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Existence conditions for solutions of bilevel vector equilibrium problems with application to traffic network problems with equilibrium constraints

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Abstract
In this paper, we introduce some strong and weak bilevel vector equilibrium problems in locally convex Hausdorff topological vector spaces and present some conditions for the existence of solutions to these problems by using the Kakutani–Fan–Glicksberg fixed-point theorem. Furthermore, as a real-world application, we obtain the existence of solutions to traffic network problems with equilibrium constraints.

Keywords Bilevel vector equilibrium problems · Traffic network problems with equilibrium constraints · Existence conditions

Mathematics Subject Classification 47J20 · 49J40

1 Introduction

Motivated from the papers [18,22] in 1994, Blum and Oettli [8] introduced equilibrium problems which motivated some optimization problems [16,20,28,43], varia-

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tional inequality problems [19,26], lower and upper bounded equilibrium problems [9,12,27], traffic network problems [13,41] and others. Usually one is interested in finding general and natural condition to guarantee the existence of solutions to optimization problems [16], lower and upper bounded equilibrium problems [9,12], equilibrium problems [21,30,37,42], variational relation problems [1,7,23,32,38] and other problems. In 2004, Mordukhovich [35] introduced and studied equilibrium problems with equilibrium constraints and these problems include bilevel optimization problems, bilevel variational inequality problems, mathematical program problems with equilibrium constraints, optimization problems with equilibrium constraints, optimization problems with variational inequality constraints, lower and upper bounded vector equilibrium problems with equilibrium constraints and traffic network problems with equilibrium constraints. Anh et al. [5], Chen et al. [11], Ding [14,15], Moudafi [36], Wangkeeree and Yimmuang [40], Chadli et al. [10] considered existence of solutions, Anh and Hung [2,3], Hung and Hai [24] considered stability properties of solutions and Chen et al. [11] considered the well-posedness of solutions. To the best of our knowledge nobody has considered existence conditions for solutions of strong and weak bilevel vector equilibrium problems and traffic network problems with equilibrium constraints so inspired by the papers of Anh and Hung [2,3], Hung and Hai [24], Yang and Pu [42], Chen et al. [11], Ding [14,15], Moudafi [36], Wangkeeree and Yimmuang [40], Chadli et al. [10], in this paper, we present existence conditions for solutions of strong and weak bilevel vector equilibrium problems with multifunctions in locally convex Hausdorff topological vector spaces and discuss applications to traffic network problems with equilibrium constraints.

The rest of the paper is organized as follows: In Sect. 2, we introduce some strong and weak bilevel vector equilibrium problems. In Sect. 3, we present some existence conditions for solutions to these problems. Traffic network problems with equilibrium constraints is discussed in Sect. 4.

2 Preliminaries

Let X, Z be real locally convex Hausdorff topological vector spaces, A be a nonempty compact subset of X and $C_1 \subset Z$ be a closed convex and pointed cone with $\mathrm{int}C_1 \neq \emptyset$, where $\mathrm{int}C_1$ is the interior of C_1. Let $K : A \rightrightarrows A$ and $F : A \times A \rightrightarrows Z$ be multifunctions.

Now, we consider the following *strong and weak vector quasi-equilibrium problems*:

(SQVEP) Find a point $\bar{x} \in K(\bar{x})$ such that

$$F(\bar{x}, y) \subset C_1, \quad \forall y \in K(\bar{x});$$

(WQVEP) Find a point $\bar{x} \in K(\bar{x})$ such that

$$F(\bar{x}, y) \not\subset -\mathrm{int}C_1, \quad \forall y \in K(\bar{x}).$$

We denote the solution sets of the problems (SQVEP) and (WQVEP) by $S_s(F)$ and $S_w(F)$, respectively.

Let P be a real locally convex Hausdorff topological vector space, $C_2 \subset P$ be a closed convex and pointed cone with int$C_2 \neq \emptyset$ and $H : A \times A \rightrightarrows P$ be a multifunction.

Also, we consider the following *strong and weak bilevel vector equilibrium problems*:

(SBVEP) Find a point $\bar{x}^* \in S_s(F)$ such that

$$H(\bar{x}^*, y^*) \subset C_2, \quad \forall y^* \in S_s(F);$$

(WBVEP) Find a point $\bar{x}^* \in S_w(F)$ such that

$$H(\bar{x}^*, y^*) \not\subset -\mathrm{int}C_2, \quad \forall y^* \in S_w(F),$$

where $S_s(F)$ and $S_w(F)$ are the solution sets of the strong and weak vector quasi-equilibrium problems, respectively.

