{p}}\right\}. \tag{3.3}$$

If, in addition, Y is a Fréchet smooth Banach space, then the condition (3.1) *can be replaced by*

$$d(0, \hat{D}^*F(x,y)(J_\varepsilon(y-\bar{y})) \geq \alpha\|x-\bar{x}\|^{p-1} \ \forall (x,y) \in \mathcal{B}(F,\bar{x},\bar{y},\delta,\varepsilon)\cap\mathcal{C}(F,\bar{x},\bar{y},\beta,p).$$

(ii) *If*

$$\begin{aligned} &d(0, \hat{D}^*F(x,y)(J_\varepsilon(y'-\bar{y})) \\ &\quad \geq \alpha d(x, F^{-1}(\bar{y}))^{p-1} \ \forall (x,y) \in \mathcal{B}(F,\bar{x},\bar{y},\delta,\varepsilon)\cap\mathcal{D}(F,\bar{x},\bar{y},\beta,p), \\ &y' \in B(y, \min\{\varepsilon, \|y-\bar{y}\|^2\})\backslash\{\bar{y}\}, \end{aligned} \tag{3.4}$$

Then F is Hölder metrically subregular of order p at $(\bar{x},\bar{y})$*, explicitly,*

$$\kappa d(x, F^{-1}(\bar{y}))^p \leq d(\bar{y}, F(x)) \ \forall x \in B(\bar{x},\delta'),$$

where κ *and* δ' *are defined as in* (3.3)*. If, in addition, Y is a Fréchet smooth Banach space, then the condition* (3.4) *can be replaced by*

$$\begin{aligned} &d(0, \hat{D}^*F(x,y)(J_\varepsilon(y-\bar{y})) \\ &\quad \geq \alpha d(x, F^{-1}(\bar{y}))^{p-1} \ \forall (x,y) \in \mathcal{B}(F,\bar{x},\bar{y},\delta,\varepsilon)\cap\mathcal{D}(F,\bar{x},\bar{y},\beta,p). \end{aligned}$$

Proof We are going to show (i), while the validity of (ii) follows similarly. In order to show that (3.2) holds, we assume to the contrary that there exists $x' \in B(\bar{x},\delta')$ such that $\kappa\|x'-\bar{x}\|^{p-1}d(x', F^{-1}(\bar{y})) > d(\bar{y}, F(x'))$. Take $y' \in F(x')$ and $\tau \in (0,\kappa)$ sufficiently close to κ such that

$$d(\bar{y}, F(x')) \leq \|y'-\bar{y}\| < \tau\lambda\|x'-\bar{x}\|^{p-1}, \tag{3.5}$$

where $\lambda = d(x', F^{-1}(\bar{y}))$. Let $\eta \in (0,1)$ be small enough such that

$$0 < \frac{4\tau\eta\delta^{p-1}+\eta}{1-2\tau\eta\delta^{p-1}-\eta}\max\{1,\delta\} < \varepsilon, 0 < \frac{\tau}{1-3\tau\eta\delta^{p-1}} < \kappa \text{ and } \frac{6\tau\eta\delta^{p-1}}{1-3\tau\eta\delta^{p-1}} < \varepsilon. \tag{3.6}$$

Define the norm on the product space $X \times Y$ as

$$\|(x,y)\|_\eta := \|x\| + \eta\|y\| \ \forall (x,y) \in X \times Y.$$

Let $\varphi : X \times Y \to [0,+\infty]$ be defined by

$$\varphi(x,y) := \|y-\bar{y}\| + \delta_{\mathrm{gph(F)}}(x,y) \ \forall (x,y) \in X \times Y.$$

Clearly, φ is lower semicontinuous (due to the closeness of gph(F)) and attains its minimum value at $(\bar{x}, \bar{y})$. According to (3.5), we easily obtain that

$$\varphi(x', y') < \inf_{(x,y)\in X\times Y} \varphi(x, y) + \tau\lambda\|x' - \bar{x}\|^{p-1}.$$

It then follows from Lemma 2.2 that there exists $(\tilde{x}, \tilde{y}) \in \text{gph(F)}$ such that

$$\|(\tilde{x}, \tilde{y}) - (x', y')\|_\eta < \frac{\lambda}{2}, \tag{3.7}$$

$$\varphi(\tilde{x}, \tilde{y}) \leq \varphi(x', y') \tag{3.8}$$

and

$$\varphi(\tilde{x}, \tilde{y}) \leq \varphi(x, y) + 2\tau\|x' - \bar{x}\|^{p-1}(\|x - \tilde{x}\| + \eta\|y - \tilde{y}\|)\ \forall (x, y) \in X \times Y \tag{3.9}$$

Note that $\lambda = d(x', F^{-1}(\bar{y}))$, by (3.7) and (3.8), one has $\|\tilde{y} - \bar{y}\| \leq \|y' - \bar{y}\|$ and

$$\|\tilde{x} - x'\| < \frac{\lambda}{2} \leq \frac{\|x' - \bar{x}\|}{2} < \frac{\delta'}{2}, \tag{3.10}$$

which leads to the fact that

$$\|\tilde{x} - \bar{x}\| \leq \|\tilde{x} - x'\| + \|x' - \bar{x}\| < \frac{3\delta'}{2} \leq \delta. \tag{3.11}$$

Using the triangle inequality, we obtain from (3.10) that

$$\begin{aligned} d(\tilde{x}, F^{-1}(\bar{y})) \geq d(x', F^{-1}(\bar{y})) - \|\tilde{x} - x'\| > \frac{\lambda}{2} > 0 \text{ and } \|\tilde{x} - \bar{x}\| \\ \geq \|x' - \bar{x}\| - \|\tilde{x} - x'\| > \frac{\|x' - \bar{x}\|}{2}. \end{aligned} \tag{3.12}$$

Then $\tilde{x} \notin F^{-1}(\bar{y})$, and hence $\tilde{y} \neq \bar{y}$. Let

$$\begin{aligned} \sigma \in \Bigg(0, \min\Bigg\{\|\tilde{x} - \bar{x}\| - \frac{\|x' - \bar{x}\|}{2}, \delta - \|\tilde{x} - \bar{x}\|, d(\tilde{x}, F^{-1}(\bar{y})) - \frac{\lambda}{2}, \frac{\eta\varepsilon}{2}, \tau\|x' - \bar{x}\|^{p-1}, \\ \eta(\tau\lambda\|x' - \bar{x}\|^{p-1} - \|y' - \bar{y}\|), \frac{\eta\|\tilde{y} - \bar{y}\|}{2}, \frac{\eta^2\|\tilde{y} - \bar{y}\|}{8\tau\eta\delta^{p-1} + 2}, \frac{\eta\|\tilde{y} - \bar{y}\|^2}{8}\Bigg\}\Bigg). \end{aligned} \tag{3.13}$$

be sufficiently small and pick any $(u, v) \in \text{gph}(F) \cap B((\tilde{x}, \tilde{y}), \sigma)$. Then $v \in F(u)$, $\|u - \tilde{x}\| < \sigma$ and $\|v - \tilde{y}\| < \sigma/\eta$. It follows from (3.12) and (3.13) that $\|u - \bar{x}\| < \|\tilde{x} - \bar{x}\| + \sigma < \delta$,

$$d(u, F^{-1}(\bar{y})) > d(\tilde{x}, F^{-1}(\bar{y})) - \sigma > \frac{\lambda}{2} > 0 \text{ and } \|u - \bar{x}\| > \|\tilde{x} - \bar{x}\| - \sigma > \frac{\|x' - \bar{x}\|}{2}. \tag{3.14}$$

Together with (3.5), (3.8) and (3.13), we have that $u \in B(\bar{x}, \delta)\backslash F^{-1}(\bar{y})$ and

$$\|v - \bar{y}\| \leq \|\tilde{y} - \bar{y}\| + \|v - \tilde{y}\| < \|\tilde{y} - \bar{y}\| + \frac{\sigma}{\eta} \leq \|y' - \bar{y}\| + \frac{\sigma}{\eta}.$$

Hence,

$$\|v-\bar{y}\| < \|y'-\bar{y}\| + \sigma/\eta < \tau\lambda\|x'-\bar{x}\|^{p-1} \le \min\{\tau\delta'^p, 2^p\tau\|u-\bar{x}\|^{p-1}d(u, F^{-1}(\bar{y}))\}.$$

Since $\tau < \kappa$, according to the definitions of κ and δ' in (3.3), we arrive at $\tau\delta'^p < \kappa\delta'^p \le \delta$ and $2^p\tau < 2^p\kappa \le \beta$. Therefore,

$$(u, v) \in \mathcal{C}(F, \bar{x}, \bar{y}, \beta, p) \text{ and } v \in F(u) \cap B(\bar{y}, \delta). \tag{3.15}$$

By (3.13), we also have that $\|\tilde{y}-\bar{y}\| < \|v-\bar{y}\| + \sigma/\eta < \|v-\bar{y}\| + \|\tilde{y}-\bar{y}\|/2$, and then $\|\tilde{y}-\bar{y}\| < 2\|v-\bar{y}\|$. Consider an arbitrary $y \in F(u)$, it follows from (3.9) and (3.13) that

$$\begin{aligned}
\|v-\bar{y}\| &< \|\tilde{y}-\bar{y}\| + \sigma/\eta \le \|y-\bar{y}\| + 2\tau\|x'-\bar{x}\|^{p-1}(\|u-\tilde{x}\| + \eta\|y-\tilde{y}\|) + \sigma/\eta \\
&< \|y-\bar{y}\| + 2\tau\|x'-\bar{x}\|^{p-1}(\sigma + \eta(\|y-\bar{y}\| + \|\bar{y}-v\| + \|v-\tilde{y}\|)) + \sigma/\eta \\
&< \|y-\bar{y}\| + 2\tau\eta\delta^{p-1}(\|y-\bar{y}\| + \|\bar{y}-v\|) + (4\tau\delta^{p-1} + 1/\eta)\sigma \\
&< \|y-\bar{y}\| + 2\tau\eta\delta^{p-1}(\|y-\bar{y}\| + \|\bar{y}-v\|) + \eta\|v-\bar{y}\|
\end{aligned}$$

