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Radial non-potential Dirichlet systems with mean curvature operator in Minkowski space

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Abstract

We deal with a multiparameter Dirichlet system having the form

$$\begin{cases} \mathcal{M}(\mathrm{u}) + \lambda_1 \mu_1(|x|) f_1(\mathrm{u}, \mathrm{v}) = 0 \ \text{ in } \mathcal{B}(R), \\ \mathcal{M}(\mathrm{v}) + \lambda_2 \mu_2(|x|) f_2(\mathrm{u}, \mathrm{v}) = 0 \ \text{ in } \mathcal{B}(R), \\ \mathrm{u}|_{\partial\mathcal{B}(R)} = 0 = \mathrm{v}|_{\partial\mathcal{B}(R)}, \end{cases}$$

where $\mathcal{M}$ stands for the mean curvature operator in Minkowski space, $\mathcal{B}(R)$ is an open ball of radius R in $\mathbb{R}^N$, the parameters λ_1, λ_2 are positive, the functions $\mu_1, \ \mu_2 : [0, R] \to [0, \infty)$ are continuous and positive and the continuous functions f_1, f_2 satisfy some sign, growth and monotonicity conditions. Among others, these type of nonlinearities, include the Lane-Emden ones. For this system we show that there exists a continuous curve Γ splitting the first quadrant into two disjoint unbounded, open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the system has zero, at least one or at least two positive radial solutions according to $(\lambda_1, \lambda_2) \in \mathcal{O}_1$, $(\lambda_1, \lambda_2) \in \Gamma$ or $(\lambda_1, \lambda_2) \in \mathcal{O}_2$, respectively. The set $\mathcal{O}_1$ is adjacent to the coordinates axes $0\lambda_1$ and $0\lambda_2$ and the curve Γ approaches asymptotically to two lines parallel to the axes $0\lambda_1$ and $0\lambda_2$. Actually, this result extends to more general radial systems the recent existence/non-existence and multiplicity result obtained in the case of Lane-Emden systems.

Keywords Minkowski curvature operator · Multiparameter system · Positive solution · Non-existence/existence · Multiplicity

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1 Introduction

In this paper we study non-existence, existence and multiplicity of positive solutions for systems having the form

$$\begin{cases} \mathcal{M}(\mathrm{u}) + \lambda_1 \mu_1(|x|) f_1(\mathrm{u}, \mathrm{v}) = 0 \ \text{ in } \mathcal{B}(R), \\ \mathcal{M}(\mathrm{v}) + \lambda_2 \mu_2(|x|) f_2(\mathrm{u}, \mathrm{v}) = 0 \ \text{ in } \mathcal{B}(R), \\ \mathrm{u}|_{\partial \mathcal{B}(R)} = 0 = \mathrm{v}|_{\partial \mathcal{B}(R)}, \end{cases} \tag{1.1}$$

where $\mathcal{B}(R) = \{x \in \mathbb{R}^N : |x| < R\}$ $(R > 0, N \geq 2)$, $\mathcal{M}$ stands for the mean curvature operator in Minkowski space

$$\mathcal{M}(\mathrm{w}) = \operatorname{div}\left(\frac{\nabla \mathrm{w}}{\sqrt{1 - |\nabla \mathrm{w}|^2}}\right),$$

the parameters λ_1, λ_2 are positive, the functions $\mu_1, \ \mu_2 : [0, R] \to [0, \infty)$ are continuous with $\mu_1(r) > 0 < \mu_2(r)$ for all $r \in (0, R]$, under the following hypothesis on the continuous functions $f_1, f_2 : [0, +\infty)^2 \to [0, +\infty)$:

(H) (i) $f_1(s, t)$, $f_2(s, t)$ are quasi-monotone nondecreasing with respect to both s and t;
(ii) there exist constants $c > 0$, $p_1, q_2 > 1$ and $q_1, p_2 > 0$ such that

$$\begin{aligned} 0 < f_1(s, t) \leq c s^{p_1} t^{q_1}, \\ 0 < f_2(s, t) \leq c s^{p_2} t^{q_2}, \end{aligned} \tag{1.2}$$

for all $s, t > 0$.