We denote the solution sets of the problems (SBVEP) and (WBVEP) by $\Psi_s(H)$ and $\Psi_w(H)$, respectively, i.e.,

$$\begin{aligned}\Psi_s(H) = \{\bar{x}^* \in & S_s(F) : H(\bar{x}^*, y^*) \subset C_2, \ \forall y^* \in S_s(F) \\ & \text{and} \ \ F(\bar{x}, y) \subset C_1, \ \forall y \in K(\bar{x})\}.\end{aligned}$$

and

$$\begin{aligned}\Psi_w(H) = \{\bar{x}^* \in & S_w(F) : H(\bar{x}^*, y^*) \not\subset -\mathrm{int}C_2, \ \forall y^* \in S_w(F) \\ & \text{and} \ \ F(\bar{x}, y) \not\subset -\mathrm{int}C_1, \ \forall y \in K(\bar{x})\}.\end{aligned}$$

First, we recall the following well-known definitions:

Definition 2.1 (see [33]) Let X, Y be two topological vector spaces, $F : X \rightrightarrows Y$ be a multifunction and let $x_0 \in X$ be a given point.

(1) F is said to be *lower semi-continuous* (lsc) at $x_0 \in X$ if $F(x_0) \cap U \neq \emptyset$ for some open set $U \subseteq Y$ implies the existence of a neighborhood N of x_0 such that $F(x) \cap U \neq \emptyset$ for all $x \in N$.
(2) F is said to be *upper semi-continuous* (usc) at $x_0 \in X$ if, for each open set $U \supseteq G(x_0)$, there is a neighborhood N of x_0 such that $U \supseteq F(x)$ for all $x \in N$.
(3) F is said to be *continuous* at $x_0 \in X$ if it is both lsc and usc at $x_0 \in X$
(4) F is said to be *closed* at x_0 if, for each of the nets $\{x_\alpha\}$ in X converging to x_0 and $\{y_\alpha\}$ in Y converging to y_0 such that $y_\alpha \in F(x_\alpha)$, we have $y_0 \in F(x_0)$.

If $A \subset X$, then F is said to be *usc* (*lsc*, *continuous*, *closed*, respectively) on the set A if F is usc (lsc, continuous, closed, respectively) for all $x \in \mathrm{dom}F \cap A$. If $A \equiv X$, then we omit "on X" in the statement.

Lemma 2.1 (see [33]) *Let X, Y be two topological vector spaces and $F : X \rightrightarrows Y$ be a multifunction. Then we have the following:*

(1) *If F is upper semi-continuous with closed values, then F is closed.*
(2) *If F is closed and $F(X)$ is compact, then F is upper semi-continuous.*

Lemma 2.2 (see [6]) *Let X, Y be two topological vector spaces and $F : X \rightrightarrows Y$ be a multifunction. Then we have the following:*

(1) *F is lower semi-continuous at $x_0 \in X$ if and only if, for each net $\{x_\alpha\} \subseteq X$ which converges to $x_0 \in X$ and for each $y_0 \in F(x_0)$, there exists $\{y_\alpha\}$ in Y such that $y_\alpha \in F(x_\alpha)$, $y_\alpha \to y_0$.*
(2) *If F has compact values, then F is upper semi-continuous at $x_0 \in X$ if and only if, for each $\{x_\alpha\} \subseteq X$ which converges to $x_0 \in X$ and for each net $\{y_\alpha\}$ in Y such that $y_\alpha \in F(x_\alpha)$, there exist $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Lemma 2.3 (see [17]) *Let A be a nonempty convex compact subset of Hausdorff topological vector space X and N be a subset of $A \times A$ such that*

(i) *for each $x \in A$, $(x, x) \notin N$;*
(ii) *for each $y \in A$, the set $\{x \in A : (x, y) \in N\}$ is open on A;*
(iii) *for each $x \in A$, the set $\{y \in A : (x, y) \in N\}$ is convex or empty.*

Then there exists $x_0 \in A$ such that $(x_0, y) \notin N$ for all $y \in A$.

Lemma 2.4 (see [29]) *Let A be a nonempty compact convex subset of a locally convex Hausdorff vector topological space X. If $F : A \rightrightarrows A$ is upper semi-continuous and, for any $x \in A$, $F(x)$ is nonempty convex closed, then there exists $x^* \in A$ such that $x^* \in F(x^*)$.*

3 Existence of solutions

In this section, we establish some existence results for strong and weak bilevel vector equilibrium problems.

Definition 3.1 Let X, Z be two topological vector spaces and $C \subset Z$ be a nonempty closed convex cone. Suppose that $F : X \times X \rightrightarrows Z$ is a multifunction.

(1) F is said to be *strongly C-quasiconvex* (in the first variable) in a convex set $A \subset X$ if, for each $y \in X$, $x_1, x_2 \in A$ and $\lambda \in [0, 1]$, $F(x_1, y) \subset C$ and $F(x_2, y) \subset C$, then

$$F(\lambda x_1 + (1 - \lambda)x_2, y) \subset C.$$

(2) F is said to be *weakly C-quasiconvex* (in the first variable) in a convex set $A \subset X$ if, for each $y \in X$, $x_1, x_2 \in A$ and $\lambda \in [0, 1]$, $F(x_1, y) \not\subset -\text{int}C$ and $F(x_2, y) \not\subset -\text{int}C$, then

$$F(\lambda x_1 + (1 - \lambda)x_2, y) \not\subset -\text{int}C.$$

Remark 3.1 In Definition 3.1, if $X = \mathbb{R}$, $C = \mathbb{R}_-$ and $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a vector function, then it follows that, if, for all $y \in X, x_1, x_2 \in X$ and $\lambda \in [0, 1]$, $F(x_1, y) \le 0$, $F(x_2, y) \le 0$, then