(the last inequality holds due to our choice of σ in (3.13) and the fact that $\|\tilde{y}-\bar{y}\| < 2\|v-\bar{y}\|$). And then

$$\|v-\bar{y}\| < \frac{1+2\tau\eta\delta^{p-1}}{1-2\tau\eta\delta^{p-1}-\eta}\|y-\bar{y}\| \ \forall y \in F(u).$$

Hence, with the help of (3.15) and the first inequality in (3.6), we have that

$$\|v-\bar{y}\| < \frac{1+2\tau\eta\delta^{p-1}}{1-2\tau\eta\delta^{p-1}-\eta}d(\bar{y}, F(u)) < \min\{(1+\varepsilon)d(\bar{y}, F(u)), d(\bar{y}, F(u)) + \varepsilon\}, \tag{3.16}$$

This implies that $v \in P^{\varepsilon}_{F(u)}(\bar{y})$. Consequently, since $u \notin F^{-1}(\bar{y})$, it follows from (3.15) that

$$(u, v) \in \mathcal{B}(F, \bar{x}, \bar{y}, \delta, \varepsilon) \cap \mathcal{C}(F, \bar{x}, \bar{y}, \beta, p) \ \forall (u, v) \in \text{gph}(F) \cap B((\tilde{x}, \tilde{y}), \sigma). \tag{3.17}$$

By (3.9) and the optimality condition, one has

$$(0, 0) \in \hat{\partial}(\varphi + 2\tau\|x'-\bar{x}\|^{p-1}(\|\cdot - \tilde{x}\| + \eta\|\cdot - \tilde{y}\|))(\tilde{x}, \tilde{y}). \tag{3.18}$$

Recall that $\sigma < \tau\|x'-\bar{x}\|^{p-1}$, it follows from Lemma 2.1 (i) that there exist $(\tilde{x}_{1\sigma}, \tilde{y}_{1\sigma}), (\tilde{x}_{2\sigma}, \tilde{y}_{2\sigma}) \in B((\tilde{x}, \tilde{y}), \sigma)$ such that $(\tilde{x}_{2\sigma}, \tilde{y}_{2\sigma}) \in \text{gph}(F)$ and

$$(0, 0) \in \hat{\partial}\|\cdot - \bar{y}\|(\tilde{x}_{1\sigma}, \tilde{y}_{1\sigma}) + \hat{\partial}\delta_{\text{gph}(F)}(\tilde{x}_{2\sigma}, \tilde{y}_{2\sigma}) + 3\tau\|x'-\bar{x}\|^{p-1}(B_{X^*} \times \eta B_{Y^*}). \tag{3.19}$$

According to (3.17), we have $(\tilde{x}_{2\sigma}, \tilde{y}_{2\sigma}) \in \mathcal{B}(F, \bar{x}, \bar{y}, \delta, \varepsilon) \cap \mathcal{C}(F, \bar{x}, \bar{y}, \beta, p)$. Note that $\|\tilde{y}_{1\sigma} - \tilde{y}\| < \sigma/\eta < \frac{\|\tilde{y}-\bar{y}\|}{2}$, one has $\tilde{y}_{1\sigma} \neq \bar{y}$. By (3.13) and (3.14), we also have the estimation $\|x'-\bar{x}\| < 2\|\tilde{x}_{2\sigma} - \bar{x}\|$ and $\|\tilde{y}-\bar{y}\| < 2\|\tilde{y}_{2\sigma} - \bar{y}\|$, and hence $\|\tilde{y}_{1\sigma} - \tilde{y}_{2\sigma}\| < 2\sigma/\eta < \min\{\varepsilon, \|\tilde{y}-\bar{y}\|^2/4\} \le \min\{\varepsilon, \|\tilde{y}_{2\sigma} - \bar{y}\|^2\}$, i.e.

$\tilde{y}_{1\sigma} \in B(\tilde{y}_{2\sigma}, \min\{\varepsilon, \|\tilde{y}_{2\sigma} - \bar{y}\|^2\})\backslash\{\bar{y}\}$. Therefore, it follows from assumption (3.1) that

$$d(0, \hat{D}^*F(\tilde{x}_{2\sigma}, \tilde{y}_{2\sigma})(J_\varepsilon(\tilde{y}_{1\sigma} - \bar{y})) \geq \alpha\|\tilde{x}_{2\sigma} - \bar{x}\|^{p-1}. \tag{3.20}$$

On the other hand, (3.19) shows that there exists $b_1^* \in J(y_{1\sigma} - \bar{y})$ and $(a_2^*, b_2^*) \in B_{X^*} \times B_{Y^*}$ such that

$$(3\tau\|x' - \bar{x}\|^{p-1}a_2^*, -b_1^* + 3\tau\|x' - \bar{x}\|^{p-1}\eta b_2^*) \in \hat{N}(\text{gph}F, (\tilde{x}_{2\sigma}, \tilde{y}_{2\sigma})).$$

Let

$$\tilde{x}^* := \frac{3\tau\|x' - \bar{x}\|^{p-1}a_2^*}{\|b_1^* - 3\tau\|x' - \bar{x}\|^{p-1}\eta b_2^*\|} \text{ and } \tilde{y}^* := \frac{b_1^* - 3\tau\|x' - \bar{x}\|^{p-1}\eta b_2^*}{\|b_1^* - 3\tau\|x' - \bar{x}\|^{p-1}\eta b_2^*\|},$$

then $(\tilde{x}^*, -\tilde{y}^*) \in \hat{N}(\text{gph}F, (\tilde{x}_{2\sigma}, \tilde{p}_{2\sigma}))$. It is easy to calculate that

$$\begin{aligned}\|\tilde{y}^* - b_1^*\| &\leq \frac{\|(1 - \|b_1^* - 3\tau\|x' - \bar{x}\|^{p-1}\eta b_2^*\|)b_1^*\|}{\|b_1^* - 3\tau\|x' - \bar{x}\|^{p-1}\eta b_2^*\|} + \frac{3\tau\|x' - \bar{x}\|^{p-1}\eta}{\|b_1^* - 3\tau\|x' - \bar{x}\|^{p-1}\eta b_2^*\|}\\ &\leq \frac{6\tau\delta^{p-1}\eta}{1 - 3\tau\delta^{p-1}\eta} < \varepsilon\end{aligned}$$

(the last inequality holds due to (3.6)). This shows that $\tilde{y}^* \in J_\varepsilon(\tilde{y}_{1\sigma} - \bar{y})$. Recall that $\|x' - \bar{x}\| < 2\|\tilde{x}_{2\sigma} - \bar{x}\|$ and $\kappa \leq \frac{\alpha}{3 \cdot 2^{p-1}}$, it follows from (3.6) that

$$\begin{aligned}d(0, \hat{D}^*F(\tilde{x}_{2\sigma}, \tilde{y}_{2\sigma})(J_\varepsilon(\tilde{y}_{1\sigma} - \bar{y})) \leq \|\tilde{x}^*\| &\leq \frac{3\tau\|x' - \bar{x}\|^{p-1}}{1 - 3\tau\eta\|x' - \bar{x}\|^{p-1}} < \frac{3\tau\|x' - \bar{x}\|^{p-1}}{1 - 3\tau\eta\delta^{p-1}}\\ &< 3\kappa\|x' - \bar{x}\|^{p-1} < 3 \cdot 2^{p-1}\kappa\|\tilde{x}_{2\sigma} - \bar{x}\|^{p-1}\\ &\leq \alpha\|\tilde{x}_{2\sigma} - \bar{x}\|^{p-1},\end{aligned}$$

which contradicts (3.20). Therefore, we conclude that (3.2) holds. If, in addition, Y is a Fréchet smooth Banach space, we apply Lemma 2.1 (i) with (3.18) to obtain the existence of $(\tilde{x}_{1\sigma}, \tilde{y}_{1\sigma}) \in B((\bar{x}, \bar{y}), \sigma)$ such that $(\tilde{x}_{1\sigma}, \tilde{y}_{1\sigma}) \in \text{gph}(F)$ and

$$(0, 0) \in \hat{\partial}(\|\cdot - \bar{y}\| + \delta_{\text{gph}(F)})(\tilde{x}_{1\sigma}, \tilde{y}_{1\sigma}) + 3\tau\|x' - \bar{x}\|^{p-1}B_{X^*} \times \eta B_{Y^*}. \tag{3.21}$$

Then it follows from (3.14) and (3.17) that $(\tilde{x}_{1\sigma}, \tilde{y}_{1\sigma}) \in \mathcal{B}(F, \bar{x}, \bar{y}, \delta, \varepsilon) \cap \mathcal{C}(F, \bar{x}, \bar{y}, \beta, p)$ and $\|x' - \bar{x}\| < 2\|\tilde{x}_{1\sigma} - \bar{x}\|$, and hence $\tilde{y}_{1\sigma} \neq \bar{y}$. Note that Y is a Fréchet smooth Banach space, applying Lemma 2.1 (ii) to (3.21) gives us

$$(0, 0) \in \{0\} \times \nabla\|\tilde{y}_{1\sigma} - \bar{y}\| + \hat{N}(\text{gph}(F), (\tilde{x}_{1\sigma}, \tilde{y}_{1\sigma})) + 3\tau\|x' - \bar{x}\|^{p-1}B_{X^*} \times \eta B_{Y^*}.$$

The rest of the proof follows similarly as in the case of Asplund spaces. The proof is completed. □

The essential work of studying pseudo metric subregularity revolves around establishing the validity of inequality (1.5). For any $\tau, \delta \in (0, +\infty)$, let $\mathcal{N}(F, \bar{x}, \bar{y}, p, \tau, \delta)$ denote the set of all $x \in B(\bar{x}, \delta)$ failing (1.5), namely

$$\mathcal{N}(F, \bar{x}, \bar{y}, p, \tau, \delta) := \{x \in B(\bar{x}, \delta) | \tau \|x - \bar{x}\|^{p-1} d(x, F^{-1}(\bar{y})) > d(\bar{y}, F(x))\}.$$