Recall, a function $g(s, t) : [0, \infty)^2 \to [0, \infty)$ is said to be *quasi-monotone non-decreasing* with respect to t (resp. s) if for fixed s (resp. t) one has

$$g(s, t_1) \leq g(s, t_2) \text{ as } t_1 \leq t_2 \quad (\text{resp. } g(s_1, t) \leq g(s_2, t) \text{ as } s_1 \leq s_2).$$

In recent years, many papers were devoted to the study of Dirichlet problems for a single equation with operator $\mathcal{M}$ in a ball in $\mathbb{R}^N$ [1–3,5,7,8,13], while at our best knowledge, for systems with such an operator the study was recently initiated in [9]. So, in [7], for systems involving Lane-Emden type perturbations of the operator $\mathcal{M}$ and having a variational structure:

$$\begin{cases} \mathcal{M}(\mathrm{u}) + \lambda \mu(|x|)(p+1)\mathrm{u}^p \mathrm{v}^{q+1} = 0, \ \text{ in } \mathcal{B}(R), \\ \mathcal{M}(\mathrm{v}) + \lambda \mu(|x|)(q+1)\mathrm{u}^{p+1} \mathrm{v}^q = 0, \ \text{ in } \mathcal{B}(R), \\ \mathrm{u}|_{\partial \mathcal{B}(R)} = 0 = \mathrm{v}|_{\partial \mathcal{B}(R)}, \end{cases} \tag{1.3}$$

where the positive exponents $p, \ q$ satisfy $\max\{p, q\} > 1$ and the function $\mu : [0, R] \to [0, \infty)$ is continuous and $\mu(r) > 0$ for all $r \in (0, R]$, it was shown that there exists $\Lambda > 0$ such that (1.3) has zero, at least one or at least two positive solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$. This result extends the corresponding one obtained in [3] in the case of a single equation.

Then, in the recent paper [8] are considered non-potential radial systems having the form

$$\begin{cases} \mathcal{M}(\mathrm{u}) + \lambda_1 \mu_1(|x|)\mathrm{u}^{p_1}\mathrm{v}^{q_1} = 0, & \text{in } \mathcal{B}(R), \\ \mathcal{M}(\mathrm{v}) + \lambda_2 \mu_2(|x|)\mathrm{u}^{p_2}\mathrm{v}^{q_2} = 0, & \text{in } \mathcal{B}(R), \\ u|_{\partial\mathcal{B}(R)} = 0 = v|_{\partial\mathcal{B}(R)}, \end{cases} \tag{1.4}$$

where λ_1, λ_2 are two positive parameters, p_1, p_2, q_1, q_2 are positive exponents with $\min\{p_1, q_2\} > 1$ and the weight functions $\mu_1, \ \mu_2 : [0, R] \to [0, \infty)$ are assumed to be continuous with $\mu_1(r) > 0 < \mu_2(r)$ for all $r \in (0, R]$. Using fixed point index estimations and lower and upper solutions method, it was proved the existence of a continuous curve Γ splitting the first quadrant into two disjoint open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the system (1.4) has zero, at least one or at least two positive, radial solutions according to $(\lambda_1, \lambda_2) \in \mathcal{O}_1$, $(\lambda_1, \lambda_2) \in \Gamma$ or $(\lambda_1, \lambda_2) \in \mathcal{O}_2$, respectively.

On the other hand, in paper [10] are studied multiparameter Dirichlet systems having the form

$$\begin{cases} \mathcal{M}(u) + \lambda_1 f_1(u, v) = 0, \text{ in } \Omega, \\ \mathcal{M}(v) + \lambda_2 f_2(u, v) = 0, \text{ in } \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega}, \end{cases}$$

where Ω is a general bounded smooth domain in $\mathbb{R}^N$ and the continuous functions f_1, f_2 satisfy some sign, growth and quasi-monotonicity conditions. For such systems it has been obtained the existence of a hyperbola like curve which separates the first quadrant in two disjoint sets, an open one $\mathcal{O}$ and a closed one $\mathcal{F}$, such that the system has zero or at least one strictly positive solution, according to $(\lambda_1, \lambda_2) \in \mathcal{O}$ or $(\lambda_1, \lambda_2) \in \mathcal{F}$. Moreover, it has been showed that inside of $\mathcal{F}$ there exists an infinite rectangle in which the parameters being, the system has at least two strictly positive solutions. The approaches are based on a lower and upper solutions method and topological degree type arguments. This result extends, in some sense, to non-radial systems the existence/non-existence and multiplicity result obtained in [8] for the radial case.

In view of the above, the aim of this paper is two fold: firstly to complete the result obtained in [8] by showing that this still remains valid for more general systems of type (1.1) and secondly, to show that, at least in the radial case, a sharper result as the one in [8] can be obtained for such more general nonlinearities.