$$F((1-\lambda)x_1 + \lambda x_2), y) \le 0.$$

This means that F is modified 0-level quasiconvex since the classical quasiconvexity says that, for each $y \in X$, $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,

$$F((1-\lambda)x_1 + \lambda x_2), y) \le \max\{F(x_1, y), F(x_2, y)\}.$$

Let X and Z be two topological vector spaces, $F : X \times X \rightrightarrows Z$ be a multifunction, $\theta \in Z$ and $C \subset Z$ be a closed convex cone. We use the following level-sets:

$$L_{\ge\theta}F := \{(x, y) \in X \times X : F(x, y) \subset \theta + C\}$$

and

$$L_{\not<\theta}F := \{(x, y) \in X \times X : F(x, y) \not\subset \theta - \text{int}C\}.$$

Now, we present some existence conditions on solution sets of the strong vector quasi-equilibrium problem (SQVEP).

Lemma 3.1 *Assume that, for the problem* (SQVEP)*,*

(i) *K is continuous on A with nonempty compact convex values;*
(ii) *for all $x \in A$, $F(x, x) \subset C_1$;*
(iii) *the set $\{y \in A : F(\cdot, y) \not\subset C_1\}$ is convex on A;*
(iv) *for all $y \in A$, $F(\cdot, y)$ is strongly C_1-quasiconvex on A;*
(v) *for all $(x, y) \in A \times A$, $L_{\ge 0}F$ is closed.*

Then problem (SQVEP) *has a solution, i.e., there exists $\bar{x} \in A$ such that $\bar{x} \in K(\bar{x})$ and*

$$F(\bar{x}, y) \subset C_1, \quad \forall y \in K(\bar{x}).$$

Moreover, the solution set of the problem (SQVEP) *is compact.*

Proof For all $x \in A$, we define a multifunction $M : A \rightrightarrows A$ by

$$M(x) = \{a \in K(x) : F(a, y) \subset C_1, \ \forall y \in K(x)\}.$$

First, we show that $M(x)$ is nonempty. Indeed, for all $y \in A$, $K(x)$ is a nonempty compact convex set. Set

$$N = \{a \in K(x) : F(a, y) \not\subset C_1\}.$$

Then we have the following:

(a) Condition (ii) implies that, for any $a \in K(x)$, $(a, a) \notin N$;
(b) Condition (iii) implies that, for any $a \in K(x)$, $\{y \in A : (a, y) \in N\}$ is convex on $K(x)$;
(c) Condition (v) implies that, for any $a \in K(x)$, $\{y \in K(x) : (a, y) \in N\}$ is open on $K(x)$.

From Lemma 2.3, there exists $a \in K(x)$ such that $(a, y) \notin N$ for all $y \in K(x)$, i.e., $F(a, y) \subset C_1$ for all $y \in K(x)$. Thus $M(x)$ is nonempty.

Next, we verify that $M(x)$ is a convex set. Let $a_1, a_2 \in M(x)$, $\lambda \in [0, 1]$ and put $a = \lambda a_1 + (1 - \lambda)a_2$. Since $a_1, a_2 \in K(x)$ and $K(x)$ is a convex set, we have $a \in K(x)$. Thus it follows that, for any $a_1, a_2 \in M(x)$,

$$F(a_1, y) \subset C_1, \quad \forall y \in K(x).$$

From condition (iv), since $F(\cdot, y)$ is strongly C_1-quasiconvex, we have

$$F(\lambda a_1 + (1 - \lambda)a_2, y) \subset C_1, \quad \forall \lambda \in [0, 1],$$

i.e., $a \in M(x)$. Therefore, $M(x)$ is convex.

Next, we prove that M is upper semi-continuous on A with nonempty compact values. Indeed, since A is a compact set, from Lemma 2.1(ii), we need only to show that M is a closed mapping. Consider a net $\{x_\alpha\} \subset A$ with $x_\alpha \to x \in A$ and let $a_\alpha \in M(x_\alpha)$ be such that $a_\alpha \to a_0$.

Now, we need to verify that $a_0 \in M(x)$. Since $a_\alpha \in K(x_\alpha)$ and K is upper semi-continuous on A with nonempty compact values, it follows that K is closed and so we have $a_0 \in K(x)$. Suppose that $a_0 \notin M(x)$. Then there exists $y_0 \in K(x)$ such that

$$F(a_0, y_0) \not\subset C_1. \tag{1}$$

It follows from the lower semi-continuity of K that there is a net $\{y_\alpha\}$ such that $y_\alpha \in K(x_\alpha)$ and $y_\alpha \to y_0$. Since $a_\alpha \in M(x_\alpha)$, we have

$$F(a_\alpha, y_\alpha) \subset C_1. \tag{2}$$

Condition (v) together with (2) yields

$$F(a_0, y_0) \subset C_1. \tag{3}$$

This is the contradiction from (1) and (3). Therefore, we conclude that $a_0 \in M(x)$. Hence M is upper semi-continuous on A with nonempty compact values.