It follows that the property of pseudo metric subregularity is equivalent to the existence of $\tau, \delta \in (0, +\infty)$ such that $\mathcal{N}(F, \bar{x}, \bar{y}, p, \tau, \delta) = \emptyset$. For given positive β, δ, since inequality (1.5) holds (with $\tau = \beta$) automatically for any $x \in B(\bar{x}, \delta) \backslash \mathcal{N}(F, \bar{x}, \bar{y}, p, \beta, \delta)$ in the case when $\mathcal{N}(F, \bar{x}, \bar{y}, p, \beta, \delta) \neq \emptyset$, one only needs to consider the candidate x in $\mathcal{N}(F, \bar{x}, \bar{y}, p, \beta, \delta)$ in order to verify the inequality (1.5). It is clear that $\mathcal{B}(F, \bar{x}, \bar{y}, \delta, \varepsilon) \cap \mathcal{C}(F, \bar{x}, \bar{y}, \beta, p) \subset \{(x, y) \in X \times Y | x \in \mathcal{N}(F, \bar{x}, \bar{y}, p, \beta, \delta), y \in P^{\varepsilon}_{F(x)}(\bar{y})\}$. In Lemma 3.1, variational conditions are provided on $\mathcal{B}(F, \bar{x}, \bar{y}, \delta, \varepsilon) \cap \mathcal{C}(F, \bar{x}, \bar{y}, \beta, p)$ to ensure the pseudo metric subregularity of F at $(\bar{x}, \bar{y})$, where the explicit quantitative relationships between β, κ, δ and δ' are calculated.

To characterize metric subregularity, Gfrerer [5] introduced the limit critical set $Cr_0 F(\bar{x}, \bar{y})$ of F at $(\bar{x}, \bar{y})$, i.e., the set of all pairs $(v, u^*) \in Y \times X^*$ such that there exist sequences $\{t_k\} \subset (0, +\infty)$, $\{(u_k, v_k^*)\} \subset S_X \times S_{Y^*}$ and $\{(v_k, u_k^*)\} \subset Y \times X^*$ satisfying $t_k \to 0$, $(v_k, u_k^*) \to (v, u^*)$ and

$$(u_k^*, -v_k^*) \in \hat{N}(\mathrm{gph}(F), (\bar{x} + t_k u_k, \bar{y} + t_k v_k)) \; \forall k \in \mathbb{N}.$$

In terms of $Cr_0 F(\bar{x}, \bar{y})$, Gfrerer proved the following interesting point-based sufficient condition for metric subregularity: Let $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$. Then, F is metrically subregular at $(\bar{x}, \bar{y})$ provided that $(0, 0) \notin Cr_0 F(\bar{x}, \bar{y})$.

For the purpose of studying pseudo metric subregularity of order p, we adopt the following definitions of general limit critical sets $Cr F(\bar{x}, \bar{y}, p)$, $\widehat{Cr} F(\bar{x}, \bar{y}, p)$ and $\widetilde{Cr} F(\bar{x}, \bar{y}, p)$ for a general multifunction $F : X \rightrightarrows Y$:

(i) $(v, u^*) \in Cr F(\bar{x}, \bar{y}, p)$ if and only if $(v, u^*) \in Y \times X^*$ with the property that there exists sequences $\{t_k\} \subset (0, +\infty)$, $\{(u_k, v_k^*)\} \subset S_X \times S_{Y^*}$ and $\{(v_k, u_k^*)\} \subset Y \times X^*$ satisfying $t_k \to 0$, $\left(v_k, \frac{u_k^*}{t_k^{p-1}}\right) \to (v, u^*)$ and

$$(u_k^*, -v_k^*) \in \hat{N}(\mathrm{gph}(F), (\bar{x} + t_k u_k, \bar{y} + t_k^p v_k)) \; \forall k \in \mathbb{N}. \tag{3.22}$$

(ii) $(v, u^*) \in \widehat{Cr} F(\bar{x}, \bar{y}, p)$ if and only if $(v, u^*) \in Y \times X^*$ with the property that there exists sequences $\{t_k\} \subset (0, +\infty)$, $\{(u_k, v_k^*)\} \subset S_X \times S_{Y^*}$ and $\{(v_k, u_k^*)\} \subset Y \backslash \{0\} \times X^*$ satisfying $\bar{x} + t_k u_k \notin F^{-1}(\bar{y})$, (3.22),

$$t_k \to 0, \left(v_k, \frac{u_k^*}{t_k^{p-1}}\right) \to (v, u^*) \text{ and } \left\langle v_k^*, \frac{v_k}{\|v_k\|} \right\rangle \to 1. \tag{3.23}$$

(iii) $(v, u^*) \in \widetilde{Cr} F(\bar{x}, \bar{y}, p)$ if and only if $(v, u^*) \in Y \times X^*$ with the property that there exists sequences $\{t_k\} \subset (0, +\infty)$, $\{(u_k, v_k^*)\} \subset S_X \times S_{Y^*}$ and $\{(v_k, u_k^*)\} \subset Y\backslash\{0\} \times X^*$ satisfying (3.22), (3.23) and $\frac{t_k^p \|v_k\|}{d(\bar{y}, F(\bar{x}+t_k u_k))} \to 1$.

Remark 3.1 In the definition of $\widetilde{Cr} F(\bar{x}, \bar{y}, p)$, the requirements of $v_k \neq 0$ and $\frac{t_k \|v_k\|}{d(\bar{y}, F(\bar{x}+t_k u_k))} \to 1$ guarantee that $\bar{y} + t_k v_k \neq \bar{y}$ and $\bar{x} + t_k u_k \notin F^{-1}(\bar{y})$ except for finitely many $k \in \mathbb{N}$.

In [25], the authors introduced a similar concept $\check{C}_P F(\bar{x}, \bar{y}, p)$ via proximal normal cone in $\mathfrak{C}^2$ type Banach space to study Hölder metric subregularity of order p. In [7, Theorem 1-(2)], upon letting $u = 0$ and $s = 1$, the author obtains that the condition $(0, 0) \notin \widehat{Cr} F(\bar{x}, \bar{y}, p)$ implies that F is pseudo metrically subregular of order p at $(\bar{x}, \bar{y})$ for $p \in [1, +\infty)$.

Remark 3.2 From the definition of critical sets, it is clear that

$$\widetilde{Cr} F(\bar{x}, \bar{y}, p) \subset \widehat{Cr} F(\bar{x}, \bar{y}, p) \subset Cr F(\bar{x}, \bar{y}, p). \tag{3.24}$$

Furthermore, we claim that the above inclusions are strict. In fact, let F be defined as in Example 1.2, $t_k = 1/k$, $v_k = 0$ and $u_k = -1$, it is easy to calculate that $\hat{N}(\mathrm{gph}(F), (t_k u_k, t_k^p v_k)) = \{(0, -1)\}$. This shows that $(0, 0) \in Cr F(0, 0, p)$. On the other hand, for any sequences $\{t_k\} \subset (0, +\infty)$, $\{(u_k, v_k^*)\} \subset S_{\mathbb{R}} \times S_{\mathbb{R}}$ and $\{(v_k, u_k^*)\} \subset \mathbb{R}\backslash\{0\} \times \mathbb{R}$ satisfying $t_k u_k \notin F^{-1}(0) = (-\infty, 0]$, (3.22) and (3.23), we have that $u_k = 1$, $v_k^* = -1$ and $t_k^p v_k \geq (t_k u_k)^p > 0$, and hence $(v_k, u_k^*) \to (v, u^*) \neq (0, 0)$. This shows that $(0, 0) \notin \widehat{Cr} F(0, 0, p)$, which justifies that the set $\widehat{Cr} F(0, 0, p)$ is strictly smaller than the set $Cr F(0, 0, p)$. For the strict inclusion $\widetilde{Cr} F(\bar{x}, \bar{y}, p) \subset \widehat{Cr} F(\bar{x}, \bar{y}, p)$, it will be shown in Example 3.1.

Now we are ready to state the following point-based sufficient conditions for pseudo metric subregularity of order p in terms of the aforementioned three types of limit critical sets.

Theorem 3.1 *Let $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$, $p \in [1, +\infty)$. Consider following statements:*

(i) $(0, 0) \notin Cr F(\bar{x}, \bar{y}, p)$;
(ii) $(0, 0) \notin \widehat{Cr} F(\bar{x}, \bar{y}, p)$;
(iii) $(0, 0) \notin \widetilde{Cr} F(\bar{x}, \bar{y}, p)$;
(iv) *F is pseudo metrically subregular of order p at $(\bar{x}, \bar{y})$.*
Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$. If, in addition, F is convex, then $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$.

