Positivity (2019) 23:599–616
https://doi.org/10.1007/s11117-018-0627-9

Positivity

Levitin–Polyak well-posedness for set optimization problems involving set order relations

Pham Thi Vui[1] · Lam Quoc Anh[2] · Rabian Wangkeeree[1]

Received: 11 March 2018 / Accepted: 26 October 2018 / Published online: 7 November 2018

Abstract

In this paper set optimization problems with three types of set order relations are concerned. We introduce various types of Levitin–Polyak (LP) well-posedness for set optimization problems and survey their relationships. After that, sufficient and necessary conditions for the reference problems to be LP well-posed are given. Furthermore, using the Kuratowski measure of noncompactness, we study characterizations of well-posedness for set optimization problems. Moreover, the links between stability and LP well-posedness of such problems are established via the study on approximating solution mappings. Tools and techniques used in this study and our results are different from existing ones in the literature.

Keywords Set order relation · Set optimization problem · Levitin–Polyak well-posedness · Stability

Mathematics Subject Classification 49J53 · 49K40 · 90C31

This research is partially supported by the Thailand Research Fund under Grant Number RSA6080077. The second author is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant Number 101.01-2017.18.

✉ Lam Quoc Anh
quocanh@ctu.edu.vn

Pham Thi Vui
ptvui@ctu.edu.vn

Rabian Wangkeeree
rabianw@nu.ac.th

1 Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

2 Department of Mathematics, Teacher College, Can Tho University, Can Tho, Vietnam

1 Introduction

The concept of well-posedness was firstly introduced by Tikhonov [36]. Beside the requirement about the uniqueness of the solution, Tikhonov well-posedness requested the convergence of each minimizing sequence to the unique solution. Therefore, well-posedness plays a vital role to make connections between stability properties and solution methods for problems related to optimization. On this topic, many research results have been devoted to a lot of important problems such as variational inequalities [7], equilibrium problems [1], inclusion problems [37] and the references therein. Several generalizations of Tikhonov well-posedness were introduced and investigated for various kinds of optimization problems [4,10,30–32]. Levitin–Polyak (LP) well-posedness is an extension of Tikhonov well-posedness and was originally proposed in [28]. Every minimizing sequence must belong to the feasible set in Tikhonov well-posedness, whereas it can be outside of the feasible region but the distance between it and this set has to approach zero in LP well-posedness. There have been many studies of LP well-posedness (see, e.g., [18,27] and the references therein).

Kuroiwa et al. [26] proposed set order relations including lower set less relation, upper set less relation and set less relation (combination of the lower and the upper set less relation). This gave a new way to formulate the solution of set-valued optimization problems which is called solutions concept based on the set approach [25], and hence the optimization problems in this approach are called set optimization problems involving set order relations [21,24]. As pointed out in [19], the set less relation is generalized and more appropriate in practical problems than both the lower and upper set less relations. Furthermore, the set less relation plays a center role in relationships with other new order relations for sets proposed in [5,19] which are more useful in set optimization. Although set optimization is a new direction in the field of optimization, it has attracted a great deal of attention of researchers with many important and interesting results [11,13,17,20]. Useful applications of set optimization in practical problems were reported, for example, the application in socio-economic [34] (to manage noise disturbance in the region surrounding the Frankfurt Airport in Germany), the application in finance [14] (to evaluate the risk of a multivariate random outcome). Moreover, relationships between set optimizations and other important problems such as variational inequalities [8], Ky Fan inequality problems (so-called equilibrium problems) [35] were investigated. For further reading and references, we refer to books [15,21].

The first introduction of well-posedness for set optimization problems was presented by Zhang et al. [38]. The authors established both sufficient and necessary conditions for set optimization problems involving the lower set less relation to be well-posed and obtained criteria as well as characterizations of well-posedness for these problems by the scalarization method. This research was generalized by Gutiérrez et al. [12] under assumptions of cone properness. After that, Dhingra and Lalitha [9] introduced a concept of well-setness for such problems and proved that it is an extension of the generalized well-posedness which was considered in [38]. Recently, well-posedness of set optimization problems involving not only the lower but also the upper set less relation have been discussed in [16,22,29]. For LP well-posedness, to the best of our knowledge there is only the paper of Khoshkhabar-amiranloo and

Khorram [22] which studied set optimization problem involving the lower set less relation. Of course such an important topic as LP well-posedness for set optimization problems must be the aim of many works. Moreover, the scalarization method which is the main tool used in papers mentioned above investigates difficultly set optimization problems involving different set order relations, some other approaches to study well-posedness for such problems should be considered.