Next, we need to prove the solution set $S_s(F) \neq \emptyset$. Indeed, since M is upper semi-continuous on A with nonempty compact values, from Lemma 2.4, there exists a point $\hat{x} \in A$ such that $\hat{x} \in M(\hat{x})$. This implies that $\hat{x} \in K(\hat{x})$ such that

$$F(\hat{x}, y) \subset C_1, \quad \forall y \in K(\hat{x}),$$

i.e., problem (SQVEP) has a solution.

Finally, we prove that $S_s(F)$ is compact and convex. In fact, since A is compact and $S_s(F) \subset A$, we need only to prove that $S_s(F)$ is closed. Consider a net $\{x_\alpha\} \subset S_s(F)$ with $x_\alpha \to x_0$. Now, we prove that $x_0 \in S_s(F)$. If $x_0 \notin S_s(F)$, there exists $y_0 \in K(x_0)$ such that $F(x_0, y_0) \not\subset C_1$. From the lower semi-continuity of K, it follows that, for any $x_0 \in K(x_0)$, there exists $x_\alpha \in K(x_\alpha)$ such that $x_\alpha \to x_0$. Since $x_\alpha \in S_s(F)$, there exists $x_\alpha \in K(x_\alpha)$ such that

$$F(x_\alpha, y_\alpha) \subset C_1, \quad \forall y \in K(x_\alpha).$$

It follows from the upper semi-continuity and compactness of K on A that there exists $y_0 \in K(x_0)$ such that $y_\alpha \to y_0$ (taking a subnet if necessary). Now condition (v) together with $(x_\alpha, y_\alpha) \to (x_0, y_0)$ gives

$$F(x_0, y_0) \subset C_1, \quad \forall y_0 \in K(x_0),$$

which is a contradiction. This means that $x_0 \in S_s(F)$. Thus $S_s(F)$ is a closed set. Therefore, $S_s(F)$ is a compact subset of A. This completes the proof. □

Definition 3.2 (see [33]) Let X, Y be two topological vector spaces, A be a nonempty subset of X, $F : A \rightrightarrows Y$ be a multifunction and $C \subset Y$ be a nonempty closed convex cone. Now F is said to be *upper C-continuous* at $x_0 \in A$ if, for any neighborhood U of the origin in Y, there is a neighborhood V of x_0 such that

$$F(x) \subset F(x_0) + U + C, \quad \forall x \in V.$$

Definition 3.3 (see [33]) Let X and Y be two topological vector spaces and A be a nonempty convex subset of X. A set-valued mapping $F : A \rightrightarrows Y$ is said to be *properly C-quasiconvex* if, for any $x, y \in A$ and $\lambda \in [0, 1]$,

$$\text{either} \quad F(x) \subset F(tx + (1-t)y) + C \quad \text{or} \quad F(y) \subset F(tx + (1-t)y) + C.$$

Remark 3.2 Yang and Pu [42] obtained some existence results for the strong vector quasi-equilibrium problem. Note, the assumptions of Theorem 3.3 in [42] are different from the assumptions of Lemma 3.1.

The following example shows that all the assumptions of Lemma 3.1 are satisfied, but Theorem 3.3 in [42] is not applicable. The reason is that F is neither upper C-continuous nor properly C-quasiconvex.

Example 3.1 Let $X = Z = \mathbb{R}$, $A = [-2, 2]$, $C = \mathbb{R}_+$ and let $K : A \rightrightarrows A$ and $F : A \times A \rightrightarrows Z$ be multifunctions defined by

$$K(x) = [0, 1],$$
$$F(x, y) = \begin{cases} \left[\frac{3}{2}, 2\right], & \text{if } x_0 = \frac{3}{2}, \\ \left[0, \frac{3}{2}\right], & \text{otherwise.} \end{cases}$$

It is clear to see that all the assumptions of Lemma 3.1 are satisfied. However, F is neither upper C-continuous nor properly C-quasiconvex at $x_0 = \frac{3}{2}$.

First, we prove that F is not upper C-continuous at $x_0 = \frac{3}{2}$. Let $U = [-\frac{1}{3}, \frac{1}{3}]$ be a neighborhood of the origin in Z, then, for any neighborhood $V = [\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon]$ of $x_0 = \frac{3}{2}$, where $\varepsilon > 0$, choose $\frac{3}{2} \neq x' \in V$ and $y = \frac{3}{2}$. Then we have

$$\begin{aligned} F(x', y) &= F\left(x', \frac{3}{2}\right) \\ &= \left[0, \frac{3}{2}\right] \\ &\not\subset F(x_0, y) + U + C \\ &= F\left(\frac{3}{2}, \frac{3}{2}\right) + \left[-\frac{1}{3}, \frac{1}{3}\right] + \mathbb{R}_+ \\ &= \left[\frac{3}{2}, 2\right] + \left[-\frac{1}{3}, \frac{1}{3}\right] + \mathbb{R}_+ \\ &= \left[\frac{7}{6}, \frac{7}{3}\right] + \mathbb{R}_+. \end{aligned}$$

