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Internal characterization of Brezis–Lieb spaces

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Abstract
In order to find an extension of Brezis–Lieb's lemma to the case of nets, we replace the almost everywhere convergence by the unbounded order convergence and introduce the pre-Brezis–Lieb property in normed lattices. Then we identify a wide class of Banach lattices in which the Brezis–Lieb lemma holds true. Among other things, it gives an extension of the Brezis–Lieb lemma for nets in L^p for $p \in [1, \infty)$.

Keywords $a.e.$-Convergence · Brezis–Lieb lemma · Banach lattice · uo-Convergence · Brezis–Lieb space · Pre-Brezis–Lieb property

Mathematics Subject Classification 46A19 · 46B42 · 46E30

1 Introduction

Let (Ω, Σ, μ) be a measure space in which, for every set $A \in \Sigma$, $\mu(A) > 0$, there exists $\Sigma \ni A_0 \subseteq A$, such that $0 < \mu(A_0) < \infty$. Given $p \in (0, \infty)$, denote by $\mathcal{L}^p = \{f : \int\limits_\Omega |f|^p \mu < \infty\}$ the vector space of p-integrable functions from Ω into $\mathbb{C}$. The Brezis–Lieb lemma [3, Thm.1] is known as the following useful refinement of the Fatou lemma.

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Theorem 1 (Brezis–Lieb's lemma for $\mathcal{L}^p$ $(0 < p < \infty)$) *Suppose* $f_n \xrightarrow{\text{a.e.}} f$ *and* $\int_\Omega |f_n|^p d\mu \leq C < \infty$ *for all* n *and some* $p \in (0, \infty)$. *Then*

$$\lim_{n\to\infty} \int_\Omega (|f_n|^p - |f_n - f|^p) d\mu = \int_\Omega |f|^p d\mu. \tag{1.1}$$

As the following example shows, Theorem 1 does not have a reasonable direct generalization for nets.

Example 1 Consider $[0, 1] \subset \mathbb{R}$ with the Lebesgue measure μ. Let Δ be the family of all finite subsets of $[0, 1]$ ordered by inclusion, and $\mathbb{I}_F$ be the indicator function of $F \in \Delta$. Then $\mathbb{I}_F \xrightarrow{\text{a.e.}} \mathbb{I}_{[0,1]}$ and $\int_0^1 |\mathbb{I}_F| d\mu = 0$, however

$$\lim_{F\to\infty} \int_0^1 (|\mathbb{I}_F| - |\mathbb{I}_F - \mathbb{I}_{[0,1]}|) d\mu = \lim_{F\to\infty} \int_0^1 (-|\mathbb{I}_{[0,1]}|) d\mu = -1 \neq 1 = \int_0^1 |\mathbb{I}_{[0,1]}| d\mu.$$

In order to avoid the collision, we restate Theorem 1 in the case of $1 \leq p < \infty$ in terms of the Banach space L^p of equivalence classes of functions from $\mathcal{L}^p(\mu)$ w.r. to μ (cf. [12, Thm.2]).

Theorem 2 (Brezis–Lieb's lemma for L^p $(1 \leq p < \infty)$) *Let* $\mathbf{f}_n \xrightarrow{\text{a.e.}} \mathbf{f}$ *in* $L^p(\mu)$ *and* $\|\mathbf{f}_n\|_p \to \|\mathbf{f}\|_p$, *where* $\|\mathbf{f}_n\|_p := \left[\int_\Omega |f_n|^p d\mu\right]^{1/p}$ *with* $f_n \in \mathcal{L}^p(\mu)$ *and* $f_n \in \mathbf{f}_n$. *Then* $\|\mathbf{f}_n - \mathbf{f}\|_p \to 0$.

Anton Schep kindly provided us with the reference [13, p.59] showing that Theorem 2 for $p > 1$ is due to Frigyes Riesz. Although in Theorem 2, we still have $a.e.$-convergent sequences in L^p, it is possible now (e.g. due to [7, Prop.3.1]) to replace the $a.e.$-convergence by the uo-convergence and restate Theorem 1 once more (cf. also [5, Prop.2.2] and [10, Prop.1.5]) as follows.

Theorem 3 (Brezis–Lieb's lemma for uo-convergent sequences in L^p) *Let* $x_n \xrightarrow{\text{uo}} x$ *in* L^p, *where* $p \in [1, \infty)$. *If* $\|x_n\|_p \to \|x\|_p$ *then* $\|x_n - x\|_p \to 0$.