Since the solvability of (1.1) is guaranteed by [8, Corrolary 2.1], the main interest concerns the non-existence, existence and multiplicity of positive radial solutions. In this direction, the techniques employed in [8] for Lane-Emden systems will be adapted for system (1.1); we point out that the growth conditions on f_1, f_2 from hypothesis (H) play a key role in the proof of the main result.

Therefore, we show that there exists a continuous curve Γ splitting the first quadrant into two disjoint unbounded, open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the system (1.1) has zero, at least one or at least two positive radial solutions according to $(\lambda_1, \lambda_2) \in \mathcal{O}_1$, $(\lambda_1, \lambda_2) \in \Gamma$ or $(\lambda_1, \lambda_2) \in \mathcal{O}_2$, respectively. The set $\mathcal{O}_1$ is adjacent to the coordinates axes $0\lambda_1$ and $0\lambda_2$ and the curve Γ approaches asymptotically to two lines parallel to the axes $0\lambda_1$ and $0\lambda_2$ (Theorem 3.1).

Also, notice that, at least when speaking about radial solutions, this result is sharper then the one obtained in [10] due to the fact that here, the curve Γ which delimitates the multiplicity of solutions is the optimal one. Moreover, the conditions in hypothesis (H) are satisfied by Lane-Emden nonlinearities, so the result in [8] is recovered.

The rest of the paper is organized as follows. In Section 2 we reduce problem (1.1) to a homogeneous mixed boundary value problem and also we recall some results concerning lower (and upper) solutions method and some fixed point index estimations proved in [8]. The main non-existence, existence and multiplicity result for the multiparameter system (1.1) is stated and proved in Section 3. An example of nonlinearities different from the ones in [8] is provided.

2 Preliminaries

As usual, when we are seeking for radial solutions of (1.1), by setting $r = |x|$ and $\mathrm{u}(x) = u(r)$, $\mathrm{v}(x) = v(r)$, the Dirichlet problem (1.1) reduces to the homogeneous mixed boundary value problem:

$$\begin{cases} [r^{N-1}\varphi(u')]' + r^{N-1}\lambda_1\mu_1(r)f_1(u,v) = 0, \\ [r^{N-1}\varphi(v')]' + r^{N-1}\lambda_2\mu_2(r)f_2(u,v) = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0). \end{cases} \tag{2.1}$$

where

$$\varphi(y) = \frac{y}{\sqrt{1-y^2}} \quad (y \in \mathbb{R},\ |y| < 1).$$

By a *solution* of (2.1) we mean a couple of nonnegative functions $(u, v) \in C^1[0, R] \times C^1[0, R]$ with $||u'||_\infty < 1$, $||v'||_\infty < 1$ and $r \mapsto r^{N-1}\varphi(u'(r))$, $r \mapsto r^{N-1}\varphi(v'(r))$ of class C^1 on $[0, R]$, which satisfies problem (2.1). Here and below, $\|\cdot\|_\infty$ stands for the usual sup-norm on $C := C[0, R]$. We say that $u \in C$ is *positive* if $u > 0$ on $[0, R)$. By a *positive solution* of (2.1) we understand a solution (u, v) with both u and v positive.

Throughout this paper, the space $C^1 := C^1[0, R]$ will be understood with the norm $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$, while the product space $C^1 \times C^1$ will be endowed with the norm $||(u, v)|| = \max\{||u||_\infty, ||v||_\infty\} + \max\{||u'||_\infty, ||v'||_\infty\}$. We consider the closed subspace

$$C^1_M := \{(u, v) \in C^1 \times C^1 : u'(0) = u(R) = 0 = v(R) = v'(0)\}$$

and its closed, convex cone

$$K := \{(u, v) \in C^1_M : u \geq 0 \leq v \text{ on } [0, R]\}.$$

Also we denote $B(\rho) := \{(u, v) \in K : \|(u, v)\| < \rho\}$.

Let us define the linear operators

$$S : C \to C, \quad Su(r) = \frac{1}{r^{N-1}} \int_0^r t^{N-1} u(t) dt \quad (r \in (0, R]), \quad Su(0) = 0;$$
$$P : C \to C^1, \quad Pu(r) = \int_r^R u(t) dt \quad (r \in [0, R]).$$

It is easy to see that P is bounded and S is compact. Hence, the nonlinear operator $P \circ \varphi^{-1} \circ S : C \to C^1$ is compact. Denoting by N_{λ_i} the Nemytskii operator associated to $\lambda_i \mu_i f_i$ $(i = 1, 2)$, i.e.,

$$N_{\lambda_i} : C \times C \to C, \; N_{\lambda_i}(u, v) = \lambda_i \mu_i(\cdot) f_i(u_+(\cdot), v_+(\cdot)) \quad (u, v \in C),$$

$(s_+ := \max\{s, 0\})$ we have that N_{λ_i} is continuous and takes bounded sets into bounded sets.