Proof From the inclusion (3.24), we immediately have $(i) \Rightarrow (ii) \Rightarrow (iii)$. For $(iii) \Rightarrow (iv)$, it suffices to show that there exist $\alpha, \beta, \varepsilon, \delta \in (0, +\infty)$ such that (3.1) holds according to Lemma 3.1. We argue with contradiction. Suppose to the contrary that for $\alpha_k = \beta_k = \delta_k = \varepsilon_k = 1/k$, there exists $(x_k, y_k) \in \mathcal{B}(F, \bar{x}, \bar{y}, 1/k, 1/k) \cap \mathcal{C}(\bar{x}, \bar{y}, 1/k, p)$, $y_k' \in B(y_k, \min\{1/k, \|y_k - \bar{y}\|^2\})\backslash\{\bar{y}\}$ such that

$$d(0, \hat{D}^* F(x_k, y_k)(J_{\frac{1}{k}}(y_k' - \bar{y})) < \frac{1}{k}\|x_k - \bar{x}\|^{p-1}.$$

Then, there exist $v_k^* \in J_{1/k}(y_k' - \bar{y})$ and $u_k^* \in \hat{D}^*F(x_k, y_k)(v_k^*)$ such that $\|u_k^*\| \leq \frac{1}{k}\|x_k - \bar{x}\|^{p-1}$. According to the definition of $\mathcal{B}(F, \bar{x}, \bar{y}, 1/k, 1/k)$ and $\mathcal{C}(\bar{x}, \bar{y}, 1/k, p)$, one has

$$x_k \in B\left(\bar{x}, \frac{1}{k}\right)\backslash F^{-1}(\bar{y}),\ y_k \in P^{\frac{1}{k}}_{F(x_k)}(\bar{y}),\ \|y_k - \bar{y}\| < \frac{1}{k}\|x_k - \bar{x}\|^{p-1}d(x_k, F^{-1}(\bar{y})). \quad (3.25)$$

This shows that $y_k \in F(x_k)$, $x_k \neq \bar{x}$ and $y_k \neq \bar{y}$. Let $t_k := \|x_k - \bar{x}\|$, $u_k := t_k^{-1}(x_k - \bar{x})$ and $v_k := t_k^{-p}(y_k - \bar{y})$. It then follows from (3.25) that $0 < t_k < \frac{1}{k}$, $(x_k, y_k) = (\bar{x} + t_k u_k, \bar{y} + t_k^p v_k) \in \mathrm{gph}(F)$, $u_k \in S_X$,

$$\|y_k - \bar{y}\| < \left(1 + \frac{1}{k}\right)d(\bar{y}, F(x_k)) \text{ and } 0 < \|v_k\| < \frac{d(x_k, F^{-1}(\bar{y}))}{kt_k} \leq \frac{1}{k}.$$

Therefore, $(u_k^*, -v_k^*) \in \hat{N}(\bar{x} + t_k u_k, \bar{y} + t_k^p v_k)$, $v_k \neq 0$, $v_k \to 0$ and $d(\bar{y}, F(\bar{x} + t_k u_k)) \leq t_k^p\|v_k\| = \|y_k - \bar{y}\| < (1 + \frac{1}{k})d(\bar{y}, F(\bar{x} + t_k u_k))$. Hence $\frac{t_k^p\|v_k\|}{d(\bar{y}, F(\bar{x}+t_k u_k))} \to 1$. Take $y_k^* \in J(y_k' - \bar{y})$ such that $\|v_k^* - y_k^*\| < 1/k$. Observing that $\|y_k' - y_k\| < \|y_k - \bar{y}\|^2$, $v_k^*, y_k^* \in S_{Y^*}$ and $\langle y_k^*, y_k' - \bar{y}\rangle = \|y_k' - \bar{y}\|$, we obtain that

$$\begin{aligned}
1 \geq \left\langle v_k^*, \frac{v_k}{\|v_k\|}\right\rangle &= \left\langle y_k^*, \frac{y_k - \bar{y}}{\|y_k - \bar{y}\|}\right\rangle + \left\langle v_k^* - y_k^*, \frac{v_k}{\|v_k\|}\right\rangle \\
&\geq \left\langle y_k^*, \frac{y_k' - \bar{y}}{\|y_k' - \bar{y}\|}\right\rangle - \left\| \frac{y_k - \bar{y}}{\|y_k - \bar{y}\|} - \frac{y_k' - \bar{y}}{\|y_k' - \bar{y}\|}\right\| - \frac{1}{k} \\
&\geq 1 - \frac{2\|y_k - y_k'\|}{\|y_k - \bar{y}\|} - \frac{1}{k} \\
&> 1 - 2\|y_k - \bar{y}\| - \frac{1}{k}.
\end{aligned}$$

Taking the limit of $k \to \infty$, we arrive at $\left\langle v_k^*, \frac{v_k}{\|v_k\|}\right\rangle \to 1$. It follows that $(0, 0) \in \widetilde{Cr}F(\bar{x}, \bar{y}, p)$, which is a contradiction with (iii). Therefore, (iv) holds. If in addition we assume that F is convex, we claim that $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$. Indeed, it suffices to show that $(iv) \Rightarrow (ii)$. We argue again with contradiction. Suppose to the contrary that (iv) holds and there exist sequences $\{t_k\} \subset (0, +\infty)$, $\{(u_k, v_k^*)\} \subset S_X \times S_{Y^*}$ and $\{(v_k, u_k^*)\} \subset Y\backslash\{0\} \times X^*$ satisfying $\bar{x} + t_k u_k \notin F^{-1}(\bar{y})$, (3.22) and (3.23) with $(v, u^*) = (0, 0)$. Let $(x_k, y_k) := (\bar{x} + t_k u_k, \bar{y} + t_k^p v_k)$, we have that $\mathrm{gph}(F) \ni (x_k, y_k) \to (\bar{x}, \bar{y})$ and $y_k \neq \bar{y}$ for all $k \in \mathbb{N}$. By ssumption (iv), there exist $\tau, \delta \in (0, +\infty)$ such that (1.5) holds. Note that $x_k \to \bar{x}$ and $\left\langle v_k^*, \frac{v_k}{\|v_k\|}\right\rangle \to 1$, without loss of generality, we may assume that $x_k \in B(\bar{x}, \delta)\backslash F^{-1}(\bar{y})$ and $\left\langle v_k^*, \frac{v_k}{\|v_k\|}\right\rangle \geq \frac{1}{2}$ for all $k \in \mathbb{N}$. Then, it follows from (1.5) that

$$\begin{aligned}
\tau t_k^{p-1}d(x_k, F^{-1}(\bar{y})) = \tau\|x_k - \bar{x}\|^{p-1}d(x_k, F^{-1}(\bar{y})) &\leq d(\bar{y}, F(x_k)) \\
&\leq \|y_k - \bar{y}\| = t_k^p\|v_k\|. \quad (3.26)
\end{aligned}$$

By the convexity of F and (3.22), one has

$$\langle u_k^*, x - x_k\rangle \le \langle v_k^*, y - y_k\rangle \;\forall (x, y) \in \mathrm{gph}(F).$$

Therefore, for any $u \in F^{-1}(\bar{y})$, we have that

$$\langle u_k^*, u - x_k\rangle \le \langle v_k^*, \bar{y} - y_k\rangle = -t_k^p\|v_k\|\left\langle v^*, \frac{v_k}{\|v_k\|}\right\rangle \le -\frac{t_k^p\|v_k\|}{2}.$$

Together with (3.26), we obtain that

$$\frac{\tau}{2}t_k^{p-1}d(x_k, F^{-1}(\bar{y})) \le \frac{t_k^p\|v_k\|}{2} \le \langle u_k^*, x_k - u\rangle \le \|u_k^*\|\|x_k - u\| \;\forall u \in F^{-1}(\bar{y}).$$

Since $x_k \notin F^{-1}(\bar{y})$ and u is chosen from $F^{-1}(\bar{y}))$ arbitrarily, we conclude that $0 < \frac{\tau}{2} \le \frac{\|u_k^*\|}{t_k^{p-1}}$, which contradicts the fact that $\frac{\|u_k^*\|}{t_k^{p-1}} \to 0$. Therefore (ii) holds and the proof is complete. □

Next we provide an example showing that in Remark 3.2, the first inclusion in (3.24) is strict, which illustrates that Theorem 3.1 ($(iii) \rightrightarrows (iv)$) is indeed an improvement of [7, Theorem 1-(2)] upon considering $u = 0$ and $s = 1$.

Example 3.1 Let $X = Y = \mathbb{R}$, $p = 2$ and $(\bar{x}, \bar{y}) = (0, 0)$. Consider the multifunction $F : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as follows:

$$F(x) := \begin{cases} \mathbb{R}, & \text{if } x \in (-\infty, 0];\\ [x^2 - \frac{1}{(n+1)^2}, +\infty) \cup (-\infty, -\frac{2}{n^2} + \frac{2}{(n+1)^2}], & \text{if } x \in (\frac{1}{n+1}, \frac{1}{n}), n \in \mathbb{N};\\ [0, +\infty) \cup (-\infty, -\frac{2}{n^2} + \frac{2}{(n+1)^2}], & \text{if } x = \frac{1}{n}, n \in \mathbb{N};\\ [x^2 - 1, +\infty) \cup (-\infty, -x^2 + 1], & \text{if } x \in [1, +\infty). \end{cases}$$

It is clear that the F is of closed graph and $F^{-1}(0) = (-\infty, 0] \cup \{\frac{1}{n} | n \in \mathbb{N}\}$. First we show that $(0, 0) \notin \widetilde{Cr}F(0, 0, 2)$. If this is not true, then there exist sequences $\{t_k\} \subset (0, +\infty)$, $\{(u_k, v_k^*)\} \subset S_X \times S_{Y^*}$ and $\{(v_k, u_k^*)\} \subset Y\backslash\{0\} \times X^*$ satisfying (3.22), (3.23) and $\frac{t_k^2\|v_k\|}{d(\bar{y}, F(\bar{x}+t_k u_k))} \to 1$ with $(v, u^*) = (0, 0)$. Let $x_k := t_k u_k$ and $y_k := t_k^2 v_k$. Without loss of generality, we may assume that $t_k \in (0, 1)$ and $0 < \frac{|y_k|}{d(0, F(x_k))} < 2$ for all $k \in \mathbb{N}$. Then there exists subsequence $\{n_k\}$ of natural numbers such that $t_k \in [\frac{1}{n_k+1}, \frac{1}{n_k})$ for all $k \in \mathbb{N}$. By Remark 3.1, we have that $x_k \notin F^{-1}(0)$, and then $u_k = 1$. Note that $F(x_k) = [x_k^2 - \frac{1}{(n_k+1)^2}, +\infty) \cup (-\infty, -\frac{2}{n_k^2} + \frac{2}{(n_k+1)^2}]$, $x_k = t_k$ and $d(0, F(x_k)) = x_k^2 - \frac{1}{(n_k+1)^2} < \frac{1}{n_k^2} - \frac{1}{(n_k+1)^2}$, one has $y_k > 0$ and $d(0, F(x_k)) \le y_k < \frac{2}{n_k^2} - \frac{2}{(n_k+1)^2}$. If $d(0, F(x_k)) < y_k$, then $(x_k, y_k) \in \mathrm{int}\,\mathrm{gph}(F)$ and then $\hat{N}(\mathrm{gph}(F), (x_k, y_k)) = \{(0, 0)\}$, which is in direct contradiction with (3.22). If $d(0, F(x_k)) = y_k$, then $\hat{N}(\mathrm{gph}(F), (x_k, y_k)) = \{(2x_k, -1)\}$, and then $u_k^* = 2x_k$ and $\frac{u_k^*}{t_k} = 2 \nrightarrow 0$, which arrives also at a contraction. This shows that $(0,0) \notin \widetilde{Cr}F(\bar{x},\bar{y},2)$. It then follows from Theorem 3.1 that F is pseudo metrically subregular of order 2 at $(0,0)$. Next, we show that $(0,0) \in \widehat{Cr}F(\bar{x},\bar{y},2)$. Let $t_n = \frac{2n+1}{2n(n+1)}, u_n = 1, v_n = -\frac{8}{2n+1}, u_n^* = 0$ and $v_n^* = -1$. It is clear that $t_n u_n \in (\frac{1}{n+1}, \frac{1}{n}), t_n^2 v_n = -\frac{2}{n^2} + \frac{2}{(n+1)^2} \in F(t_n u_n), \langle v_n^*, v_n/\|v_n\|\rangle = 1, (v_n, u_n^*) \to (0,0)$ and $(u_n^*, -v_n^*) = (0,1) \in \hat{N}(\mathrm{gph}(F), (t_n u_n, t_n^2 v_n))$. This shows that $(0,0) \in \widehat{Cr}F(0,0,2)$. Together with (3.24), we conclude that $\widetilde{Cr}F(0,0,2)$ is strictly smaller than $\widehat{Cr}F(0,0,2)$.