Motivated and inspired by works mentioned above, in this paper, without using the scalarization method, we investigate different types of LP well-posedness for set optimization problems involving several kinds of set order relations. More precisely, we concern set optimization problems involving three types of set order relations. Then, we introduce concepts of LP well-posedness for such problems and discuss relationships among them. Moreover, necessary and/or sufficient conditions for these concepts of well-posedness are investigated. Applying Kuratowski measure of noncompactness, we study characterizations of such concepts. Finally, approximating solution mappings and their stability are studied to build the connection between stability of approximating problem and LP well-posedness of the set optimization problem.

The outline of this paper is given as follows. In Sect. 2, we recall some definitions and results needed in what follows. Sect. 3 introduces various kinds of LP well-posedness for set optimization problems and investigates their relationships. Furthermore, sufficient and/or necessary conditions of pointwise LP well-posedness for such problems are also obtained. In this section, characterizations of these types of pointwise LP well-posedness are surveyed by using measure of noncompactness. In the last section, Sect. 4, we study sufficient conditions for such problem to be metrically LP well-posed and their relationships.

2 Preliminaries

Let X be a normed space and Y be a real Hausdorff topological linear space. Let K be a closed convex pointed cone in Y with $\text{int}K \neq \emptyset$, where $\text{int}K$ denotes the interior of K. The space Y is endowed with an order relation induced by cone K in the following way

$$x \leq_K y \Leftrightarrow y - x \in K,$$
$$x <_K y \Leftrightarrow y - x \in \text{int}K.$$

The cone K induces various set orderings in Y. These such orderings as the following were presented in [19,21,25]. Let $\mathscr{P}(Y)$ be the family of all nonempty subsets of Y. For $A, B \in \mathscr{P}(Y)$, lower set less relation, upper set less relation and set less relation, respectively, are defined by

$$A \leq^l B \text{ if and only if } B \subset A + K,$$
$$A \leq^u B \text{ if and only if } A \subset B - K,$$
$$A \leq^s B \text{ if and only if } A \subset B - K \text{ and } B \subset A + K.$$

Remark 2.1 The relationship between the lower set less relation $\leq^l$ and the upper set less relation $\leq^u$ was given by Remark 2.6.10 in [21] as the following

$$A \leq^l B \Leftrightarrow -B \leq^u -A.$$

Definition 2.1 [19] The binary relation $\leq$ is said to be

(i) compatible with the addition if and only if $A \leq B$ and $D \leq E$ imply $A + D \leq B + E$ for all $A, B, D, E \in \mathscr{P}(Y)$;
(ii) compatible with the multiplication with a nonnegative real number if and only if $A \leq B$ implies $\lambda A \leq \lambda B$ for all scalars $\lambda \geq 0$ and all $A, B \in \mathscr{P}(Y)$;
(iii) compatible with the conlinear structure of $\mathscr{P}(Y)$ if and only if it is compatible with both the addition and the multiplication with a nonnegative real number.

Proposition 2.1 [19]

(i) *The order relations $\leq^l$, $\leq^u$ and $\leq^s$ are pre-order (i.e., these relations are reflexive and transitive).*
(ii) *The order relations $\leq^l$, $\leq^u$ and $\leq^s$ are compatible with the conlinear structure of $\mathscr{P}(Y)$.*
(iii) *In general, the order relations $\leq^l$, $\leq^u$ and $\leq^s$ are not antisymmetric; more precisely, for arbitrary sets $A, B \in \mathscr{P}(Y)$ we have*

$$(A \leq^l B \quad and \quad B \leq^l A) \Leftrightarrow A + K = B + K,$$
$$(A \leq^u B \quad and \quad B \leq^u A) \Leftrightarrow A - K = B - K,$$
$$(A \leq^s B \quad and \quad B \leq^s A) \Leftrightarrow (A + K = B + K \quad and \quad A - K = B - K).$$

For $\alpha \in \{l, u, s\}$, we say that

$$A \sim^\alpha B \text{ if and only if } A \leq^\alpha B \text{ and } B \leq^\alpha A.$$

Let $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty values on X. For each $\alpha \in \{l, u, s\}$, we consider the following set optimization problem

$$(\mathrm{P}_\alpha) \qquad \alpha\text{-Min}F(x)$$
$$\text{subject to} \quad x \in M,$$

where M is a nonempty closed subset of X. A point $\bar{x} \in M$ is said to be an α-minimal solution of (P_α) if and only if for any $x \in M$ such that $F(x) \leq^\alpha F(\bar{x})$ then $F(\bar{x}) \leq^\alpha F(x)$. The set of all α-minimal solutions of (P_α) is denoted by $S_{\alpha\text{-Min}F}$.