If A is a subset of K, we set

$$\mathcal{K}(A) := \{T \mid T : A \to K \text{ is a compact operator}\}.$$

Also, given a bounded open (in K) subset $\mathcal{O}$ of K, we denote by $i(T, \mathcal{O})$ the fixed point index of the operator $T \in \mathcal{K}(\overline{\mathcal{O}})$ on $\mathcal{O}$ with respect to K [6].

The following proposition follows from Propositions 2.1 and 2.2 in [8].

Proposition 2.1 (i) *A couple of functions $(u, v) \in K$ is a solution of (2.1) iff it is a fixed point of the compact nonlinear operator*

$$\mathcal{D}_{\lambda_1,\lambda_2} : K \to K, \quad \mathcal{D}_{\lambda_1,\lambda_2} = \left(P \circ \varphi^{-1} \circ S \circ N_{\lambda_1}, P \circ \varphi^{-1} \circ S \circ N_{\lambda_2}\right).$$

In addition, for all $(u, v) \in K$, it holds

$$\|\mathcal{D}_{\lambda_1,\lambda_2}(u, v)\| < R + 1. \tag{2.2}$$

(ii) *For all $d \geq R + 1$ it holds*

$$i(\mathcal{D}_{\lambda_1,\lambda_2}, B(d)) = 1. \tag{2.3}$$

In particular, problem (2.1) always has a solution.

The following two lemmas are immediate consequences of [8, Lemma 3.2] and [7, Lemma 2.4].

Lemma 2.1 *Assume (H). If there is some $M > 0$ such that either*

$$\lim_{s \to 0_+} \frac{f_1(s, t)}{s} = 0 \text{ uniformly with } t \in [0, M], \tag{2.4}$$

or

$$\lim_{t\to 0_+}\frac{f_2(s,t)}{t}=0 \text{ uniformly with } s\in[0,M], \tag{2.5}$$

then there exists $\rho_0=\rho_0(\lambda_1,\lambda_2)>0$ *such that*

$$i(\mathcal{D}_{\lambda_1,\lambda_2}, B(\rho))=1 \text{ for all } 0<\rho\le\rho_0.$$

Lemma 2.2 *Under assumption* (H)*, if* (u,v) *is a nontrivial solution of problem* (2.1)*, then* (u,v) *is a positive solution with both u and v strictly decreasing.*

Definition 2.1 A *lower solution* of (2.1) is a couple of nonnegative functions $(\alpha_u,\alpha_v)\in C^1\times C^1$, such that $\|\alpha_u'\|_\infty<1$, $\|\alpha_v'\|_\infty<1$, the mappings $r\mapsto r^{N-1}\varphi(\alpha_u'(r))$, $r\mapsto r^{N-1}\varphi(\alpha_v'(r))$ are of class C^1 on $[0,R]$ and satisfies

$$\begin{cases}[r^{N-1}\varphi(\alpha_u')]'+r^{N-1}\lambda_1\mu_1(r)f_1(\alpha_u,\alpha_v)\ge 0,\\ [r^{N-1}\varphi(\alpha_v')]'+r^{N-1}\lambda_2\mu_2(r)f_2(\alpha_u,\alpha_v)\ge 0,\\ \alpha_u(R)=0,\quad \alpha_v(R)=0.\end{cases} \tag{2.6}$$

An *upper solution* $(\beta_u,\beta_v)\in C^1\times C^1$ is defined by reversing the first two inequalities in (2.6) and asking $\beta_u(R)\ge 0,\ \beta_v(R)\ge 0$ instead of $\alpha_u(R)=0,\ \alpha_v(R)=0$.

The following lemma is an immediate consequence of [8, Lemma 3.1].