Notice that Theorem 3 is a result of the Banach lattice theory which does not involve the measure theory directly. This observation motivates us to investigate those Banach lattices in which the statement of Theorem 3 holds true. We call them by the σ-Brezis–Lieb spaces. After introducing a geometrical property of normed lattices in Definition 2, we prove Theorem 4 which is the main result of the present paper. Theorem 4 gives an internal geometric characterization of σ-Brezis–Lieb's spaces and implies immediately the following result.

Proposition 1 *Let* $\mathbf{f}_\alpha \xrightarrow{\text{uo}} \mathbf{f}$ *in* $L^p(\mu)$ $(1 \leq p < \infty)$, *and* $\|\mathbf{f}_\alpha\|_p \to \|\mathbf{f}\|_p$. *Then* $\|\mathbf{f}_\alpha - \mathbf{f}\|_p \to 0$.

It is worth mentioning that Proposition 1 may serve as a net-extension of the Brezis–Lieb lemma (in its form of Theorem 3).

2 Brezis–Lieb spaces

In this section all normed lattices are considered over the real field $\mathbb{R}$. Recall that a net v_α in a vector lattice E *uo*-converges to $v \in E$ whenever, for every $u \in E_+$, the net $|v_\alpha - v| \wedge u$ converges in order to 0. For the further theory of vector lattices, we refer to [1,2] and, for the unbounded order convergence, to [7,8].

Definition 1 A normed lattice $(E, \|\cdot\|)$ is said to be a *Brezis–Lieb space* (*shortly*, BL*-space*) (resp. σ*-Brezis–Lieb space* σ-BL*-space*)) if, for any net x_α (resp. for any sequence x_n) in X such that $\|x_\alpha\| \to \|x_0\|$ (resp. $\|x_n\| \to \|x_0\|$) and $x_\alpha \xrightarrow{\text{uo}} x_0$ (resp. $x_n \xrightarrow{\text{uo}} x_0$), we have $\|x_\alpha - x_0\| \to 0$ (resp. $\|x_n - x_0\| \to 0$).

Trivially, any BL-space is a σ-BL-space, and any finite-dimensional normed lattice is a BL-space. Furthermore, by [7, Thm.3.2], any regular sublattice F of any BL-space (σ-BL-space) E is itself a BL-space (σ-BL-space). Taking into account the fact that the $a.e.$-convergence for sequences in L^p coincides with the *uo*-convergence [7, Prop.3.1], Theorem 3 says exactly that L^p is a σ-BL-space for $1 \leq p < \infty$. The following result is due to Vladimir Troitsky who also kindly provided us with its proof.

Proposition 2 *A Banach lattice* E *with countable sup property and a weak unit* w *is a* BL*-space iff* E *is a* σ-BL*-space.*

Proof By [7, Cor.3.5], $x_\alpha \xrightarrow{\text{uo}} x$ iff $|x_\alpha - x| \wedge w \xrightarrow{\text{o}} 0$. Suppose that E is a σ-Brezis–Lieb space. Suppose that $x_\alpha \xrightarrow{\text{uo}} x$ and $\|x_\alpha\| \to \|x\|$, yet $\|x_\alpha - x\| \not\to 0$. Then there exists $\varepsilon > 0$ such that for every α one can find $\beta \geq \alpha$ with $\|x_\beta - x\| \geq \varepsilon$.

It follows from $|x_\alpha - x| \wedge w \xrightarrow{\text{o}} 0$ that there is a net $(u_\gamma)_{\gamma \in \Gamma}$ such that $u_\gamma \downarrow 0$ and for every γ there exists α_0 such that $|x_\alpha - x| \wedge w \leq u_\gamma$ whenever $\alpha \geq \alpha_0$. Since E has countable sup property, we can find an increasing sequence γ_n in Γ such that $u_{\gamma_n} \downarrow 0$. For each n, find α_n such that $|x_\alpha - x| \wedge w \leq u_{\gamma_n}$ for all $\alpha \geq \alpha_n$.