Remark 2.2 It can be seen that if $\bar{x} \in S_{\alpha\text{-Min}F}$ and $F(\bar{x}) \sim^\alpha F(\hat{x})$ for some $\hat{x} \in M$, then $\hat{x} \in S_{\alpha\text{-Min}F}$.

We recall the following definitions of semicontinuity for a set-valued mapping and their properties used in the sequel.

Definition 2.2 ([3], p. 38, 39) Let $F : X \rightrightarrows Y$ be a set-valued mapping.

(i) F is said to be upper semicontinuous at $x_0 \in X$ if and only if for any open subset U of Y with $F(x_0) \subset U$, there is a neighborhood N of x_0 such that $F(x) \subset U$ for every $x \in N$.
(ii) F is said to be lower semicontinuous at $x_0 \in X$ if and only if for any open subset U of Y with $F(x_0) \cap U \neq \emptyset$, there is a neighborhood N of x_0 such that $F(x) \cap U \neq \emptyset$ for all $x \in N$.
(iii) F is said to be lower (upper) semicontinuous on a subset S of X if and only if it is lower (upper) semicontinuous at every $x \in S$.

Lemma 2.1 *Let $F : X \rightrightarrows Y$ be a set-valued mapping.*

(i) ([3], p. 39) *F is lower semicontinuous at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subset X$ converging to x_0 and for any $y \in F(x_0)$, there exist $y_\alpha \in F(x_\alpha)$ such that $\{y_\alpha\}$ converges to y.*
(ii) ([2]) *If $F(x_0)$ is compact, then F is upper semicontinuous at $x_0 \in X$ if and only if for any net $\{x_\alpha\}$ converging to x_0 and for any $y_\alpha \in F(x_\alpha)$, there exist $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $\{y_\beta\}$ converges to y_0. If, in addition, $F(x_0) = \{y_0\}$ is a singleton, then for the above nets, $\{y_\beta\}$ converges to y_0.*

Now we recall the concepts of Hausdorff distance and Hausdorff convergence of sequence of sets. If S is a nonempty subset of X and $x \in X$, then the distance d between x and S is defined by

$$d(x, S) := \inf_{u \in S} \|x - u\| .$$

If S_1 and S_2 are two nonempty subsets of X, then Hausdorff distance between S_1 and S_2, denoted by $H(S_1, S_2)$, is defined by

$$H(S_1, S_2) := \max\{H^*(S_1, S_2), H^*(S_2, S_1)\},$$

where $H^*(S_1, S_2) := \sup_{x \in S_1} d(x, S_2)$.

Definition 2.3 ([23], p. 359) Let $\{A_n\}$ be a sequence of subsets of X. We say that A_n converge to $A \subset X$ in the sense of the Hausdorff metric, denoted by $A_n \to A$, if and only if $H(A_n, A) \to 0$ as $n \to \infty$.

Next, we recall the concept of Kuratowski measure of noncompactness and it's properties used in the sequel.

Definition 2.4 ([33], Definition 2.1) Let M be a nonempty subset of X. The Kuratowski measure of noncompactness μ of the set M is defined by

$$\mu(M) := \inf \left\{ \varepsilon > 0 \mid M \subset \bigcup_{i=1}^{n} M_i, \operatorname{diam} M_i < \varepsilon, i = 1, \cdots, n \text{ for some } n \in \mathbb{N} \right\},$$

where $\operatorname{diam} M_i := \sup\{d(x, y) \mid x, y \in M_i\}$ is the diameter of M_i.

Lemma 2.2 ([33], Proposition 2.3) *The following assertions are true:*

(i) $\mu(M) = 0$ *if* M *is compact*;

(ii) $\mu(M) \leq \mu(N)$ *whenever* $M \subset N$;

(iii) *if* $\{M_n\}$ *is a sequence of closed subsets in* X *satisfying* $M_{n+1} \subset M_n$ *for every* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \mu(M_n) = 0$, *then* $K := \bigcap_{n\in\mathbb{N}} M_n$ *is nonempty compact and* $\lim_{n\to\infty} H(M_n, K) = 0$.

