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On the convergence of sequence of maximal monotone operators of type (D) in Banach spaces

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Abstract
Within the setting of general real Banach spaces, we prove that the sequence of maximal monotone operators of type (D) graphically converges provided, their corresponding class of representative functions converge epigraphically. Moreover, we provide a condition to guarantee that the lower limit of a sequence of maximal monotone operators of type (D) is a maximal monotone operator of type (D) in real Banach spaces.

Keywords Maximal monotone operator · Monotone operator of type (D) · Representative function · Epi-convergence

Mathematics Subject Classification 47H05 · 49J52 · 47N10

1 Introduction

Monotone operators are an important class of operators used in the study of modern nonlinear analysis and various classes of optimization problems. The theory of monotone operators (multifunctions) were first introduced by Minty [20] and later it was used substantially in proving existence results in partial differential equations by Felix Browder and his school [1,2,4–6,8,14,15,27]. In particular, maximal monotone operators have found their plethora of applications in partial differential equations, optimization problems, variational inequalities and mathematical economics.

García and Lassonde [10], first studied the sequential lower limit of maximal monotone operators in reflexive Banach spaces. Further, they have applied it to prove the representability of the variational sum [3,22,23] and the variational composition [21]. In the recent years, Bueno et al. [7] studied the lower limit of a sequence of maximal

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monotone operators of type (D) in general Banach spaces and they prove that the lower limit of a sequence of maximal monotone operators of type (D) is representable. The type (D) operator was first introduced by Gosséz [13]. This operator plays a significant role to recover most of the results from the reflexive spaces to the non-reflexive spaces.

In the first part of the article, we provide a sufficient condition for graphical convergence of a sequence of maximal monotone operators of type (D). The representability of the lower limit of a sequence of maximal monotone operators of type (D) was established by Bueno et al. [7] in general Banach spaces. Here, we generalize the result of García and Lassonde [11, Theorem 2.2] to general Banach spaces by considering the sequence of maximal monotone operators of type (D). In Theorem 8, we prove that the lower limit of a sequence of maximal monotone operators of type (D) is representable through representative functions of the corresponding sequence of maximal monotone operators. Indeed, the representative function of the lower limit of a sequence of maximal monotone operators of type (D) is exactly wherever the corresponding sequence of representative functions converge epigraphically. Finally, we have established the maximal monotonicity of type (D) of the lower limit of a sequence of maximal monotone operators of type (D) provided their representative functions and conjugate representative functions have epi-convergence.

The remainder of this note is organized as follows. In Sect. 2, we present some basic notions and axillary results from convex analysis and monotone operator theory. A sufficient condition for convergence of a sequence of maximal monotone operators of type (D) is established in Sect. 3. Finally, the representability and maximal monotonicity of type (D) of the lower limit of a sequence of maximal monotone operators of type (D) through the convergence (in the sense of epi-convergence) of their representative functions are presented in Sect. 4.

2 Basic notations and auxiliary results

In this note, X will be denoted as a real Banach space with the norm, $\|.\|$. X^* is the topological dual of X. X and X^* are paired by $\langle x, x^*\rangle = x^*(x)$ for $x \in X$ and $x^* \in X^*$. The norm on the product space $X \times X^*$ is defined as $\|(x, x^*)\| = \|x\| + \|x^*\|$, for every $(x, x^*) \in X \times X^*$. Weak and weak star convergence are denoted by the notation $\overset{w}{\to}$ and $\overset{w^*}{\to}$ respectively. The dual of $X \times X^*$ is defined as $X^* \times X^{**}$ and the dual pairing is defined as $x^*(x) + x^{**}(x^*)$.