Next, we show that F is not properly C-quasiconvex at $x_0 = \frac{3}{2}$. Let $y = \frac{3}{2}, \lambda = \frac{3}{2}$ and $x_1 = 1, x_2 = 0$. Then we have

$$\begin{aligned} F(x_1, y) = F\left(0, \frac{3}{2}\right) &= \left[0, \frac{3}{2}\right] \\ &\not\subset F(x_1\lambda + (1-\lambda)x_2, y) + C \\ &= F\left(\frac{3}{2}, \frac{3}{2}\right) + \mathbb{R}_+ = \left[\frac{3}{2}, 2\right] + \mathbb{R}_+ \end{aligned}$$

and

$$\begin{aligned} F(x_2, y) = F\left(0, \frac{3}{2}\right) &= \left[0, \frac{3}{2}\right] \\ &\not\subset F(x_1\lambda + (1-\lambda)x_2, y) + C \\ &= F\left(\frac{3}{2}, \frac{3}{2}\right) + \mathbb{R}_+ = \left[\frac{3}{2}, 2\right] + \mathbb{R}_+. \end{aligned}$$

Thus Lemma 3.1 can be applied, but Theorem 3.3 in [42] is not applicable.

To establish an existence results for problem (WQVEP), we can easily get the following corresponding result with Lemma 3.1 (we omit the proof).

Lemma 3.2 *Assume that for the problem* (WQVEP),

(i) *K is continuous on A with nonempty compact convex values;*
(ii) *for all $x \in A$, $F(x, x) \not\subset -\text{int}C_1$;*

(iii) *the set* $\{y \in A : F(\cdot, y) \subset -\text{int}C_1\}$ *is convex on* A*;*
(iv) *for all* $y \in A$*,* $F(\cdot, y)$ *is weakly* C*-quasiconvex on* A*;*
(v) *for all* $(x, y) \in A \times A$*,* $L_{\not\subset 0}F$ *is closed.*

Then problem (WQVEP) *has a solution, i.e., there exists* $\bar{x} \in A$ *such that* $\bar{x} \in K(\bar{x})$ *and*

$$F(\bar{x}, y) \not\subset -\text{int}C_1, \quad \forall y \in K(\bar{x}).$$

Moreover, the solution set of the problem (WQVEP) *is compact.*

Now we investigate sufficient optimality conditions for problem (SBVEP).

Theorem 3.1 *Suppose that all the conditions in Lemma 3.1 are satisfied,* $S_s(F)$ *is convex and the following additional conditions hold:*

(i′) *for all* $x^* \in A$*,* $H(x^*, x^*) \subset C_2$*;*
(ii′) *the set* $\{y^* \in A : H(\cdot, y^*) \not\subset C_2\}$ *is convex on* A*;*
(iii′) *for all* $y^* \in A$*,* $H(\cdot, y^*)$ *is strongly* C_2*-quasiconvex on* A*;*
(iv′) *for all* $y^* \in A$*,* $L_{\geq 0}H(\cdot, y^*)$ *is closed on* A*.*

Then problem (SBVEP) *has a solution, i.e., there exists* $\bar{x}^* \in A$ *such that* $\bar{x}^* \in S_s(F)$ *and*

$$H(\bar{x}^*, y^*) \subset C_2, \quad \forall y^* \in S_s(F).$$

Moreover, the solution set of the problem (SBVEP) *is compact.*

Proof For all $x^* \in A$, we define a multifunction $R : A \rightrightarrows A$ by

$$R(x^*) = \{b \in S_s(F) \mid H(b, y^*) \subset C_2, \quad \forall y^* \in S_s(F)\}.$$

First, we prove that $R(x^*)$ is nonempty. Indeed, for all $y^* \in A$, $S_s(F)$ is a nonempty compact convex set. Set

$$P = \{b \in S_s(F) : H(b, y^*) \not\subset C_2\}.$$

Then we have the following:

(a) Condition (i′) implies that, for any $b \in S_s(F)$, $(b, b) \notin P$;
(b) Condition (ii′) implies that, for any $b \in S_s(F)$, $\{y^* \in A : (b, y) \in P\}$ is convex on $S_s(F)$;
(c) Condition (iv′) implies that, for any $b \in S_s(F)$, $\{y^* \in S_s(F) : (b, y^*) \in P\}$ is open on $S_s(F)$.

From Lemma 2.3, there exists $b \in S_s(F)$ such that $(b, y^*) \notin P$ for all $y^* \in S_s(F)$, i.e., $H(b, y^*) \subset C_2$ for all $y^* \in S_s(F)\}$. Thus it follows that $R(x^*)$ is nonempty.