Since $\|x_\alpha\| \to \|x\|$, we have $\Big|\|x_\alpha\| - \|x\|\Big| < 1$ for all sufficiently large α. So we can choose $\beta_1 \geq \alpha_1$ such that $\Big|\|x_{\beta_1}\| - \|x\|\Big| < 1$ and $\|x_{\beta_1} - x\| \geq \varepsilon$. Similarly, choose β_2 such that $\beta_2 > \beta_1$ and $\beta_2 \geq \alpha_2$ with $\Big|\|x_{\beta_2}\| - \|x\|\Big| < \frac{1}{2}$ and $\|x_{\beta_2} - x\| \geq \varepsilon$. Proceeding inductively, we get a strictly increasing sequence β_n such that $\beta_n \geq \alpha_n$, $\|x_{\beta_n} - x\| \geq \varepsilon$, and $\Big|\|x_{\beta_n}\| - \|x\|\Big| < \frac{1}{n}$ for every n. It follows that $\|x_{\beta_n}\| \to \|x\|$. Also, it follows from $\beta_n \geq \alpha_n$ that $|x_{\beta_n} - x| \wedge w \leq u_{\gamma_n}$, so that $x_{\beta_n} \xrightarrow{\text{uo}} x$. Therefore, $\|x_{\beta_n} - x\| \to 0$, which is a contradiction. □

Now, we consider examples of Banach lattices which are not σ-Brezis–Lieb spaces.

Example 2 The Banach lattice $(c_0, \|\cdot\|_\infty)$ is not a σ-BL-space. To see this, take $x_n = e_{2n} + \sum_{k=1}^{n} \frac{1}{k} e_k$ and $x = \sum_{k=1}^{\infty} \frac{1}{k} e_k$ in c_0. Clearly, $\|x\| = \|x_n\| = 1$ for all n and $x_n \xrightarrow{\text{uo}} x$, however $1 = \|x - x_n\|$ does not converge to 0.

We do not know whether or not for an arbitrary lattice norm $\|\cdot\|$ in c_0, which is equivalent to $\|\cdot\|_\infty$, the Banach lattice $(c_0, \|\cdot\|)$ is not a σ-BL-space.

Example 3 Since c_0 is an order ideal in c and in ℓ^∞, c_0 is regular there, and hence, both Banach lattices $(c, \|\cdot\|_\infty)$ and $(\ell^\infty, \|\cdot\|_\infty)$ are not σ-BL-spaces. Accordingly to the fact, that c_0 is a regular sublattice of c and to the last sentence of Example 2, it is also unknown whether or not the Banach lattice $(c, \|\cdot\|)$ is not a σ-BL-space for an arbitrary lattice norm $\|\cdot\|$ that is equivalent to $\|\cdot\|_\infty$.

In opposite to c, the Banach lattice ℓ^∞ is Dedekind complete. Let $\|\cdot\|$ be any lattice norm in ℓ^∞ that is equivalent to $\|\cdot\|_\infty$. Clearly, the norm $\|\cdot\|$ is not order continuous. Therefore, by Theorem 4, $(\ell^\infty, \|\cdot\|)$ is not a σ-BL-space.

A slight change of an infinite-dimensional BL-space can turn it into a normed lattice which is not even a σ-BL-space.

Example 4 Let E be an infinite-dimensional normed lattice. Let $F = \mathbb{R} \oplus_\infty E$. Take a disjoint sequence y_n in E such that $\|y_n\|_E = 1$ for all n. Then $y_n \xrightarrow{\text{uo}} 0$ in E [7, Cor.3.6]. Let $x_n = (1, y_n) \in F$. Then $\|x_n\|_F = \sup(1, \|y_n\|_E) = 1$ and $x_n = (1, y_n) \xrightarrow{\text{uo}} (1, 0) =: x$ in F, however

$$\|x_n - x\|_F = \|(0, y_n)\|_F = \|y_n\|_E = 1,$$

and so x_n does not converge to x in $(F, \|\cdot\|_F)$. Therefore $F = \mathbb{R} \oplus_\infty E$ is not a σ-BL-space.

In order to characterize BL-spaces, we introduce the following definition.

Definition 2 A normed lattice $(E, \|\cdot\|)$ is said to have the *pre-Brezis–Lieb property* (*shortly, pre-BL-property*), whenever $\limsup_{n\to\infty} \|u_0 + u_n\| > \|u_0\|$ for any disjoint normalized sequence $(u_n)_{n=1}^\infty$ in E_+ and for any $u_0 \in E_+$, $u_0 > 0$.

Every finite dimensional normed lattice E has the pre-BL-property. It is easy to see that the Banach lattices c_0, c, and ℓ^∞ w.r. to the supremum norm $\|\cdot\|_\infty$ do not have the pre-BL-property. The modification of the norm in an infinite-dimensional Banach lattice E with the pre-BL-property, as in Example 4, turns it into the Banach lattice $F = \mathbb{R} \oplus_\infty E$ without the pre-BL-property. Indeed, take a disjoint normalized sequence $(y_n)_{n=1}^\infty$ in E_+. Let $u_0 = (1, 0)$ and $u_n = (0, y_n)$ for $n \geq 1$. Then $(u_n)_{n=0}^\infty$ is a disjoint normalized sequence in F_+ with $\limsup_{n\to\infty} \|u_0 + u_n\| = 1 = \|u_0\|$.