Let $f : X \to]-\infty, +\infty]$ be a function, its domain is defined as $\operatorname{dom} f := f^{-1}(\mathbb{R})$ and $\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$. f is said to be proper if $\operatorname{dom} f \neq \phi$. We denote $\Gamma(X)$ as the set of all lower semi-continuous convex functions from X into $]-\infty, +\infty]$. The Fenchel-conjugate of f is $f^* := X^* \to [-\infty, +\infty]$, given by

$$f^*(x^*) = \sup_{x \in X}[\langle x, x^*\rangle - f(x)].$$

Let f be any proper convex function then the subdifferential operator of f is defined as $\partial f : X \rightrightarrows X^* : x \mapsto \{x^* \in X^* |\ \langle y - x, x^*\rangle + f(x) \leq f(y), \forall y \in X\}$. Similarly, for $\epsilon \geq 0$, the ϵ-subdifferetial of f is defined by

$$\partial_\epsilon f = \{(x, x^*) : f(y) \geq f(x) + \langle y - x, x^* \rangle - \epsilon, \forall y \in X\}.$$

The duality map $J : X \to X^*$ is defined as $J := \partial(\frac{1}{2}\|.\|^2)$ and $J_\epsilon := \partial_\epsilon(\frac{1}{2}\|.\|^2)$. Using $f(x) = \frac{1}{2}\|x\|^2$ in the above definitions, we get

$$x^* \in J(x) \Leftrightarrow \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = \langle x, x^* \rangle$$

or equivalently,

$$J(x) = \{x^* \in X^* | \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

and

$$J_\epsilon(x) = \left\{x^* \in X^* | \ \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \leq \langle x, x^* \rangle + \epsilon\right\}.$$

Let $A : X \rightrightarrows X^*$ be a set-valued operator (also known as multifunction or point-to-set mapping) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$. The domain of A is denoted as $\text{dom}A := \{x \in X | \ Ax \neq \phi\}$ and the range of A is $\text{ran}A := \{x^* \in Ax | \ x \in \text{dom}A\}$. The graph of A is denoted as $\text{gra}A = \{(x, x^*) \in X \times X^* | \ x^* \in Ax\}$. The set-valued mapping $A : X \rightrightarrows X^*$ is said to be monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in \text{gra}A.$$

Let $A : X \rightrightarrows X^*$ be monotone and $(x, x^*) \in X \times X^*$. We say that (x, x^*) is monotonically related to $\text{gra}A$ if

$$\langle x - y, x^* - y^* \rangle \geq 0, \ \forall (y, y^*) \in \text{gra}A.$$

And a set-valued mapping A is said to maximal monotone if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). In the other words A is maximal monotone if any $(x, x^*) \in X \times X^*$ is monotonically related to $\text{gra}A$ belongs to $\text{gra}\ A$. A monotone operator $A : X \rightrightarrows X^*$ is representable [19] if there exists a function $f \in \Gamma(X \times X^*)$ such that

$$f(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*$$

and

$$f(x, x^*) = \langle x, x^* \rangle \iff (x, x^*) \in \text{gra}\ A.$$

In this case, f is called as a representative function of the operator A. We denote $\mathcal{H}(A)$ as the class of all representative functions for monotone operator A.

For our convenience, we recall some fundamental properties of maximal monotone operators. Let $A : X \rightrightarrows X^*$ be maximally monotone. We say A is of dense type or type (D) [12] if for every $(x^{**}, x^*) \in X^{**} \times X^*$ with

$$\inf_{(a,a^*)\in \text{gra } A} \langle a - x^{**}, a^* - x^* \rangle \geq 0,$$

there exists a bounded net $(a_\alpha, a_\alpha^*)_{\alpha\in\Gamma}$ in gra A such that $(a_\alpha, a_\alpha^*)_{\alpha\in\Gamma}$ converges to (x^{**}, x^*) with respect to (weak$^*\times$ strong) and A is said to be of type negative infimum (NI) [25] if

$$\inf_{(a,a^*)\in \text{gra } A} \langle a - x^{**}, a^* - x^* \rangle \leq 0, \quad \forall \ (x^{**}, x^*) \in X^{**} \times X^*.$$

By Simons [26, Theorem 36.3(a)] and Marques Alves and Svaiter [16, Theorem 4.4] we see that these two operators coincide. For a maximal monotone operator $A : X \rightrightarrows X^*$ we will define $\tilde{A} : X^{**} \rightrightarrows X^*$ as

$$\tilde{A} = \{(x^{**}, x^*) \in X^{**} \times X^* : (x^{**}, x^*) \text{ is monotonically related to gra } A\}.$$

When A is of type (D) $\tilde{A}$ is the unique maximal monotone extension on $X^{**} \times X^*$.