Next, we show that $R(x^*)$ is a convex set. Let $b_1, b_2 \in R(x^*)$ and $\lambda \in [0, 1]$ and put $b = \lambda b_1 + (1-\lambda)b_2$. Since $b_1, b_2 \in S_s(F)$ and $S_s(F)$ is a convex set, we have $b \in S_s(F)$. Thus it follows that, for all $b_1, b_2 \in R(x^*)$,

$$H(b_1, y^*) \subset C_2, \quad \forall y^* \in R(x^*).$$

From condition (iii′), since $H(\cdot, y^*)$ is strongly C_2-quasiconvex, we have

$$H(\lambda b_1 + (1-\lambda)b_2, y^*) \subset C_2, \quad \forall \lambda \in [0, 1],$$

i.e., $b \in R(x^*)$. Thus, $R(x^*)$ is convex.

Next, we prove that R is upper semi-continuous on A with nonempty compact values. Indeed, since A is a compact set, from Lemma 2.1 (ii), we need only to show that R is a closed mapping. Consider a net $\{x^*_\alpha\} \subset A$ with $x^*_\alpha \to x^* \in A$ and let $b_\alpha \in R(x^*_\alpha)$ be such that $b_\alpha \to b_0$.

Now, we need to show that $b_0 \in R(x^*)$. Since $b_\alpha \in S_s(F)$ and $S_s(F)$ is compact, we have $b_0 \in S_s(F)$. Suppose that $b_0 \notin R(x^*)$. Then there exists $y^* \in S_s(F)$ such that

$$H(b_0, y^*) \not\subset C_2. \tag{4}$$

On the other hand, since $b_\alpha \in R(x^*_\alpha)$, we have

$$H(b_\alpha, y^*) \subset C_2, \quad \forall y^* \in S_s(F). \tag{5}$$

Now condition (iii′) together with (5) gives

$$H(b_0, y^*) \subset C_2, \tag{6}$$

which is a contradiction from (4) and (6). Thus $b_0 \in R(x^*)$. Hence R is upper semi-continuous on A with nonempty compact values.

Next, we prove that the solution set $S_s(H)$ is nonempty. In fact, since R is upper semi-continuous on A with nonempty compact values, from Lemma 2.4, there exists a point $\hat{x}^* \in A$ such that $\hat{x}^* \in R(\hat{x}^*)$. Hence there exists $\hat{x}^* \in S_s(F)$ such that

$$H(\hat{x}^*, y^*) \subset C_2, \quad \forall y^* \in S_s(F),$$

i.e., the problem (SBVEP) has a solution.

Finally, we prove that $S_s(H)$ is compact and convex. Consider a net $\{x^*_\alpha\} \subset S_s(H)$ with $x^*_\alpha \to x^*_0$. Now, we prove that $x^*_0 \in S_s(H)$. Indeed, from the closedness of $S_s(F)$, there exists $x^*_\alpha \in S_s(F)$ such that $x^*_\alpha \to x^*_0$. Since $x^*_\alpha \in S_s(H)$, there exists $x_\alpha \in S_s(F)$ such that

$$H(x^*_\alpha, y^*) \subset C_2, \quad \forall y^* \in S_s(F).$$

Now condition (iv') together with $x^*_\alpha \to x^*_0$ yields

$$H(x^*_0, y^*) \subset C_2, \quad \forall y^* \in S_s(F),$$

so $x^*_0 \in S_s(H)$. Thus $S_s(H)$ is a closed set. Since $S_s(H) \subset S_s(F)$ and $S_s(F)$ is compact it follows that $S_s(H)$ is a compact subset of A. This completes the proof. □

For problem (WBVEP), we obtain a similar conclusion as in Theorem 3.1.

Theorem 3.2 *Suppose that all the conditions in Lemma 3.2 are satisfied, $S_w(F)$ is convex and the following additional conditions hold:*

(i′) *for all $x^* \in A$, $H(x^*, x^*) \not\subset -\text{int}C_2$;*
(ii′) *the set $\{y^* \in A : H(\cdot, y^*) \subset -\text{int}C_2\}$ is convex on A;*
(iii′) *for all $y^* \in A$, $H(\cdot, y^*)$ is weakly C_2-quasiconvex on A;*
(iv′) *for all $y^* \in A$, $L_{\not\leq 0}H(\cdot, y^*)$ is closed on A.*

Then problem (WBVEP) *has a solution, i.e., there exists $\bar{x}^* \in A$ such that $\bar{x}^* \in S_w(F)$ and*

$$H(\bar{x}^*, y^*) \not\subset -\text{int}C_2, \quad \forall y^* \in S_w(F).$$

Moreover, the solution set of the problem (WBVEP) *is compact.*

Remark 3.3 Our main results, Theorems 3.1 and 3.2 , are new and completely different from the results obtained by Moudafi [36], Ding [14,15], Wangkeeree and Yimmuang [40], Chen et al. [11] and Chadli et al. [10].

4 Application to traffic network problems with equilibrium constraints

We discuss the traffic network problems, which was considered by many authors (see e.g. [4,13,25,31,34,39,41] and the references therein).