Remarkably, it is not a coincidence. The following theorem identifies BL-spaces among σ-Dedekind complete Banach lattices.

Theorem 4 *For a σ-Dedekind complete Banach lattice E, the following conditions are equivalent:*

(1) *E is a Brezis–Lieb space;*
(2) *E is a σ-Brezis–Lieb space;*
(3) *E has the pre-Brezis–Lieb property, and the norm in E is order continuous.*

Proof The implication (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): We show first that E has the pre-BL-property. If not, then there exist a disjoint normalized sequence $(u_n)_{n=1}^{\infty}$ in E_+ and $0 < u_0 \in E_+$ with $\limsup_{n\to\infty} \|u_0 + u_n\| = \|u_0\|$. Since $\|u_0 + u_n\| \geq \|u_0\|$, then $\lim_{n\to\infty} \|u_0 + u_n\| = \|u_0\|$. Denote $v_n := u_0 + u_n$. By [7, Cor.3.6], $u_n \xrightarrow{\text{uo}} 0$ and hence $v_n \xrightarrow{\text{uo}} u_0$. Since E is a σ-BL-space and $\lim_{n\to\infty} \|v_n\| = \|u_0\|$, then $\|v_n - u_0\| \to 0$, a contradiction to $\|v_n - u_0\| = \|u_0 + u_n - u_0\| = \|u_n\| = 1$. Notice that, in this part of the proof of (2) $\Rightarrow$ (3), the σ-Dedekind completeness of E was not used.

Assume that the norm in E is not order continuous. Then, by the Fremlin–Meyer–Nieberg theorem (see e.g. [2, Thm.4.14]) there exists $y \in E_+$ and a disjoint sequence $e_k \in [0, y]$ such that $\|e_k\| \not\to 0$. Without lost of generality, we may assume $\|e_k\| = 1$ for all $k \in \mathbb{N}$. By the σ-Dedekind completeness of E, for any sequence $\alpha_n \in \mathbb{R}_+$, there exist the following vectors

$$x_0 = \bigvee_{k=1}^{\infty} e_k, \quad x_n = \alpha_{2n} e_{2n} + \bigvee_{k=1, k\neq n, k\neq 2n}^{\infty} e_k \quad (\forall n \in \mathbb{N}). \tag{2.1}$$

Now, we choose $\alpha_{2n} \geq 1$ in (2.1) such that $\|x_n\| = \|x_0\|$ for all $n \in \mathbb{N}$. Clearly, $x_n \xrightarrow{\text{uo}} x_0$. Since E is a σ-BL-space, then $\|x_n - x_0\| \to 0$, violating

$$\|x_n - x_0\| = \|(\alpha_{2n} - 1)e_{2n} - e_n\| = \|(\alpha_{2n} - 1)e_{2n} + e_n\| \geq \|e_n\| = 1.$$

The obtained contradiction shows that the norm in E is order continuous.

(3) $\Rightarrow$ (1): If E is not a Brezis–Lieb space, then there exists a net $(x_\alpha)_{\alpha\in A}$ in E such that $x_\alpha \xrightarrow{\text{uo}} x$ and $\|x_\alpha\| \to \|x\|$, but $\|x_\alpha - x\| \not\to 0$. Then $|x_\alpha| \xrightarrow{\text{uo}} |x|$ and $\||x_\alpha|\| \to \||x|\|$.

Notice that $\||x_\alpha| - |x|\| \not\to 0$. Indeed, if $\||x_\alpha| - |x|\| \to 0$, then, for any $\varepsilon > 0$, $(|x_\alpha|)_{\alpha\in A}$ is eventually in $[-|x|, |x|] + \varepsilon B_E$. Thus $(|x_\alpha|)_{\alpha\in A}$ is almost order bounded. Since E is order continuous and $x_\alpha \xrightarrow{\text{uo}} x$, then by [8, Pop.3.7.], $\|x_\alpha - x\| \to 0$, that is impossible. Therefore, without lost of generality, we may assume $x_\alpha \in E_+$ and, by normalizing, also $\|x_\alpha\| = \|x\| = 1$ for all α.