A sequence of sets $\{S_n\}_{n\in\mathbb{N}}$ is said to be converge to a set S, denoted as $S_n \longrightarrow S$ if

1. for every $x \in S$, there exists a sequence $\{x_n\}$ with $\lim x_n = x$ and $x_n \in S_n$ for n sufficiently large;
2. the cluster points of every sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in S_n$ for n sufficiently large belongs to S.

A sequence of functions $\{f_n\}$ is said to epi-converge to f, if epi $f_n \longrightarrow$ epi f or equivalently, if for any $x \in X$,

1. $\liminf_n f_n(x_n) \geq f(x)$ for every sequence $x_n \to x$;
2. $\limsup_n f_n(x_n) \leq f(x)$ for some sequence $x_n \to x$.

For more on epi-convergence, one may refer [24]. Let us collect some fundamental facts required for proving main results.

Fact 1 ([9], Theorem 3.10) *Let $A : X \rightrightarrows X^*$ be a maximal monotone operator. Then the Fitzpatrick function associated with A is defined as*

$$F_A : X \times X^* \to]-\infty, +\infty] : (x, x^*) \mapsto \sup_{(a,a^*)\in \text{gra } A} (\langle x, a^* \rangle + \langle a, x^* \rangle - \langle a, a^* \rangle)$$

is the minimal convex function f on $X \times X^*$ such that $f(x, x^*) \geq \langle x, x^* \rangle, \forall (x, x^*) \in X \times X^*$ and $f(x, x^*) = \langle x, x^* \rangle$ for $(x, x^*) \in$ gra A.

Fact 2 ([16], Theorem 3.6) *Let $A : X \rightrightarrows X^*$ be a maximal monotone operator. Then A is of type (D) if and only if* $\text{ran}(A + J_\epsilon(. - x_0)) = X^*$, *for all $x_0 \in X$ and $\epsilon > 0$.*

Fact 3 ([7], Proposition 3.2) *Let X be a real Banach space, $\{A_n : X \rightrightarrows X^*\}$ be a sequence of maximal monotone operators of type (D) and let (ϵ_n) be any sequence of positive numbers convergent to zero. If $A = \liminf A_n$ then $(x, x^*) \in$ gra A if and only if $x = \lim x_n$, where x_n is a solution of*

$$x^* \in A_n(x_n) + J_{\epsilon_n}(x_n - x).$$

Fact 4 ([17], Theorem 4.2) *Let $f : X \times X^* \to \mathbb{R} \cup \{\infty\}$ be proper lower-semicontinuous convex function such that*

$$f(x, x^*) \geq \langle x, x^* \rangle$$

and

$$f^*(x^*, x^{**}) \geq \langle x^* x^{**})\rangle.$$

Define

$$M_f = \{(x, x^*) \in X \times X^* : f(x, x^*) = \langle x, x^* \rangle\}.$$

Then,

1. $M_f := \{(x, x^*) \in X \times X^* : f^*(x^*, x) = \langle x, x^* \rangle\}.$
2. M_f *is maximal monotone.*
3. *Let F_{M_f} be the Fitzpatrick function of M_f that is*

$$F_{M_f}(x, x^*) = \sup_{(y, y^*) \in M_f} [\langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle].$$

Then

$$F_{M_f}(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*$$

and

$$F^*_{M_f}(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

Fact 5 ([18], Theorem 1.2) *Let X be a real Banach space and $A : X \rightrightarrows X^*$, Then the following conditions are equivalent:*

1. *A is type MA, that is A is maximal monotone and there exists some $h \in \mathcal{H}(A)$ such that $h^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle$ (and $h \geq \langle x, .x^* \rangle$) for all $(x^*, x^{**}) \in X^* \times X^{**}$ (and for all $(x, x^*) \in (X \times X^*)$).*
2. *A is of type NI.*

Fact 6 ([7], Lemma 3.1) *Let $A_n : X \rightrightarrows X^*$ be a sequence of maximal monotone operators of type (D). For any $(x, x^*) \in X \times X^*$, let $(x_n)_n$ be the sequence of solutions of the inclusion*

$$x^* \in A_n(x_n) + J_{\epsilon_n}(x_n - x),$$

for any $\epsilon_n > 0$ converging to 0. Then $(x_n)_n$, $(x_n^*)_n$ and $(w_n^*)_n$ are bounded, where $x_n^* \in A_n(x_n)$ and $w_n^* \in J_{\epsilon_n}(x_n - x)$.