Consider a transportation network $G = (M, N)$, where M denotes the set of nodes and N denotes the set of arcs. Let $Q = (Q_1, Q_2, \ldots, Q_n)$ be the set of origin-destination pairs (O/D pairs in short). Assume that the pair $Q_i, i = 1, 2, \ldots, n$ is connected by a set S_i of paths and S_i contains $s_i \geq 1$ paths. Let $L = (L_1, L_2, \ldots, L_m)$ be the paths vector flow, where $m = \sum_{i=1}^{n} s_i$. Let the capacity restriction be

$$L \in A = \{L \in \mathbb{R}^m : 0 \leq \omega_p \leq L_p \leq \Omega_p, p = 1, 2, \ldots, m\},$$

where ω_p and Ω_p are given real numbers, $A \subseteq \mathbb{R}^m$ a nonempty set. Assume further that the travel cost on the path flow $L_p, p = 1, 2, \ldots, m$, depends on the whole path vector flow L and $T_p(L) \geq 0$. Then, we have the path cost vector $T(L) = (T_1(L), T_2(L), \ldots, T_m(L))$.

A path flow vector $\bar{L}$ is said to be an equilibrium flow vector if

$$\forall Q_i, \forall \xi \in S_i, \forall \tau \in S_i \text{ such that } [T_\xi(\bar{L}) < T_\tau(\bar{L})] \Rightarrow [\bar{L}_\xi = \Omega_\xi \ \text{ or } \ \bar{L}_\tau = \omega_\tau].$$

Suppose the travel demand ψ_i of the O/D pair $Q_i, i = 1, 2, \ldots, n$, depend on the equilibrium flows $\bar{L}$. Hence, considering all the O/D pairs, we have a mapping $\psi : \mathbb{R}^m_+ \to \mathbb{R}^n_+$. We use the Kronecker notation

$$\phi_{i\tau} = \begin{cases} 1 & \text{if } \tau \in S_i, \\ 0 & \text{if } \tau \notin S_i. \end{cases}$$

and

$$\phi = \{\phi_{i\tau}\}, \quad i = 1, 2, \ldots n, \text{ and } \tau = 1, 2, \ldots m.$$

Then, the path vector flows meetings the travel demands are called the feasible path vector flows and form the constraint set

$$K(\bar{L}) = \{L \in A, \phi L = \psi(\bar{L})\}.$$

A path vector flow $\bar{L} \in K(\bar{L})$ is an equilibrium flow if and only if it is a solution of the following quasivariational inequality.
(TNP) finding $\bar{L} \in K(\bar{L})$ such that

$$\langle T(\bar{L}), L - \bar{L}\rangle \geq 0, \forall L \in K(\bar{L}).$$

The following example describes the traffic network problem (TNP).

Example 4.1 Let $m = n = 2$ and a traffic network (see Fig. 1) consists of three nodes $\{1, 2, 3\}$, three arcs $\{(\overrightarrow{12}), (\overrightarrow{13}), (\overrightarrow{32})\}$ and two O/D pairs $Q_1 = (1, 2)$, $Q_2 = (1, 3)$. The O/D pairs $Q_i, i = 1, 2$ is connected by a set S_i of paths, where $S_1 = \{\tau_1 = (12)\}$, $S_2 = \{\tau_2 = (13)\}$. Then, $s_1 = 1, s_2 = 1, m = \sum_{i=1}^{2} s_i = s_1 + s_2 = 2$ and

$$\phi_{i\tau} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let the capacities on paths be $\Omega_p = 1$, $p = 1, 2$. Hence,

$$A = \{L \in \mathbb{R}^2_+ : 0 \leq L_p \leq 1, p = 1, 2\}.$$

Let the cost of a flow on path p be equal to this flow, this means that

$$T_p(L) = \{L_p\}, p = 1, 2, T(L) = \{L\}.$$

Assume the demand $\psi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is defined by $\psi(L) = (\psi_1(L), \psi_2(L))$. We have

$$\begin{aligned} K(\bar{L}) =& \{L \in A : \phi L = \psi \bar{L}\} \\ =& \left\{(L_1, L_2) \in \mathbb{R}^2_+ : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}. \end{aligned}$$

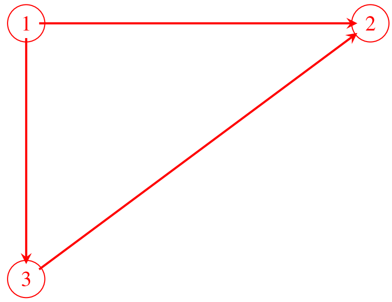

Fig. 1 Illustration of the traffic network

It is easy to see that $\bar{L} = (1, 1) \in K(\bar{L})$ satisfied

$$\langle T(\bar{L}), L - \bar{L} \rangle \geq 0, \forall L \in K(\bar{L}).$$

Thus, it follows that $\bar{L} = (1, 1)$ is a solution of the traffic network problem (TNP).

Next, we establish the traffic network problems with equilibrium constraints.

Let $X = \mathbb{R}^m$, $Z = \mathbb{R}^n$, $P = \mathbb{R}$, $C_2 = \mathbb{R}_+$ and A, K, C_1 be as in Sect. 2, let $L(X, P)$ be the space of all linear continuous operators from X into P, and $T : A \to L(X, P)$ be a vector function. We consider the following traffic network problems with equilibrium constraints.