From Example 3.1 and Remark 3.2, it is easy to observe that condition (i) in Theorem 3.1 is only sufficient but not necessary for pseudo metric subregularity.

It is worth to mention that condition (i) in Theorem 3.1 is not sufficient for pseudo metric regularity (for definition, see [7, Definition 1]). For instance, consider a multifunction $F : \mathbb{R} \rightrightarrows \mathbb{R}$ such that $F(x) = [x^p, +\infty)$ $(p \in [1,+\infty))$ for all $x \in \mathbb{R}$. It is easy to verify that $(0,0) \notin CrF(0,0,p)$, but F is not pseudo metrically regular at $(0,0)$.

Next, we show that statement (i) in Theorem 3.1 is a characterization for stability of pseudo metric subregularity under $C^{1,p}$ and p-bounded smooth perturbations.

Theorem 3.2 *Let $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ and $p \in [1,+\infty)$. Then, the following statements are equivalent:*

(i) *$(0,0) \notin CrF(\bar{x},\bar{y},p)$;*
(ii) *F is pseudo metrically subregular of order p stable at $(\bar{x},\bar{y})$ under p-bounded smooth perturbation.*
(iii) *F is pseudo metrically subregular of order p stable at $(\bar{x},\bar{y})$ under $C^{1,p}$ perturbation.*

Proof We first show that $(i) \Rightarrow (ii)$. Suppose to the contrary that (i) holds but F is not pseudo metrically subregular of order p stable at $(\bar{x},\bar{y})$ under p-bounded smooth perturbation, i.e., there exist $f_k \in C^1(X,Y,\bar{x})$ and $\delta_k \in (0,1/k)$ such that $f_k(\bar{x}) = 0$,

$$\|\nabla f_k(x)\| \le \frac{1}{k}\|x-\bar{x}\|^{p-1} \ \forall x \in B(\bar{x},\delta_k) \tag{3.27}$$

and $F + f_k$ is not pseudo metrically subregular of order p at $(\bar{x},\bar{y})$. Note that $(\bar{x},\bar{y}) \in \mathrm{gph}(F+f_k)$, it follows from Theorem 3.1 that $(0,0) \in \widetilde{Cr}(F+f_k)(\bar{x},\bar{y},p)$. Then, for every k there exist sequences $\{t_{k_i}\} \subset (0,+\infty), \{(u_{k_i}, v_{k_i}^*)\} \subset S_X \times S_{Y^*}$ and $\{(v_{k_i}, u_{k_i}^*)\} \subset Y\backslash\{0\} \times X^*$ satisfying $t_{k_i} \to 0$, $\left(v_{k_i}, \frac{u_{k_i}^*}{t_{k_i}^{p-1}}\right) \to (0,0)$, as $i \to \infty$, and

$$(u_{ki}^*, -v_{ki}^*) \in \hat{N}(\mathrm{gph}(F+f_k), (\bar{x}+t_{ki}u_{ki}, \bar{y}+t_{ki}^p v_{ki})) \ \forall i \in \mathbb{N}. \tag{3.28}$$

For each k we can find some index i_k such that $t_{k_{i_k}} \le \delta_k, \|v_{k_{i_k}}\| \le 1/k$ and $\|u_{k_{i_k}}^*\| \le \frac{t_{k_{i_k}}^{p-1}}{k}$. Let $t_k := t_{k_{i_k}}, u_k := u_{k_{i_k}}, v_k := v_{k_{i_k}} - \frac{f(\bar{x}+t_{k_{i_k}}u_{k_{i_k}})}{t_{k_{i_k}}^p}, u_k^* := u_{k_{i_k}}^* - \nabla f(\bar{x} +$

$t_{ki_k}u_{ki_k})^*v^*_{ki_k}$ and $v^*_k := v^*_{ki_k}$. It is clear that $(u_k, v^*_k) \in S_X \times S_{Y^*}$, $\bar{y} + t_k^p v_k \in F(\bar{x} + t_k u_k)$. From (2.1) and (3.28), we obtain that

$$\begin{aligned}(u^*_k, -v^*_k) &\in \hat{N}(\mathrm{gph}(F), (\bar{x} + t_{ki_k}u_{ki_k}, \bar{y} + t^p_{ki_k}v_{ki_k} - f(\bar{x} + t_{ki_k}u_{ki_k}))) \\ &= \hat{N}(\mathrm{gph}(F), (\bar{x} + t_k u_k, \bar{y} + t_k^p v_k)).\end{aligned}$$

Since $\|\bar{x} + t_k u_k - \bar{x}\| = t_k < \delta_k$, it follows from (3.27) that $\|\nabla f_k(\bar{x} + t_k u_k)\| \le \frac{1}{k}\|t_k u_k\|^{p-1} = \frac{t_k^{p-1}}{k}$, for each $k \in \mathbb{N}$. By the mean value theorem, there exists $\theta \in (0, 1)$ such that

$$\frac{\|f(\bar{x} + t_k u_k) - f(\bar{x})\|}{t_k^p} = \frac{\|\nabla f(\bar{x} + \theta t_k u_k) t_k u_k\|}{t_k^p} \le \frac{1}{k}.$$

Therefore, we calculate that

$$\frac{\|u^*_k\|}{t_k^{p-1}} \le \frac{\|u^*_{ki_k}\|}{t_k^{p-1}} + \frac{\|\nabla f(\bar{x} + t_k u_k)^* v^*_k\|}{t_k^{p-1}} \le \frac{\|u^*_{ki_k}\|}{t_k^{p-1}} + \frac{1}{k} \to 0$$

and $\|v_k\| \le \|v_{ki_k}\| + \frac{\|f(\bar{x}+t_k u_k)\|}{t_k^p} \le \|v_{ki_k}\| + \frac{1}{k} \to 0$ as $k \to \infty$. This shows that $(0, 0) \in CrF(\bar{x}, \bar{y}, p)$, which is a direct contradiction with the assumption. Hence, $(i) \Rightarrow (ii)$ is true.

$(ii) \Rightarrow (iii)$ follows straightly from Definition 1.1.

For $(iii) \Rightarrow (i)$, we argue with contradiction. Assume that (iii) is true but $(0, 0) \in CrF(\bar{x}, \bar{y}, p)$, i.e., there exist sequences $\{t_k\} \subset (0, +\infty)$, $\{(u_k, v^*_k)\} \subset S_X \times S_{Y^*}$ and $\{(v_k, u^*_k)\} \subset Y \times X^*$ satisfying $t_k \to 0$, $\left(v_k, \frac{u^*_k}{t_k^{p-1}}\right) \to (0, 0)$ and (3.22). Taking subsequences if necessary, we may assume that $t_k \in (0, 1)$, $t_{k+1} < \frac{t_k}{4}$, $\|v_k\| < \frac{1}{k^2}$ and $\|u^*_k\| < \frac{t_k^{p-1}}{2k}$. Let $(x_k, y_k) := (\bar{x} + t_k u_k, \bar{y} + t_k^p v_k)$. By (3.22), there are numbers $\rho_k \in (0, \frac{t_k}{2k})$ such that

$$\begin{aligned}&\langle (u^*_k, -v^*_k), (x, y) - (x_k, y_k)\rangle \le \frac{t_k^{p-1}}{2k}(\|x - x_k\| + \|y - y_k\|)\ \forall (x, y) \in \mathrm{gph}(F) \cap (B(x_k, \rho_k) \\ &\quad \times B(y_k, \rho_k)),\end{aligned}$$

and thus we arrive at the estimates

$$\begin{aligned}\langle -v^*_k, y - y_k\rangle &\le \langle -u^*_k, x - x_k\rangle + \frac{t_k^{p-1}}{2k}(\|x - x_k\| + \|y - y_k\|) \\ &\le \frac{t_k^{p-1}}{k}(\|x - x_k\| + \|y - y_k\|)\ \forall (x, y) \in \mathrm{gph}(F) \cap (B(x_k, \rho_k) \times B(y_k, \rho_k)).\end{aligned} \tag{3.29}$$

For each $k \in \mathbb{N}$, pick continuous linear functions $q_k^* \in S_{X^*}$ and $p_{ki}^* \in S_{X^*}$, for each $i < k$ such that

$$\langle q_k^*, \bar{x} - x_k \rangle = \|x_k - \bar{x}\| = t_k, \langle p_{ki}^*, x_i - x_k \rangle = \|x_i - x_k\|.$$