Lemma 2.3 *Assume that* (2.1) *has a lower solution* (α_u,α_v) *and* $f_1(s,t)$ *(resp.* $f_2(s,t)$*) is quasi-monotone nondecreasing with respect to t (resp. s) and let*

$$\mathcal{A}_\alpha=\mathcal{A}_{(\alpha_u,\alpha_v)}:=\{(u,v)\in K:\alpha_u\le u,\ \alpha_v\le v\}.$$

Then, the following hold true:

(i) *problem* (2.1) *has always a solution in* $\mathcal{A}_\alpha$*;*
(ii) *if* (2.1) *has an unique solution* (u_0,v_0) *in* $\mathcal{A}_\alpha$ *and there exists* $\rho_0>0$ *such that* $\overline{B}((u_0,v_0),\rho_0):=\{(u,v)\in K\ :\ \|(u-u_0,v-v_0)\|\le\rho_0\}\subset\mathcal{A}_\alpha$*, then*

$$i(\mathcal{D}_{\lambda_1,\lambda_2}, B((u_0,v_0),\rho))=1,\quad \text{for all } 0<\rho\le\rho_0.$$

3 Non-existence, existence and multiplicity

In this section, under hypothesis (H), we study the existence and multiplicity of positive solutions for system (1.1). We employ here the technique used in [8] for the study of a system involving Lane-Emden nonlinearities to the more general system (1.1). For this, we consider the corresponding radial problem (2.1).
Setting

$$\mathcal{S}:=\{(\lambda_1,\lambda_2):\ \lambda_1,\lambda_2>0 \text{ and (2.1) has at least one positive solution}\},$$

we know that $\mathcal{S}$ is nonempty and unbounded in both directions of the axes $0\lambda_1$ and $0\lambda_2$ (see [8, Theorem 2.3]).

Lemma 3.1 *Assume (H). Then, the followings are true:*

(i) *There exist $\lambda_1^*, \lambda_2^* > 0$, such that $\mathcal{S} \subset [\lambda_1^*, +\infty) \times [\lambda_2^*, +\infty)$ and for all $(\lambda_1, \lambda_2) \in (0, +\infty)^2 \setminus ([\lambda_1^*, +\infty) \times [\lambda_2^*, +\infty))$, problem (2.1) has only the trivial solution.*

(ii) *If $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathcal{S}$, then $[\overline{\lambda}_1, +\infty) \times [\overline{\lambda}_2, +\infty) \subset \mathcal{S}$.*

(iii) *If $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathcal{S}$, then for all $(\lambda_1, \lambda_2) \in (\overline{\lambda}_1, +\infty) \times (\overline{\lambda}_2, +\infty)$, problem (2.1) has at least two positive solutions.*

Proof This follows the outline of the proof of Lemma 4.1 in [8].
(i) Let $\lambda_1, \lambda_2 > 0$ and (u, v) be a positive solution of (2.1). It follows from Lemma 2.2 that u and v are both strictly decreasing. Integrating the first equation in (2.1) on $[0, r]$, one obtains

$$-r^{N-1}\varphi(u'(r)) = \lambda_1 \int_0^r t^{N-1}\mu_1(t) f_1(u(t), v(t))dt, \quad \text{for all } r \in [0, R].$$

Since u, v are strictly decreasing on $[0, R]$ and using (1.2), we deduce

$$\begin{aligned} -r^{N-1}u'(r) &\le -r^{N-1}\varphi(u'(r)) \\ &\le \lambda_1 \int_0^r t^{N-1}\mu_1(t) c u^{p_1}(t) v^{q_1}(t)dt \\ &\le \lambda_1 \mu_1^M c u^{p_1}(0) v^{q_1}(0) r^N/N, \end{aligned}$$

where $\mu_i^M := \max_{[0,R]} \mu_i \quad (i = 1, 2)$. Integrating on $[0, R]$ we get

$$u(0) \le \lambda_1 \mu_1^M c u^{p_1}(0) v^{q_1}(0) R^2/(2N). \tag{3.1}$$

Analogously, one has

$$v(0) \le \lambda_2 \mu_2^M c u^{p_2}(0) v^{q_2}(0) R^2/(2N). \tag{3.2}$$

From $0 < u(0), v(0) < R$ and $p_1, q_2 > 1$ we obtain

$$\lambda_i > 2N/(\mu_i^M c R^{p_i+q_i+1}) > 0 \quad (i = 1, 2). \tag{3.3}$$

Consider now the nonempty sets

$$\begin{aligned} \mathcal{S}_1 :=& \{\lambda_1 > 0 : \ \exists\, \lambda_2 > 0 \text{ such that } (\lambda_1, \lambda_2) \in \mathcal{S}\}, \\ \mathcal{S}_2 :=& \{\lambda_2 > 0 : \ \exists\, \lambda_1 > 0 \text{ such that } (\lambda_1, \lambda_2) \in \mathcal{S}\} \end{aligned}$$