3 A sufficient condition for convergence of sequence of maximal monotone operators of type (D)

In the following theorem, we establish a sufficient condition for the convergence of a sequence of maximal monotone operators of type (D) in Banach spaces.

Theorem 7 *Let $A_n : X \rightrightarrows X^*$ be a sequence of maximal monotone operator of type (D), $A : X \rightrightarrows X^*$ be a maximal monotone operator of type (D) and $f_n \in \mathcal{H}(A_n)$, $f \in \mathcal{H}(A)$. If f_n epi-converges to f then A_n converges to A.*

Proof Suppose f_n epi-converges to f. In order to prove $A_n \to A$, we need to verify the following:

1. for any $(x, x^*) \in \operatorname{gra} A$, there exists a sequence (x_n, x_n^*) with $\lim(x_n, x_n^*) = (x, x^*)$ and $(x_n, x_n^*) \in \operatorname{gra} A_n$ for n sufficiently large;
2. the cluster points of every sequence $(x_n, x_n^*) \in \operatorname{gra} A_n$ for n sufficiently large belongs to $\operatorname{gra} A$.

Let $(x, x^*) \in \operatorname{gra} A$, we will find a sequence $(x_n, x_n^*) \in \operatorname{gra} A_n$ such that $\lim_n(x_n, x_n^*) = (x, x^*)$. Since $f \in \mathcal{H}(A)$, $f(x, x^*) = \langle x, x^*\rangle$. Thus, $((x, x^*), \langle x, x^*\rangle) \in \operatorname{epi} f$. By definition of epi-convergence of f_n, there exist a sequence $(y_n, y_n^*, t_n) \in \operatorname{epi} f_n$ such that

$$(y_n, y_n^*) \to (x, x^*) \tag{1}$$

and

$$t_n \to \langle x, x^*\rangle. \tag{2}$$

Since A_n is maximal monotone operator of type (D). Then by Fact 2,

$$\operatorname{ran}\Big(A_n + J_{\frac{1}{n}}(. - y)\Big) = X^*, \forall n \in \mathbb{N}, \forall y \in X.$$

Since, $(y_n, y_n^*)_{n\in\mathbb{N}} \in X \times X^*$,

$$y_n^* \in \operatorname{ran}\Big(A_n + J_{\frac{1}{n}}(. - y_n)\Big), \forall n \in \mathbb{N}.$$

Thence, there exists $z_n \in X$ such that $y_n^* = z_n^* + w_n^*$, where $z_n^* \in A_n(z_n)$ and $w_n^* \in J_{\frac{1}{n}}(z_n - y_n)$. Thus, $y_n^* - z_n^* \in J_{\frac{1}{n}}(z_n - y_n)\ \forall n \in \mathbb{N}$. Therefore, by definition of $J_{\frac{1}{n}}$, we get

$$\begin{aligned}
\frac{1}{2}\|z_n - y_n\|^2 + \frac{1}{2}\|y_n^* - z_n^*\|^2 &\le \langle z_n - y_n, y_n^* - z_n^*\rangle + \frac{1}{n} \\
&\le \langle z_n, y_n^*\rangle + \langle y_n, z_n^*\rangle - \langle z_n, z_n^*\rangle - \langle y_n, y_n^*\rangle + \frac{1}{n}. \\
&\le F_{A_n}(y_n, y_n^*) - \langle y_n, y_n^*\rangle + \frac{1}{n} \\
&\le f_n(y_n, y_n^*) - \langle y_n, y_n^*\rangle + \frac{1}{n}, \text{ (by maximality of } A_n).
\end{aligned}$$

Since $(y_n, y_n^*, t_n) \in \text{epi } f_n$,

$$\frac{1}{2}\|y_n - z_n\|^2 + \frac{1}{2}\|y_n^* - z_n^*\|^2 \leq t_n - \langle y_n, y_n^* \rangle + \frac{1}{n} \tag{3}$$