We define the function $\xi_k : X \to \mathbb{R}_+$ and the mapping $f : X \to Y$ as

$$\xi_k(x) := 4t_k^{-2} \left(\langle q_k^*, x - x_k \rangle^2 + \sum_{i=1}^{k-1} \langle p_{ki}^*, x - x_k \rangle^2 \right) \ \forall x \in X, k \in \mathbb{N}$$

and

$$f(x) := \sum_{k=1}^{\infty} f_k(x) := \sum_{k=1}^{\infty} -(1 - \xi_k(x))_+^{1+p} \left(t_k^p v_k - \frac{t_k^{p-1} \rho_k z_k}{\sqrt{k}} \right) \ \forall x \in X,$$

respectively, where $(1 - \xi_k(x))_+ := \max\{1 - \xi_k(x), 0\}$ and $\{z_k\} \subset S_Y$ such that $\langle v_k^*, z_k \rangle \geq 1/2$ for all $k \in \mathbb{N}$. It is clear that

$$\sum_{k=1}^{\infty} \|f_k(x)\| \leq \sum_{k=1}^{\infty} \left\| t_k^p v_k - \frac{t_k^{p-1} \rho_k z_k}{\sqrt{k}} \right\| = \sum_{k=1}^{\infty} \left\| v_k - \frac{\rho_k z_k}{t_k \sqrt{k}} \right\| t_k^p \leq \sum_{k=1}^{\infty} \left(\frac{1}{k^2} + \frac{1}{2k^{\frac{3}{2}}} \right) t_k^p < \infty,$$

which indicates that f is well defined on X. Next we prove that the mapping $F + f$ is not pseudo metrically subregular of order p at $(\bar{x}, \bar{y})$. To this end, we fix an arbitrary $x \in B(x_k, \rho_k/2)$, then for any $k \in \mathbb{N}$, it follows from the proof (part 2) of [5, Theorem 3.2, p. 1447] that

$$0 \leq \xi_k(x) \leq \frac{16\rho_k^2}{15t_k^2} \leq \frac{4}{15k^2} \to 0 \qquad \text{as } k \to \infty \tag{3.30}$$

and $\xi_l(x) \geq 1$ for all $l \neq k$. Hence we obtain that

$$f(x) = -(1 - \xi_k(x))^{1+p} \left(t_k^p v_k - \frac{t_k^{p-1} \rho_k z_k}{\sqrt{k}} \right) \ \forall x \in B(x_k, \rho_k/2)$$

and therefore $f(x_k) = -t_k^p v_k + \frac{t_k^{p-1} \rho_k z_k}{\sqrt{k}}$ (since $\xi_k(x_k) = 0$). Using the fact that $(x_k, y_k) \in \text{gph}(F)$, we arrive at

$$d(\bar{y}, (F + f)(x_k)) \leq \|\bar{y} - y_k - f(x_k)\| = \left\| \frac{t_k^{p-1} \rho_k z_k}{\sqrt{k}} \right\| = \frac{t_k^{p-1} \rho_k}{\sqrt{k}}. \tag{3.31}$$

By (3.30), we immediately obtain the bound

$$0 \le t_k^p(1-(1-\xi_k(x))^{1+p}) = t_k^p((1+p)\xi_k(x) + o(\xi_k(x))) \le 2(1+p)t_k^p\xi_k(x) \\ \le \frac{32}{30k}(1+p)t_k^{p-1}\rho_k \tag{3.32}$$

for sufficiently large $k \in \mathbb{N}$, where $o : \mathbb{R}_+ \to \mathbb{R}_+$ shares the property that $o(t)/t \to 0$ as $t \downarrow 0$. Now we claim that $d(x_k, (F+f)^{-1}(\bar{y})) \ge \frac{\rho_k}{2}$ for sufficiently large $k \in \mathbb{N}$. Indeed, if this is not true, then for any sufficiently large $K \in \mathbb{N}$, there exists some $k \ge K$ and $x \in B(x_k, \rho_k/2)$ such that $\bar{y} \in (F+f)(x)$. Then $\bar{y} - f(x) \in F(x)$. By (3.32), we calculate that

$$\begin{aligned}\|\bar{y} - f(x) - y_k\| &= \left\| -t_k^p v_k + (1-\xi_k(x))^{1+p}\left(t_k^p v_k - \frac{t_k^{p-1}\rho_k z_k}{\sqrt{k}}\right)\right\| \\ &\le t_k^p(1-(1-\xi_k(x))^{1+p})\|v_k\| + (1-\xi_k(x))^{1+p}\frac{t_k^{p-1}\rho_k}{\sqrt{k}} \\ &\le \left(\frac{32(1+p)}{30k^3} + \frac{1}{\sqrt{k}}\right)t_k^{p-1}\rho_k < \frac{\rho_k}{2},\end{aligned}$$

(the last inequality holds since k is sufficiently large), which indicates that $(x, \bar{y} - f(x)) \in \operatorname{gph}(F) \cap B(x_k, \rho_k/2) \times B(y_k, \rho_k/2)$. Hence it follows from (3.29) that

$$\langle -v_k^*, \bar{y} - f(x) - y_k\rangle \le \frac{t_k^{p-1}}{k}(\|x - x_k\| + \|\bar{y} - f(x) - y_k\|) \le \frac{t_k^{p-1}\rho_k}{k}. \tag{3.33}$$

On the other hand, for sufficiently large k, by using the fact that $\frac{1}{2} < (1 - \frac{4}{15k^2})^{1+p} < (1-\xi_k(x))^{1+p} \le 1$ and inequality (3.32), we arrive at

$$\begin{aligned}\langle -v_k^*, \bar{y} - f(x) - y_k\rangle &= \left\langle -v_k^*, ((1-\xi_k(x))^{1+p} - 1)t_k^p v_k - (1-\xi_k(x))^{1+p}\frac{t_k^{p-1}\rho_k z_k}{\sqrt{k}}\right\rangle \\ &\ge -t_k^p(1-(1-\xi_k(x))^{1+p})\|v_k\| + (1-\xi_k(x))^{1+p}\frac{t_k^{p-1}\rho_k}{\sqrt{k}}\langle v_k^*, z_k\rangle \\ &\ge -\frac{32t_k^{p-1}\rho_k}{30k^3} + \frac{t_k^{p-1}\rho_k}{4\sqrt{k}},\end{aligned}$$

which is a contradiction to (3.33) due to the fact that $-\frac{32}{30k^3} + \frac{1}{4\sqrt{k}} > \frac{1}{k}$ for sufficiently large $k \in \mathbb{N}$. Together with (3.31) we conclude that

$$\|x_k - \bar{x}\|^{p-1} d(x_k, (F+f)^{-1}(\bar{y})) = t_k^{p-1} d(x_k, (F+f)^{-1}(\bar{y})) \ge \frac{\sqrt{k}}{2} d(\bar{y}, (F+f)(x_k))$$

for $k \in \mathbb{N}$ large enough. This shows that $F + f$ is not pseudo metrically subregular of order p at $(\bar{x}, \bar{y})$.

To complete the proof, it remains to demonstrate that $f \in C^{1,p}(X, Y, \bar{x})$ with $f(\bar{x}) = 0$ and $\nabla f(\bar{x}) = 0$. It is clear that each of the function ξ_k is continuously differentiable at any $x \in X$ with derivative

$$\langle \nabla \xi_k(x), u\rangle = 8t_k^{-2}\left(\langle q_k^*, x - x_k\rangle\langle q_k^*, u\rangle + \sum_{i=1}^{k-1} 16^{i-k}\langle p_{ki}^*, x - x_k\rangle\langle p_{ki}^*, u\rangle\right)\ \forall u \in X$$

and by using the Cauchy-Schwarz inequality, we also have $t_k^2\|\nabla \xi_k(x)\| \leq 8t_k\sqrt{\xi_k(x)}$ (for details see the proof (part 2) of [5, Theorem 3.2, p. 1449]). And then, each f_k is also continuously differentiable at any $x \in X$ with derivative

$$\nabla f_k(x)u = (1+p)(1-\xi_k(x))_+^p\langle \nabla \xi_k(x), u\rangle\left(t_k^p v_k - \frac{t_k^{p-1}\rho_k z_k}{\sqrt{k}}\right)\ \forall u \in X,$$

and consequently

$$\begin{aligned}\sum_{i=1}^{\infty}\|\nabla f_k(x)\| &\leq \textstyle\sum_{i=1}^{\infty}(1+p)(1-\xi_k(x))_+^p\|\nabla \xi_k(x)\|\left\|t_k^p v_k - \frac{t_k^{p-1}\rho_k z_k}{\sqrt{k}}\right\| \\ &\leq \textstyle\sum_{i=1}^{\infty} 8(1+p)(1-\xi_k(x))_+^p\sqrt{\xi_k(x)}\,t_k^{p-1}\left(v_k + \frac{\rho_k}{t_k}\frac{\|z_k\|}{\sqrt{k}}\right) \\ &\leq \textstyle\sum_{i=1}^{\infty} 8(1+p)t_k^{p-1}\left(\frac{1}{k^2} + \frac{1}{k^{\frac{3}{2}}}\right) < +\infty\ \forall x \in X \qquad (3.34)\end{aligned}$$

(the fact that $0 \leq (1-\xi_k(x))_+^p\sqrt{\xi_k(x)} \leq 1$ for all $x \in X$ has been used in the above inequality). Hence f is continuously differentiable thanks to the uniform convergence of $\sum_{i=1}^{\infty}\|\nabla f_k(x)\|$ with respect to x. Notice that $\langle q_k^*, \bar{x} - x_k\rangle = t_k$ for all $k \in \mathbb{N}$, one has $\xi_k(\bar{x}) \geq 4$, and then $(1-\xi_k(\bar{x}))_+ = 0$. Hence, we have that $f(\bar{x}) = 0$ and $\nabla f(\bar{x}) = 0$. We also have the estimates

$$\frac{t_k^2}{4}\xi_k(x) \geq \langle q_k^*, x - x_k\rangle^2 = (\langle q_k^*, x - \bar{x}\rangle + t_k)^2 \geq t_k^2 - 2t_k\|x - \bar{x}\|\ \forall x \in X,$$

and consequently

$$1 - \xi_k(x) \leq -3 + 8t_k^{-1}\|x - \bar{x}\| < 8t_k^{-1}\|x - \bar{x}\|\ \forall x \in X. \qquad (3.35)$$