and let

$$(0 <)\ \lambda_i^* := \inf \mathcal{S}_i \ (< +\infty) \quad (i = 1, 2).$$

It follows that $\mathcal{S} \subset [\lambda_1^*, +\infty) \times [\lambda_2^*, +\infty)$ and for all $\lambda_1, \lambda_2 \in (0, +\infty)^2 \setminus ([\lambda_1^*, +\infty) \times [\lambda_2^*, +\infty))$, problem (2.1) has only the trivial solution (see Lemma 2.2).
(ii) Let $(\lambda_1^0, \lambda_2^0) \in [\overline{\lambda}_1, +\infty) \times [\overline{\lambda}_2, +\infty)$ be arbitrarily chosen and $(\overline{u}, \overline{v})$ be a positive solution for (2.1) with $\lambda_1 = \overline{\lambda}_1$ and $\lambda_2 = \overline{\lambda}_2$. Then, $(\overline{u}, \overline{v})$ is a lower solution of (2.1) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$. From Proposition 2.3 (i) and the fact that $(\overline{u}, \overline{v})$ is positive, we obtain $(\lambda_1^0, \lambda_2^0) \in \mathcal{S}$.
(iii) From (ii) we get that $(\overline{\lambda}_1, +\infty) \times (\overline{\lambda}_2, +\infty) \subset \mathcal{S}$ and let $(\lambda_1^0, \lambda_2^0) \in (\overline{\lambda}_1, +\infty) \times (\overline{\lambda}_2, +\infty)$. It remains to show that problem (2.1) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$ has a second positive solution. For this, let $(\overline{u}, \overline{v})$ be the lower solution constructed as above. We fix (u_0, v_0) a positive solution of (2.1) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$ such that $(u_0, v_0) \in \mathcal{A} := \mathcal{A}_{(\overline{u}, \overline{v})}$.

Now, we *claim* that there exists $\varepsilon > 0$ such that $\overline{B}((u_0, v_0), \varepsilon) \subset \mathcal{A}$. By using the quasi-monotonicity of the functions f_1 and f_2, for all $r \in [0, R/2]$, we have

$$\begin{aligned}\overline{u}(r) &= \int_r^R \varphi^{-1}\left(\frac{1}{t^{N-1}} \int_0^t s^{N-1}[\overline{\lambda}_1 \mu_1(s) f_1(\overline{u}(s), \overline{v}(s))]ds\right) dt \\ &< \int_r^R \varphi^{-1}\left(\frac{1}{t^{N-1}} \int_0^t s^{N-1}[\lambda_1^0 \mu_1(s) f_1(u_0(s), v_0(s))]ds\right) dt \\ &= u_0(r).\end{aligned}$$

Analogously we obtain that $\overline{v}(r) < v_0(r)$ on $[0, R/2]$. So, we can find $\varepsilon_1 > 0$ such that if $(u, v) \in K$ then

$$\|u - u_0\|_\infty \le \varepsilon_1 \Rightarrow \overline{u} \le u \text{ and } \|v - v_0\|_\infty \le \varepsilon_1 \Rightarrow \overline{v} \le v \text{ on } [0, R/2]. \quad (3.4)$$

On the other hand, for $r \in [R/2, R]$ one obtains $u_0'(r) < \overline{u}'(r)$ and $v_0'(r) < \overline{v}'(r)$. Thus, there is some $\varepsilon_2 \in (0, \varepsilon_1)$ such that if $(u, v) \in K$, then

$$\|u' - u_0'\|_\infty \le \varepsilon_2 \Rightarrow \overline{u}' > u' \text{ and } \|v' - v_0'\|_\infty \le \varepsilon_2 \Rightarrow \overline{v}' > v' \text{ on } [R/2, R].$$

From

$$u(r) = -\int_r^R u'(s)ds > -\int_r^R \bar{u}'(s)ds = \bar{u}(r)$$

we have that $u > \overline{u}$ (and, similarly $v > \overline{v}$) on $[R/2, R)$. This means that

$$\|u' - u_0'\|_\infty \le \varepsilon_2 \Rightarrow \overline{u} \le u \text{ and } \|v' - v_0'\|_\infty \le \varepsilon_2 \Rightarrow \overline{v} \le v \text{ on } [R/2, R]. \quad (3.5)$$

The claim follows from (3.4) and (3.5), by taking $\varepsilon \in (0, \varepsilon_2)$.