Note that,

$$\begin{aligned} |t_n - \langle y_n, y_n^* \rangle| &= |t_n - \langle x, x^* \rangle + \langle x, x^* \rangle - \langle y_n, y_n^* \rangle| \\ &\leq |t_n - \langle x, x^* \rangle| + |\langle y_n, y_n^* \rangle - \langle x, x^* \rangle|. \end{aligned}$$

By (1) and (2),

$$|t_n - \langle y_n, y_n^* \rangle| \to 0.$$

Thus, from (3), $\|y_n - z_n\| \to 0$ and $\|y_n^* - z_n^*\| \to 0$. Therefore,

$$\begin{aligned} \|z_n - x\| &= \|z_n - y_n + y_n - x\| \\ &\leq \|z_n - y_n\| + \|y_n - x\| \to 0. \end{aligned}$$

Similarly, $\|z_n^* - x^*\| \to 0$. Which proves that there exists a sequence $(z_n, z_n^*) \in \text{gra } A_n$ such that $(z_n, z_n^*) \to (x, x^*)$.

Finally, let $(x, x^*) \in X \times X^*$ be a cluster point of every sequence $(x_n, x_n^*) \in \text{gra } A_n$. Then we prove that $(x, x^*) \in \text{gra } A$. By assumption, there exists a subsequence $(x_{n_k}, x_{n_k}^*) \in \text{gra } A_{n_k}$ such that

$$(x_{n_k}, x_{n_k}^*) \to (x, x^*).$$

Let us take $n_k = n$ for our simplicity. Then, $f_n(x_n, x_n^*) = \langle x_n, x_n^* \rangle$.

Thus, $((x_n, x_n^*), \langle x_n, x_n^* \rangle) \in \text{epi } f_n$. Since, $(x_n, x_n^*) \to (x, x^*)$ and $\text{epi } f_n \to \text{epi } f$, we have $((x, x^*), \langle x, x^* \rangle) \in \text{epi } f$, i.e., $f(x, x^*) \leq \langle x, x^* \rangle$. Since f is a representative function of A we conclude that $(x, x^*) \in \text{gra } A$. □

4 Lower limit of sequence of maximal monotone operators of type (D)

Let us denote,

$$\mathcal{F}(X \times X^*) := \{f \in \Gamma(X \times X^*) : f(x, x^*) \geq \langle x, x^* \rangle, \forall (x, x^*) \in X \times X^*\},$$

and

$$\begin{aligned} \mathcal{F}^*(X^* \times X^{**}) := \{f \in \mathcal{F}(X \times X^*) : f^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \\ \forall (x^*, x^{**}) \in X^* \times X^{**}\}. \end{aligned}$$

Also we denote,

$$M_f := \{(x, x^*) \in X \times X^* : f(x, x^*) = \langle x, x^* \rangle\}.$$

Proposition 1 *Let X be a real Banach space. Then*

1. *Let $f_n \in \mathcal{F}(X \times X^*)$. If f_n epi-converges to f, then $f \in \mathcal{F}(X \times X^*)$.*
2. *Let $f_n \in \mathcal{F}^*(X^* \times X^{**})$. If f_n^* epi-converges to f^*, then $f \in \mathcal{F}^*(X^* \times X^{**})$.*

Proof (1) Let $f_n \in \mathcal{F}(X \times X^*)$ and f_n epi-converge to f. First, we will show that f is convex. Let $(x, x^*), (y, y^*) \in X \times X^*$ and let $\alpha \in [0, 1]$. Then by definition of epi-convergence, there exists sequences (x_n, x_n^*) and (y_n, y_n^*) such that $(x_n, x_n^*) \to (x, x^*)$, $(y_n, y_n^*) \to (y, y^*)$ and

$$\limsup_n f_n(x_n, x_n^*) \leq f(x, x^*);$$
$$\limsup_n f_n(y_n, y_n^*) \leq f(y, y^*).$$

Note that $(\alpha(x_n, x_n^*) + (1-\alpha)(y_n, y_n^*)) \to (\alpha(x, x^*) + (1-\alpha)(y, y^*))$,

$$f((\alpha(x, x^*) + (1-\alpha)(y, y^*))) \leq \liminf_n f_n((\alpha(x_n, x_n^*) + (1-\alpha)(y_n, y_n^*))).$$