Passing to a subnet, denoted by x_α again, we may assume

$$\|x_\alpha - x\| > C > 0 \quad (\forall \alpha \in A). \tag{2.2}$$

Notice that $x \geq (x - x_\alpha)^+ = (x_\alpha - x)^- \xrightarrow{\text{uo}} 0$, and hence $(x_\alpha - x)^- \xrightarrow{\text{o}} 0$. The order continuity of the norm ensures

$$\|(x_\alpha - x)^-\| \to 0. \tag{2.3}$$

Denoting $w_\alpha = (x_\alpha - x)^+$ and using (2.2) and (2.3), we may also assume

$$\|w_\alpha\| = \|(x_\alpha - x)^+\| > C \quad (\forall \alpha \in A). \tag{2.4}$$

In view of (2.4), we obtain

$$2 = \|x_\alpha\| + \|x\| \geq \|(x_\alpha - x)^+\| = \|w_\alpha\| > C \quad (\forall \alpha \in A). \tag{2.5}$$

From $x_\alpha \xrightarrow{\text{uo}} x$ we have $w_\alpha \xrightarrow{\text{uo}} (x - x)^+ = 0$. It follows from [4, Thm.3.2] that there exists an increasing sequence of indices α_n and a disjoint sequence z_n such that

$$\|w_{\alpha_n} - z_n\| \to 0 \tag{2.6}$$

Without loss of generality, replacing z_n with $|z_n|$, we may assume $z_n \geq 0$. Passing to further increasing sequence of indices, we may assume that

$$\|w_{\alpha_n}\| \to M \in [C, 2] \quad (n \to \infty).$$

Now

$$\lim_{n\to\infty} \left\| M^{-1}x + \|z_n\|^{-1} z_n \right\| = M^{-1} \lim_{n\to\infty} \|x + z_n\| = [\text{by } (2.6)] =$$
$$M^{-1} \lim_{n\to\infty} \|x + w_{\alpha_n}\| = [\text{by } (2.3)] = M^{-1} \lim_{n\to\infty} \|x + (x_{\alpha_n} - x)\| =$$
$$M^{-1} \lim_{n\to\infty} \|x_{\alpha_n}\| = M^{-1} = \|M^{-1}x\|,$$

violating the pre-Brezis–Lieb property for $u_0 = M^{-1}x$ and $u_n = \|z_n\|^{-1} z_n$, $n \geq 1$. The obtained contradiction completes the proof. □

Anton Schep kindly informed us that a special case of Theorem 4 is due to Nakano [11, Thm.33.6].

Since every order continuous Banach lattice is Dedekind complete, the following result is a direct consequence of Theorem 4.

Corollary 1 *For an order continuous Banach lattice E, the following conditions are equivalent:*

(1) *E is a BL-space;*
(2) *E is a σ-BL-space;*
(3) *E has the pre-BL-property.*

Corollary 1 applied to the order continuous Banach lattices L^p $(1 \leq p < \infty)$ gives Proposition 1.

It follows from Theorem 4 that if E is a σ-Dedekind complete Banach lattice, then E is a σ-BL-space iff E has the pre-BL-property, and the norm in E is order continuous. This result is related to the following fact mentioned on page 28 of [9], where the same condition with weak convergence replaced with uo-convergence which is used in Definition 1: A Banach lattice $(E, \|\cdot\|)$ is order continuous iff there is an equivalent norm $\|\cdot\|_1$ on E so that

$$E \ni x_n \xrightarrow{w} x \text{ and } \|x_n\|_1 \to \|x\|_1 \Rightarrow \|x_n - x\|_1 \to 0.$$

Relationship between weak and *uo*-convergence have been studied in [14].

We do not know whether or not implication (2) $\Rightarrow$ (3) of Theorem 4 holds true without the assumption that the Banach lattice E is σ-Dedekind complete. More precisely:

Question 1 *Does every σ-Brezis–Lieb Banach lattice have an order continuous norm?*

In the proof of (2) $\Rightarrow$ (3) of Theorem 4, the σ-Dedekind completeness of E has been used only for showing that E has an order continuous norm. So, any σ-Brezis–Lieb Banach lattice has the pre-BL-property. Therefore, for answering in positive the question of possibility to drop σ-Dedekind completeness assumption in Theorem 4, it suffices to answer in positive the following question which is formally weaker than Question 1.

Question 2 *Does the pre-BL-property imply order continuity of the norm in the underlying Banach lattice?*

In the end of the paper, we mention one more question closely related to the question in the last sentence of Example 2.

Question 3 *Does the pre-BL-property of a Banach lattice E ensure that E is a KB-space?*

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