Next, if (2.1) has a second solution contained in $\mathcal{A}$, then it is nontrivial and the proof is complete. If not, by Lemma 2.3 we infer that

$$i(\mathcal{D}_{\lambda_1^0, \lambda_2^0}, B((u_0, v_0), \rho)) = 1 \text{ for all } 0 < \rho \le \varepsilon,$$

where $\mathcal{D}_{\lambda_1^0,\lambda_2^0}$ stands for the fixed point operator associated to problem (2.1) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$. Also, from Proposition 2.1 (ii) we have

$$i(\mathcal{D}_{\lambda_1^0,\lambda_2^0}, B(\rho)) = 1 \text{ for all } \rho \geq R+1,$$

and from Lemma 2.1 we get

$$i(\mathcal{D}_{\lambda_1^0,\lambda_2^0}, B(\rho)) = 1 \text{ for all } \rho > 0 \text{ sufficiently small.}$$

Let $\rho_1, \rho_2 > 0$ be sufficiently small and $\rho_3 \geq R+1$ be such that $\overline{B}((u_0, v_0), \rho_1) \cap \overline{B}(\rho_2) = \emptyset$ and $\overline{B}((u_0, v_0), \rho_1) \cup \overline{B}(\rho_2) \subset B(\rho_3)$. From the additivity-excision property of the fixed point index it follows that

$$i(\mathcal{D}_{\lambda_1^0,\lambda_2^0}, B(\rho_3)\backslash[\overline{B}((u_0, v_0), \rho_1) \cup \overline{B}(\rho_2)]) = -1.$$

Therefore, $\mathcal{D}_{\lambda_1^0,\lambda_2^0}$ has a fixed point $(u, v) \in B(\rho_3)\backslash[\overline{B}((u_0, v_0), \rho_1) \cup \overline{B}(\rho_2)]$. But this means that (2.1) has a second positive solution. □

Now, for $\theta \in (0, \pi/2)$, we denote

$$\mathcal{L}(\theta) := \{\lambda > 0 : \ (\lambda \cos\theta, \lambda \sin\theta) \in \mathcal{S}\},$$

which is a nonempty set, and we rewrite problem (2.1) in the form

$$\begin{cases} [r^{N-1}\varphi(u')]' + r^{N-1}\lambda \cos\theta \ \mu_1(r) f_1(u, v) = 0, \\ [r^{N-1}\varphi(v')]' + r^{N-1}\lambda \sin\theta \ \mu_2(r) f_2(u, v) = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0), \end{cases} \tag{3.6}$$

where $\lambda > 0$ is a real parameter.

Proposition 3.1 *There exists a continuous function* $\Lambda : (0, \pi/2) \to (0, \infty)$ *such that*

$$\lim_{\theta\to 0} \Lambda(\theta) \sin\theta - \lambda_2^* = 0 = \lim_{\theta\to\pi/2} \Lambda(\theta) \cos\theta - \lambda_1^* \tag{3.7}$$

and the followings hold true:

(i) $\Lambda(\theta) \in \mathcal{L}(\theta)$, *for every* $\theta \in (0, \pi/2)$;
(ii) *system* (2.1) *has at least two positive solutions, for all* $(\lambda_1, \lambda_2) \in (\Lambda(\theta)\cos\theta, +\infty) \times (\Lambda(\theta)\sin\theta, +\infty)$.

Proof This follows the outline of the proof of Proposition 4.1 in [8]. For each $\theta \in (0, \pi/2)$, let

$$\Lambda(\theta) := \inf \mathcal{L}(\theta). \tag{3.8}$$

Note that $\Lambda(\theta)$ is $< \infty$ because $\mathcal{L}(\theta) \neq \emptyset$ and > 0 by Lemma 3.1 (i). We first prove statements (i) and (ii).
(i) Let $\{\lambda^k\} \subset \mathcal{L}(\theta)$ be a decreasing sequence converging to $\Lambda(\theta)$ and $(u_k, v_k) \in K$ with $u_k > 0 < v_k$ on $[0, R)$ be such that

$$u_k = P \circ \varphi^{-1} \circ S \circ [\lambda^k \cos\theta \; \mu_1 f_1(u_k, v_k)],$$
$$v_k = P \circ \varphi^{-1} \circ S \circ [\lambda^k \sin\theta \; \mu_2 f_2(u_k, v_k)].$$