Therefore,

$$f((\alpha(x, x^*) + (1-\alpha)(y, y^*))) \leq \limsup_n f_n((\alpha(x_n, x_n^*) + (1-\alpha)(y_n, y_n^*))).$$

Since f_n is convex for each $n \in \mathbb{N}$, we get

$$f((\alpha(x, x^*) + (1-\alpha)(y, y^*))) \leq \alpha f(x, x^*) + (1-\alpha) f(y, y^*).$$

This proves that f is convex and by definition of epi-convergence, $f \in \Gamma(X \times X^*)$. Now, by definition of epi-convergence, for any $(x, x^*) \in X \times X^*$ there exists $(x_n, x_n^*) \in X \times X^*$ such that $(x_n, x_n^*) \to (x, x^*)$ and

$$\limsup_n f_n(x_n, x_n^*) \leq f(x, x^*).$$

Thus,

$$f(x, x^*) = \lim_n f_n(x_n, x_n^*) \geq \langle x_n, x_n^* \rangle.$$

Hence, $f(x, x^*) \geq \langle x, x^* \rangle$. This shows that $f \in \mathcal{F}(X \times X^*)$.

The proof of (2) is same as the proof of (1). □

The following theorem states that the lower limit of a sequence of maximal monotone operators of type (D) is representable, and the representable function is the epi-limit of the corresponding representative functions of the sequence of maximal monotone operators.

Theorem 8 *Let $f_n \in \Gamma(X \times X^*)$ with $A_n = M_{f_n}$ a maximal monotone of type (D). If f_n epi-converges to f then $M_f = \liminf A_n$.*

Proof Since f_n epi-converges to f, by Proposition 1, $f \in \mathcal{F}(X \times X^*)$. Let gra $A = \liminf A_n$. First we show that gra $A \subseteq M_f$. Let $(x, x^*) \in$ gra A. By definition of $\liminf A_n$, for every $n \in \mathbb{N}$, there exists $(x_n, x_n^*) \in A_n$ such that $(x_n, x_n^*) \to (x, x^*)$. Since, gra $A_n = M_{f_n}$, for every $n \in \mathbb{N}$. Thus, $f_n(x_n, x_n^*) = \langle x_n, x_n^* \rangle$. By definition of epi-converges of f_n,

$$\liminf_n f_n(x_n, x_n^*) \geq f(x, x^*).$$

Therefore, using the definition of representable function, we get

$$f(x, x^*) \leq \liminf \langle x_n, x_n^* \rangle = \langle x, x^* \rangle.$$

Hence, $(x, x^*) \in M_f$.

Finally, we show that $M_f \subseteq$ gra A. Let $(x, x^*) \in M_f$. For each $n \in \mathbb{N}$, let (x_n) is a solution of $x^* \in A_n(x_n) + J_{\frac{1}{n}}(x_n - x)$. According to Fact 3, it is sufficient to show that $\lim x_n = x$. Since (x_n) is a solution of $x^* \in A_n(x_n) + J_{\frac{1}{n}}(x_n - x)$, there exists $x_n^* \in A_n(x_n)$ and $w_n^* \in J_{\frac{1}{n}}(x_n - x)$ such that $x^* = x_n^* + w_n^*$, $\forall n \in \mathbb{N}$. Let (y_n, y_n^*) be a sequence in $X \times X^*$ such that $(y_n, y_n^*) \to (x, x^*)$ and

$$f(x, x^*) \geq \limsup_n f_n(y_n, y_n^*).$$

Since f_n is representative function of A_n and A_n is maximal monotone for each $n \in \mathbb{N}$, then by Fact 1

$$F_{A_n}(y_n, y_n^*) \leq f_n(y_n, y_n^*).$$

Thus,

$$\limsup F_{A_n}(y_n, y_n^*) \leq \limsup f_n(y_n, y_n^*) \leq f(x, x^*). \tag{4}$$