Let $\varepsilon \in (0, 1)$ be arbitrarily given. Since $\sum_{k=1}^{\infty} 8^p(1+p)\left(\frac{1}{k^2} + \frac{1}{k^{\frac{3}{2}}}\right) < \infty$, there exists an index K such that $\sum_{k=K+1}^{\infty} 8^p(1+p)\left(\frac{1}{k^2} + \frac{1}{k^{\frac{3}{2}}}\right) < \varepsilon$. Let $\delta = \frac{1}{2}\min\{t_k : k = 1, \cdots, K\}$. Fixing any $1 \leq k \leq K$ and $x \in B(\bar{x}, \delta)$, we have $\langle q_k^*, x - x_k\rangle = \langle q_k^*, \bar{x} - x_k\rangle + \langle q_k^*, x - \bar{x}\rangle \geq t_k - \delta \geq t_k/2$, which indicates that $\xi_k(x) \geq 1$. This shows that $(1-\xi_k(x))_+ = 0$, and then $\nabla f_k(x) = 0$ for all $k = 1, \cdots, K$ and all $x \in B(\bar{x}, \delta)$. Together with (3.34),(3.35) and $\nabla f(\bar{x}) = 0$, we conclude that

$$\begin{aligned}
&\|\nabla f(x) - \nabla f(\bar{x})\| \\
&\leq \sum_{k=1}^{\infty} \|\nabla f_k(x)\| = \sum_{k=K+1}^{\infty} \|\nabla f_k(x)\| \\
&\leq \sum_{k=K+1}^{\infty} 8(1+p)(1-\xi_k(x))_+^p \sqrt{\xi_k(x)} t_k^{p-1} \left(\frac{1}{k^2} + \frac{1}{k^{\frac{3}{2}}}\right) \\
&\leq \sum_{k=K+1}^{\infty} 8(1+p)(1-\xi_k(x))_+ \sqrt{\xi_k(x)} (8t_k^{-1}\|x-\bar{x}\|)^{p-1} t_k^{p-1} \left(\frac{1}{k^2} + \frac{1}{k^{\frac{3}{2}}}\right) \\
&\leq \sum_{k=K+1}^{\infty} 8^p(1+p)\|x-\bar{x}\|^{p-1} \left(\frac{1}{k^2} + \frac{1}{k^{\frac{3}{2}}}\right) < \varepsilon\|x-\bar{x}\|^{p-1} \ \forall x \in B(\bar{x}, \delta).
\end{aligned}$$

This shows that $f \in C^{1,p}(X, Y, \bar{x})$, which justifies that $(iii) \Rightarrow (i)$. The proof is complete. □

Remark 3.3 In Theorem 3.2, the approach of proving $(iii) \Rightarrow (i)$ is inspired from Gfrerer [5, Theorem 3.2-(2)] and it can be simplified by setting $\xi_k(x) := 4t_k^2\|x - x_k\|^2$ when X is a Fréchet smooth Banach space. In the case of $p = 1$, the equivalence $(i) \Leftrightarrow (ii)$ in Theorem 3.2 goes back to the statement $(i) \Leftrightarrow (ii)$ in [8, Theorem 2.8], which gives a characterization for the stability of metric subregularity under small C^1 perturbation.

It's well understood that Hölder strong metric subregularity is stronger than pseudo metric subregularity and it naturally possess the stability under small Lipschitz function perturbation, see [19] and the references therein. The following example illustrates that, even in finite dimensional spaces, condition (i) in Theorem 3.2 implies the stability of pseudo metric subregularity, but not necessarily imply the property of Hölder strong metric subregularity.

Example 3.2 Let $X = \mathbb{R}^2, Y = \mathbb{R}, p = 2$ and $(\bar{x}, \bar{y}) = ((0, 0), 0)$. Consider the multifunction $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ satisfying $F(s, t) = [s^2+2s+t^2, +\infty)$ for all $(s, t) \in \mathbb{R}^2$. At first, we show that $(0, (0, 0)) \notin CrF(\bar{x}, \bar{y}, p)$. Otherwise, there exists sequences $\{\tau_k\} \subset (0, +\infty)$, $\{(u_k, v_k^*)\} = \{((s_k, t_k), v_k^*)\} \subset S_{\mathbb{R}^2} \times S_{\mathbb{R}}$ and $\{(v_k, (s_k^*, t_k^*))\} \subset \mathbb{R}\backslash\{0\} \times \mathbb{R}^2$ satisfying $\tau_k \to 0$, $\left(v_k, \frac{(s_k^*, t_k^*)}{\tau_k}\right) \to (0, (0, 0))$ and

$$((s_k^*, t_k^*), -v_k^*) \in \hat{N}(\mathrm{gph}(F), (\tau_k(s_k, t_k), \tau_k^2 v_k)) \ \forall k \in \mathbb{N}. \tag{3.36}$$

Then, it is easy to see that $(\tau_k(s_k, t_k), \tau_k^2 v_k) \in \mathrm{bd}(\mathrm{gph}(F))$ and $\hat{N}(\mathrm{gph}(F), (\tau_k(s_k, t_k), \tau_k^2 v_k)) = \{\lambda((2\tau_k s_k + 2, 2\tau_k t_k), -1) : \lambda \geq 0\}$, for all $k \in \mathbb{N}$. Note that $v_k^* \in S_{\mathbb{R}}$, one has $v_k^* = 1$, and then, it follows from (3.36) that $(s_k^*, t_k^*) = (2\tau_k s_k + 2, 2\tau_k t_k)$, for all $k \in \mathbb{N}$. Hence, $\frac{(s_k^*, t_k^*)}{\tau_k} = \frac{(2\tau_k s_k + 2, 2\tau_k t_k)}{\tau_k} \to (0, 0)$, which contradicts the fact that

$(s_k, t_k) \in S_{\mathbb{R}^2}$. Therefore, $(0, (0, 0)) \notin CrF(\bar{x}, \bar{y}, p)$ and it follows from Theorem 3.2 that F is pseudo metrically subregular of order 2 stable at $((0, 0), 0)$ under $C^{1,2}$ perturbation. It is clear that $F^{-1}(0) = \{(s, t) \in \mathbb{R}^2 : s^2 + 2s + t^2 \leq 0\} \neq \{(0, 0)\}$, which indicates that F is not Hölder strongly metrically subregular of order 2 at $((0, 0), 0)$.

4 Pseudo weak sharp minimizer

In this section, we mainly consider the pseudo weak sharp minimizer of a proper lower semicontinuous function f and its relation with pseudo metric subregularity of the subdifferential mapping $\hat{\partial} f$. Recall that for a lower semicontinuous function f on a Banach space X, $\bar{x} \in \mathrm{dom}(f)$ is said to be a q-order weak sharp minimizer ($q \in (0, +\infty)$) if there exist $\kappa, r, \delta \in (0, +\infty)$ such that $f(\bar{x}) = \inf_{u \in B[\bar{x}, r]} f(u)$ and

$$\kappa d(x, S(f, \bar{x}, r))^q \leq f(x) - f(\bar{x}) \;\; \forall x \in B_X(\bar{x}, \delta),$$

where $S(f, \bar{x}, r) := \{x \in X : f(x) = \inf_{u \in B[\bar{x}, r]} f(u)\}$. It is well recognized that $\bar{x} \in S(f, \bar{x}, r) \subset (\hat{\partial} f)^{-1}(0)$, which induce the following weaker notion of pseudo weak sharp minimizer. Let $p \in [1, +\infty)$, $r \in (0, +\infty]$, $f : X \to \mathbb{R} \cup \{+\infty\}$ and $\bar{x} \in \mathrm{dom}(f)$, we say that $\bar{x}$ is a p-order pseudo weak sharp minimizer of f if there exist $\kappa, \delta \in (0, +\infty)$ such that

$$\kappa \|x - \bar{x}\|^{p-1} d(x, (\hat{\partial} f)^{-1}(0))^2 \leq f(x) - \inf_{u \in B[\bar{x}, r]} f(u) \;\; \forall x \in B(\bar{x}, \delta).$$

It is clear that $\bar{x}$ is a p-order pseudo weak sharp minimizer of f, it must be a minimizer of f. In terms of Hölder metric subregularity of the subdifferential mapping, the authors [18,19,28,29] get the Hölder weak sharp minimizer for a proper lower semicontinuous function f. Under the pseudo metric subregularity of the subdifferential mapping $\hat{\partial} f$, we have the following result:

Theorem 4.1 *Let X be an Asplund space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $p \in [1, +\infty)$, $r \in (0, +\infty]$ and let $\bar{x} \in (\hat{\partial} f)^{-1}(0)$. Then the following statements hold:*

(i) *Suppose that there exist $\kappa, \delta \in (0, +\infty)$ such that*

$$\kappa \|x - \bar{x}\|^{p-1} d(x, (\hat{\partial} f)^{-1}(0)) \leq d(0, \hat{\partial} f(x)) \;\; \forall x \in B(\bar{x}, \delta). \tag{4.1}$$

Then

$$\tau \|x - \bar{x}\|^{p-1} d(x, (\hat{\partial} f)^{-1}(0))^2 \leq f(x) - \inf_{u \in B[\bar{x}, r]} f(u) \;\; \forall x \in B(\bar{x}, \eta), \tag{4.2}$$

where $\tau := \frac{p\kappa}{(1+p)^{1+p}}$ and $\eta := \frac{1+p}{1+2p} \min\{r, \delta\}$.

(ii) *Suppose that f is convex and that there exist $\tau, \eta \in (0, +\infty)$ such that* (4.2) *holds, then*

$$\tau \|x - \bar{x}\|^{p-1} d(x, (\hat{\partial} f)^{-1}(0)) \leq d(0, \hat{\partial} f(x)) \; \forall x \in B(\bar{x}, \eta). \tag{4.3}$$

Consequently, under the convexity assumption on f, $\bar{x}$ is a p-order pseudo weak sharp minimizer of f if and only if $\hat{\partial} f$ is pseudo metrically subregular of order p at $(\bar{x}, 0)$.