From (2.2) and Arzela-Ascoli theorem we obtain that there exists $(u, v) \in K$ such that, passing eventually to a subsequence, $\{(u_k, v_k)\}$ converges to (u, v) in $C \times C$ – with the usual product topology. Hence, $u \geq 0 \leq v$ and

$$u = P \circ \varphi^{-1} \circ S \circ [\Lambda(\theta) \cos\theta \; \mu_1 f_1(u, v)],$$
$$v = P \circ \varphi^{-1} \circ S \circ [\Lambda(\theta) \sin\theta \; \mu_2 f_2(u, v)].$$

From (3.1) and (3.2) we have that

$$u_k(0) \leq \lambda^k \cos\theta \; \mu_1^M c u_k^{p_1}(0) v_k^{q_1}(0) R^2/(2N)$$

and

$$v_k(0) \leq \lambda^k \sin\theta \; \mu_2^M c u_k^{p_2}(0) v_k^{q_2}(0) R^2/(2N),$$

which, taking into account that $0 < u_k(0), v_k(0) < R$, imply

$$u_k^{p_1 - 1}(0) > \frac{2N}{\lambda^k \mu_1^M c R^{q_1+2} \cos\theta}$$

and

$$v_k^{q_2 - 1}(0) > \frac{2N}{\lambda^k \mu_2^M c R^{p_2+2} \sin\theta}.$$

These ensure that there is a constant $c_1 > 0$ such that $u_k(0), v_k(0) > c_1$ for all k. This leads to $u(0), v(0) \geq c_1$, hence by Lemma 2.2 we get $u > 0 < v$ on $[0, R)$. Consequently, $\Lambda(\theta) \in \mathcal{L}(\theta)$.
(ii) This follows from statement (iii) in Lemma 3.1.

The continuity of Λ and the equalities in (3.7) can be proved in the same manner as it is done in the proof of Proposition 4.1 in [8]. □

Theorem 3.1 *Assume (H). Then, there exist* $\lambda_1^*, \lambda_2^* > 0$ *and a continuous function* $\Lambda : (0, \pi/2) \to (0, +\infty)$, *generating the curve*

$$(\Gamma) \begin{cases} \lambda_1(\theta) = \Lambda(\theta)\cos\theta \\ \lambda_2(\theta) = \Lambda(\theta)\sin\theta \end{cases}, \qquad \theta \in (0, \pi/2)$$

such that

(i) $\Gamma \subset [\lambda_1^*, +\infty) \times [\lambda_2^*, +\infty)$;
(ii) *the following asymptotic behaviors hold*

$$\lim_{\theta \to \pi/2} \lambda_2(\theta) = +\infty = \lim_{\theta \to 0} \lambda_1(\theta), \tag{3.9}$$

$$\lim_{\theta \to 0} \lambda_2(\theta) - \lambda_2^* = 0 = \lim_{\theta \to \pi/2} \lambda_1(\theta) - \lambda_1^*; \tag{3.10}$$

(iii) Γ *separates the first quadrant* $(0, +\infty) \times (0, +\infty)$ *in two disjoint sets* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *such that problem (1.1) has zero, at least one or at least two radial positive solutions, according to* $(\lambda_1, \lambda_2) \in \mathcal{O}_1$, $(\lambda_1, \lambda_2) \in \Gamma$ *or* $(\lambda_1, \lambda_2) \in \mathcal{O}_2$.

Proof This follows from Lemma 3.1 and Proposition 3.1. □

Example 3.1 Let $p_1, q_2 > 1$ and $q_1, p_2 > 0$. The conclusion of Theorem 3.1 is obtained for the following choices of f_1 and f_2 in problem (1.1):

(i) $f_1(\mathrm{u,v}) = \mathrm{u}^{p_1}\mathrm{v}^{q_1}$ and $f_2(\mathrm{u,v}) = \mathrm{u}^{p_2}\mathrm{v}^{q_2}$ – Lane-Emden type nonlinearities;
(ii) $f_1(\mathrm{u,v}) = \mathrm{u}^{p_1} \ln(1 + \mathrm{v}^{q_1})$ and $f_2(\mathrm{u,v}) = \mathrm{v}^{q_2} \ln(1 + \mathrm{u}^{p_2})$;
(iii) $f_1(\mathrm{u,v}) = \mathrm{u}^{p_1}\mathrm{v}^{q_1}\mathrm{arctg}(\mathrm{v})$ and $f_2(\mathrm{u,v}) = \mathrm{u}^{p_2}\mathrm{v}^{q_2}\mathrm{arctg}(\mathrm{u})$.

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