Since $x^* = x_n^* + w_n^*$, $x^* - w_n^* = x_n^* \in A_n(x_n)$, by definition of F_{A_n}, we get

$$\begin{aligned}
F_{A_n}(y_n, y_n^*) &= \sup_{(a_n, a_n^*) \in \text{gra } A_n} [\langle y_n, a_n^* \rangle + \langle a_n, y_n^* \rangle - \langle a_n, a_n^* \rangle] \\
&\geq \langle y_n, x_n^* \rangle + \langle x_n, y_n^* \rangle - \langle x_n, x_n^* \rangle \\
&= \langle y_n - x_n, x_n^* \rangle + \langle x_n, y_n^* \rangle \\
&= \langle y_n - x_n, x_n^* \rangle - \langle y_n - x_n, y_n^* \rangle + \langle y_n - x_n, y_n^* \rangle + \langle x_n, y_n^* \rangle \\
&= \langle y_n - x_n, x_n^* - y_n^* \rangle + \langle y_n, y_n^* \rangle \\
&= \langle y_n - x_n, x^* - w_n^* - y_n^* \rangle + \langle y_n, y_n^* \rangle \\
&= \langle y_n - x_n, x^* - y_n^* \rangle + \langle x_n - x, w_n^* \rangle + \langle x - y_n, w_n^* \rangle + \langle y_n, y_n^* \rangle.
\end{aligned}$$

Since $w_n^* \in J_{\frac{1}{n}}(x_n - x)$,

$$\frac{1}{2}\|x_n - x\|^2 + \frac{1}{2}\|w_n^*\|^2 \leq \langle x_n - x, w_n^* \rangle + \frac{1}{n}.$$

Note that

$$\frac{1}{2}\|x_n - x\|^2 \leq \frac{1}{2}\|x_n - x\|^2 + \frac{1}{2}\|w_n^*\|^2 \leq \langle x_n - x, w_n^* \rangle + \frac{1}{n}.$$

Thus,

$$F_{A_n}(y_n y_n^*) \geq -\|y_n - x_n\|\|x^* - y_n^*\| + \frac{1}{2}\|x_n - x\|^2 - \frac{1}{n} - \|x - y_n\|\|w_n^*\| + \langle y_n, y_n^* \rangle.$$

By Fact 6, (x_n) and (w_n^*) are bounded. Therefore,

$$\limsup F_{A_n}(y_n, y_n^*) \geq \limsup \left(\frac{1}{2}\|x_n - x\|^2\right) + \langle x, x^* \rangle.$$

By appealing Eq. (4),

$$\langle x, x^* \rangle \geq \limsup \frac{1}{2}\|x_n - x\|^2 + \langle x, x^* \rangle.$$

Hence, $\limsup \frac{1}{2}\|x_n - x\|^2 \leq 0$. This proves that $x_n \to x$. □

Remark 1 The representability of the lower limit of a sequence of maximal monotone operators of type (D) was established by Bueno et al. [7] in general Banach spaces. In the above Theorem 8, we establish that the lower limit of a sequence of maximal monotone operators of type (D) is representable, and the representable function is the epi-limit of the corresponding representative functions of the sequence of maximal monotone operators.

Finally, we prove that the limit of a sequence of maximal monotone operators of type (D) is a maximal monotone operator and moreover, we prove that it is of type (D).

Corollary 1 *Let $A_n : X \rightrightarrows X^*$ be a sequence of maximal monotone operators of type (D) and let $f_n \in \mathcal{F}(X \times X^*)$ with $A_n = M_{f_n}$. If f_n epi-converges to f and f_n^* epi-converges to f^*, then $M_f = \liminf A_n$ is a maximal monotone operator. Moreover, M_f is of type (D).*

Proof By Theorem 8, $M_f = \liminf A_n$. Since f_n^* epi-converges to f^*, by Proposition 1,

$$f^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle.$$

Again by epi-converges of f_n and Proposition 1, we get

$$f(x, x^*) \geq \langle x, x^* \rangle.$$

Hence, by Fact 4, M_f is a maximal monotone. Again, by Fact 4, we have

$$F^*_{M_f}(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

By Fact 5, M_f is of type (NI) and hence, of type (D). □

5 Conclusion

Theorem 7 establishes a sufficient condition for convergence of a sequence of maximal monotone operators of type (D) in general Banach spaces. Theorem 8 and Corollary 1 are the generalization of the results of [11, Theorem 2.2] to a Banach spaces by assuming that the sequence of maximal monotone operators is of type (D).

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