Proof (i) Suppose to the contrary that (4.2) is not true, namely there exists $x_0 \in B(\bar{x}, \eta)$ such that

$$f(x_0) < \inf_{u \in B[\bar{x}, r]} f(u) + \tau \|x_0 - \bar{x}\|^{p-1} d(x_0, (\hat{\partial} f)^{-1}(0))^2.$$

This implies that $\|x_0 - \bar{x}\|^{p-1} d(x_0, (\hat{\partial} f)^{-1}(0))^2 > 0$. Take some $\tau' \in (0, \tau)$ sufficiently close to τ such that

$$f(x_0) < \inf_{u \in B[\bar{x}, r]} f(u) + \tau' \|x_0 - \bar{x}\|^{p-1} d(x_0, (\hat{\partial} f)^{-1}(0))^2,$$

then, it follows from the Ekeland variational principle (Lemma 2.2) that there exists $\hat{x} \in B[\bar{x}, r]$ such that

$$\|\hat{x} - x_0\| < \frac{p}{1+p} d(x_0, (\hat{\partial} f)^{-1}(0)) \tag{4.4}$$

and

$$f(\hat{x}) \leq f(x) + \frac{\tau'(1+p)}{p} \|x_0 - \bar{x}\|^{p-1} d(x_0, (\hat{\partial} f)^{-1}(0)) \|x - \hat{x}\| \; \forall x \in B(\bar{x}, r). \tag{4.5}$$

This implies that $0 \in \hat{\partial}(f + \frac{\tau'(1+p)}{p} \|x_0 - \bar{x}\|^{p-1} d(x_0, (\hat{\partial} f)^{-1}(0)) \| \cdot - \hat{x}\|)(\hat{x})$. For any

$$\sigma \in \left(0, \min\left\{ \frac{p}{1+p} d(x_0, (\hat{\partial} f)^{-1}(0)) - \|\hat{x} - x_0\|, \frac{(\tau - \tau')(1+p)}{p} \|x_0 - \bar{x}\|^{p-1} d(x_0, (\hat{\partial} f)^{-1}(0)) \right\}\right),$$

it follows from Lemma 2.1 (i) that there exists $\tilde{x} \in B(\hat{x}, \sigma)$ such that

$$0 \in \hat{\partial} f(\tilde{x}) + \left(\frac{\tau'(1+p)}{p} \|x_0 - \bar{x}\|^{p-1} d(x_0, (\hat{\partial} f)^{-1}(0)) + \sigma \right) B_X. \tag{4.6}$$

Then by (4.4) and the choice of σ, it is easy to see that $\|\tilde{x}-x_0\| \le \|\hat{x}-x_0\|+\|\hat{x}-\tilde{x}\| < \|\hat{x}-x_0\|+\sigma < \frac{p}{1+p}d(x_0, (\hat{\partial} f)^{-1}(0))$. Therefore, we have

$$d(\tilde{x}, (\hat{\partial} f)^{-1}(0)) \ge d(x_0, (\hat{\partial} f)^{-1}(0)) - \|\tilde{x}-x_0\| \ge \frac{1}{1+p}d(x_0, (\hat{\partial} f)^{-1}(0))$$

and

$$\|\tilde{x}-\bar{x}\| \ge \|x_0-\bar{x}\| - \|\tilde{x}-x_0\| \ge \|x_0-\bar{x}\| - \frac{p}{1+p}d(x_0, (\hat{\partial} f)^{-1}(0)) \ge \frac{1}{1+p}\|x_0-\bar{x}\|.$$

Together with (4.6), we obtain that

$$\begin{aligned} d(0, \hat{\partial} f(\tilde{x})) &\le \frac{\tau'(1+p)}{p}\|x_0-\bar{x}\|^{p-1}d(x_0, (\hat{\partial} f)^{-1}(0)) + \sigma \\ &< \frac{\tau(1+p)}{p}\|x_0-\bar{x}\|^{p-1}d(x_0, (\hat{\partial} f)^{-1}(0)) \\ &\le \frac{\tau(1+p)^{1+p}}{p}\|\tilde{x}-\bar{x}\|^{p-1}d(\tilde{x}, (\hat{\partial} f)^{-1}(0)). \end{aligned} \tag{4.7}$$

On the other hand, it is easy to see that

$$\begin{aligned} \|\tilde{x}-\bar{x}\| &\le \|x_0-\bar{x}\| + \|\tilde{x}-x_0\| \le \|x_0-\bar{x}\| + \frac{p}{1+p}d(x_0, (\hat{\partial} f)^{-1}(0)) \\ &\le \frac{1+2p}{1+p}\|x_0-\bar{x}\| < \frac{(1+2p)\eta}{1+p}. \end{aligned}$$

By the definition of η, it follows that $\tilde{x} \in B(\bar{x}, \min\{r, \delta\})$. Hence, by (4.1), one has

$$\kappa\|\tilde{x}-\bar{x}\|^{p-1}d(\tilde{x}, (\hat{\partial} f)^{-1}(0)) \le d(0, \hat{\partial} f(\tilde{x})).$$

Together with (4.7), we have $\kappa < \frac{\tau(1+p)^{1+p}}{p}$, which contradicts the definition of τ and completes the proof of part (i).

(ii) Pick any $x \in B(\bar{x}, \eta)$ and take a sequence $\{x_k\}$ in $(\hat{\partial} f)^{-1}(0)$ such that $d(x, (\hat{\partial} f)^{-1}(0)) = \lim_{k\to\infty}\|x_k - x\|$. Since $\bar{x} \in (\hat{\partial} f)^{-1}(0)$, it follows from the convexity of f that $f(\bar{x}) = f(x_k) = \inf_{u\in B[\bar{x},r]} f(u)$ for all $k \in \mathbb{N}$. Let $x^* \in \hat{\partial} f(x)$, then

$$f(x) - f(\bar{x}) = f(x) - f(x_k) \le \langle x^*, x - x_k\rangle \le \|x^*\|\|x - x_k\| \ \forall k \in \mathbb{N}.$$

This and (4.2) imply that

$$\tau\|x-\bar{x}\|^{p-1}d(x, (\hat{\partial} f)^{-1}(0))^2 \le f(x) - f(\bar{x}) \le \|x^*\|\|x - x_k\| \ \forall k \in \mathbb{N}.$$

Letting $k \to \infty$, we obtain that $\tau\|x-\bar{x}\|^{p-1}d(x, (\hat{\partial} f)^{-1}(0)) \le \|x^*\|$. Since x^* is arbitrarily chosen from $\hat{\partial} f(x)$, we conclude that (4.3) holds. The proof is complete. □

Under the stability assumption of pseudo metric subregularity of the subdifferential mapping, we obtain the following result involving the stability of pseudo weak sharp minimizer.

Corollary 4.1 *Let $p \in [1, +\infty)$, $r \in (0, +\infty]$, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function, and $\bar{x} \in (\hat{\partial} f)^{-1}(0)$. And let $\hat{\partial} f$ be pseudo metrically subregular of order p stable at $(\bar{x}, 0)$ under $C^{1,p}$ perturbation. Then, for any twice smooth function $g : X \to \mathbb{R}$ with $\nabla g \in C^{1,p}(X, \mathbb{R}, \bar{x})$, $\nabla g(\bar{x}) = 0$ and $\nabla^2 g(\bar{x}) = 0$, we have that $\bar{x}$ is a p-order pseudo weak sharp minimizer of $f + g$.*

Proof Pick any twice smooth function $g : X \to \mathbb{R}$ with $\nabla g \in C^{1,p}(X, \mathbb{R}, \bar{x})$, $\nabla g(\bar{x}) = 0$ and $\nabla^2 g(\bar{x}) = 0$, one has $\hat{\partial}(f + g)(\bar{x}) = \hat{\partial} f(\bar{x}) + \nabla g(\bar{x}) = \hat{\partial} f(\bar{x})$. And then, $\bar{x} \in (\hat{\partial}(f + g))^{-1}(0)$. By the assumption, we have $\hat{\partial}(f + g)$ is pseudo metrically subregular of order p at $(\bar{x}, 0)$. Then, the result directly follows from Theorem 4.1 (i). The proof is complete. □

5 Concluding remarks

The major efforts of this paper are dedicated to investigating the stability of pseudo-metric subregularity of order p under small smooth perturbations. Limit critical sets involving order p are employed as the basic tool to characterize sufficient conditions as well as equivalent description for pseudo-metric subregularity. In Example 1.2, it is pointed out that the property of Hölder metric subregularity is also not stable under small $C^{1,p}$ perturbation. Motivated by Theorem 3.1, to study Hölder metric subregularity of order p, we may adopt the following limit critical set $Cr'F(\bar{x}, \bar{y}, p)$: for $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$, $p \in [1, +\infty)$ and $(v, u^*) \in Y \times X^*$, we define that $(v, u^*) \in Cr'F(\bar{x}, \bar{y}, p)$ if there exist sequences $\{t_k\} \subset (0, +\infty)$, $\{(u_k, v_k^*)\} \subset S_X \times S_{Y^*}$ and $\{(v_k, u_k^*)\} \subset Y \backslash \{0\} \times X^*$ satisfying (3.22) and

$$t_k \to 0, \left(v_k, \frac{u_k^*}{d(\bar{x} + t_k u_k, F^{-1}(\bar{y}))^{p-1}} \right) \to (v, u^*).$$

Similar to the proof of Theorem 3.1 (by applying Lemma 3.1 (ii) instead of Lemma 3.1 (i)), it can also be shown that $(0, 0) \notin Cr'F(\bar{x}, \bar{y}, p)$ is a sufficient condition for Hölder metric subregularity. Then it is natural to propose the following open question: Is $(0, 0) \notin Cr'F(\bar{x}, \bar{y}, p)$ a characterization for the stability of Hölder metric subregularity under small $C^{1,p}$ perturbations?

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