(STNPEC) finding $\bar{L}^* \in S_s(F)$ such that

$$\langle T(\bar{L}^*), E^* - \bar{L}^* \rangle \geq 0, \forall E^* \in S_s(F).$$

(WTNPEC) finding $\bar{L}^* \in S_w(F)$ such that

$$\langle T(\bar{L}^*), E^* - \bar{L}^* \rangle \geq 0, \forall E^* \in S_w(F),$$

where $S_s(F)$ and $S_w(F)$ are the solution sets of (SQVEP) and (WQVEP), respectively. We denote the solution sets of (STNPEC) and (WTNPEC) by $\Psi_s(T)$ and $\Psi_w(T)$, respectively.

Next, we discuss sufficient optimality conditions for (STNPEC).

Theorem 4.1 *Suppose that all the conditions in Lemma 3.1 are satisfied, $\Psi_s(T)$ is convex and the following additional conditions hold:*

(i′) *for all $L^* \in A$, $\langle T(L^*), L^* - L^* \rangle \geq 0$;*
(ii′) *the set $\{E^* \in A, \langle T(.), E^* - . \rangle \not\geq 0\}$ is convex on A;*
(iii′) *for all $E^* \in A$, the function $L^* \mapsto \langle T(L^*), E^* - L^* \rangle$ is $\mathbb{R}_+$-quasiconvex on A;*
(iv′) *for all $E^* \in A$, the function $L^* \mapsto \langle T(L^*), E^* - L^* \rangle$ is continuous on A.*

Then, (STNPEC) *has a solution, i.e., there exists $\bar{L}^* \in A$ such that $\bar{L}^* \in S_s(F)$ and*

$$\langle T(\bar{L}^*), E^* - \bar{L}^* \rangle \geq 0, \forall E^* \in S_s(F).$$

Moreover, the solution set of the (STNPEC) *is compact.*

Proof Setting $X = \mathbb{R}^m$, $Z = \mathbb{R}^n$, $P = \mathbb{R}$, $C_2 = \mathbb{R}_+$ and $H(L^*, E^*) = \langle T(L^*), E^* - L^* \rangle$, problem (STNPEC) becomes a particular case of (SBVEP), so Theorem 4.1 is a direct consequence of Theorem 3.1. □

The following example shows that all the assumptions of Theorem 4.1 are satisfied.

Example 4.2 Let $X = Z = \mathbb{R}$, $A = [-1, 1]$, $C_1 = \mathbb{R}_+$ and let $K : A \rightrightarrows A$ and $F : A \times A \rightrightarrows Z$ be multifunctions defined by

$$\begin{aligned} K(L) =& [0, 1], \\ F(L, E) =& 2L + 3. \end{aligned}$$

It is clear to see that all the assumptions of Lemma 3.1 are satisfied. The solution set of the strong vector quasi-equilibrium problem is

$$\begin{aligned} S_s(F) =& \{L \in A \cap K(L) : F(L, E) \subset C_1, \ \forall E \in K(L)\} \\ =& \{L \in [-1, 1] \cap [0, 1] : 2L + 3 \geq 0, \ \forall E \in [0, 1]\} \\ =& [0, 1]. \end{aligned}$$

Next, we choose the travel cost $T(L) = [\frac{1}{2}, 1]$. Then, the solution set of the traffic network problem with equilibrium constraints is

$$\begin{aligned} \Psi_s(T) =& \{L^* \in A \cap S_s(F) : \langle T(L^*), E^* - L^* \rangle \geq 0, \ \forall E^* \in S_s(F) \\ & \text{and} \ \ F(L, E) \subset C_1, \ \forall E \in K(L)\} = \{0\}. \end{aligned}$$

Hence, all the assumptions of Theorem 4.1 hold and so the traffic network problem with equilibrium constraints (STNPEC) has a solution and the solution set of the (STNPEC) is compact.

Finally, applying Theorem 3.2, we also obtain the following result immediately.

Theorem 4.2 *Suppose that all the conditions in Lemma 3.2 are satisfied, $\Psi_w(T)$ is convex and the following additional conditions hold:*

(i′) *for all $L^* \in A$, $\langle T(L^*), L^* - L^* \rangle \geq 0$;*
(ii′) *the set $\{E^* \in A, \langle T(.), E^* - . \rangle \ngeq 0\}$ is convex on A;*
(iii′) *for all $E^* \in A$, the function $L^* \mapsto \langle T(L^*), E^* - L^* \rangle$ is $\mathbb{R}_+$-quasiconvex on A;*
(iv′) *for all $E^* \in A$, the function $L^* \mapsto \langle T(L^*), E^* - L^* \rangle$ is continuous on A.*

Then, (WTNPEC) *has a solution, i.e., there exists $\bar{L}^* \in A$ such that $\bar{L}^* \in S_w(F)$ and*

$$\langle T(\bar{L}^*), E^* - \bar{L}^* \rangle \geq 0, \forall E^* \in S_w(F).$$

Moreover, the solution set of the (WTNPEC) *is compact.*

Remark 4.1 Note in the literature there are no results on existence conditions for solutions of traffic network problems with equilibrium constraints, so as a result Theorems 4.1 and 4.2 are new.

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