Lemma 2.3 *Let* X *be a normed space and* A, B *be subsets of* X. *If* A *is compact and* B *is closed, then* $A + B$ *is closed.*

Proof Assume that $\{a_n + b_n\}$, $a_n \in A, b_n \in B$, converges to c for some $c \in X$. We show that $c \in A + B$. In fact, since A is compact, there exist a subsequence $\{a_{n_k}\}$ of sequence $\{a_n\}$ and $a \in A$ such that $\{a_{n_k}\}$ converges to a. We have

$$\left\|b_{n_k} - c + a\right\| = \left\|(b_{n_k} + a_{n_k}) - c + (a - a_{n_k})\right\| \leq \left\|b_{n_k} + a_{n_k} - c\right\| + \left\|a - a_{n_k}\right\|.$$

We obtain that $\{b_{n_k}\}$ converges to $c - a$. Since B is closed, we get $c - a \in B$. Hence, there exists $b \in B$ such that $b = c - a$. Then, $c = a + b \in A + B$. So, $A + B$ is closed. □

Lemma 2.4 *Let* M *be a nonempty subset of a normed space* X. *Then, for every* $x, y \in X$, $|d(x, M) - d(y, M)| \leq \|x - y\|$.

Proof Let $x, y \in X$, we have $\|x - y\| + d(y, M) = \|x - y\| + \inf_{z\in M} \|y - z\| = \inf_{z\in M}\{\|x - y\| + \|y - z\|\} \geq \inf_{z\in M} \|x - z\| = d(x, M)$. Hence, $\|x - y\| \geq d(x, M) - d(y, M)$. Similarly, we also get $\|x - y\| \geq d(y, M) - d(x, M)$. We conclude that $|d(x, M) - d(y, M)| \leq \|x - y\|$. □

3 Pointwise *LP* well-posedness and generalized pointwise *LP* well-posedness

Motivated by the study [22] on the pointwise *LP* well-posedness for (P_l), we are going to establish characterizations of this type of well-posedness for (P_α) without using the scalarization method. Consider the problem (P_α), for a given $\bar{x} \in S_{\alpha\text{-Min}F}$, the LP approximating solution mapping at $\bar{x}$, $S_{\alpha\text{-Min}F}(\bar{x}, \cdot) : \{\bar{x}\} \times \mathbb{R}_+ \rightrightarrows M$ is defined by

$$S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon) := \{x \in X \mid d(x, M) \leq \varepsilon,\ F(x) \leq^{\alpha} F(\bar{x}) + \varepsilon e\},$$

for each $\varepsilon \in \mathbb{R}_+$.

Inspired by ideas in [22] (Definition 5.1), we extend some notions for the problem (P_l) in [22] to the problem (P_α) and propose some new concepts for the problem (P_α). Let $e \in \mathrm{int}K$.

Definition 3.1 Let $\bar{x} \in S_{\alpha\text{-Min}F}$ be given. A sequence $\{x_n\} \subset X$ is said to be a *LP*-minimizing sequence for the problem (P_α) at $\bar{x}$ if and only if there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+\backslash\{0\}$ converging to 0 such that

$$d(x_n, M) \leq \varepsilon_n, \quad F(x_n) \leq^{\alpha} F(\bar{x}) + \varepsilon_n e.$$

The equivalence of this concept is given by the following result.

Proposition 3.1 *$\{x_n\} \subset X$ is a LP-minimizing sequence for the problem (P_α) at $\bar{x} \in S_{\alpha\text{-}MinF}$ if and only if there exists a sequence $\{d_n\} \subset K\backslash\{0\}$ converging to 0 such that*

$$d(x_n, M) \to 0, \quad F(x_n) \leq^{\alpha} F(\bar{x}) + d_n.$$

Proof We only prove the assertion for the case $\alpha = s$; the proofs of the assertion for the cases $\alpha = l$ and $\alpha = u$ are similar. Let $\{x_n\} \subset X$ and $\{d_n\} \subset K\backslash\{0\}$ converging to 0 such that $d(x_n, M) \to 0$ and $F(x_n) \leq^{s} F(\bar{x}) + d_n$, i.e.,

$$F(\bar{x}) + d_n \subset F(x_n) + K, \quad F(x_n) \subset F(\bar{x}) + d_n - K. \tag{1}$$

Since $e - K$ is a neighborhood of the origin 0 in Y, there exists $\varepsilon > 0$ such that $\varepsilon B(0, 1) \subset e - K$ where $B(x, r)$ is the closed ball centered x with radius r. For a given $n \in \mathbb{N}$, we have $d_n \in \|d_n\| B(0, 1) \subset \|d_n\| \varepsilon^{-1}(e - K) = \|d_n\| \varepsilon^{-1}e - K$. For each $n \in \mathbb{N}$, taking $\varepsilon_n = \|d_n\| \varepsilon^{-1}$, then $\{\varepsilon_n\} \subset \mathbb{R}_+\backslash\{0\}$ converges to 0 and $\varepsilon_n e - d_n \in K$. It follows from (1) that

$$F(\bar{x}) + \varepsilon_n e \subset F(x_n) + K, \quad F(x_n) \subset F(\bar{x}) + \varepsilon_n e - K,$$

i.e., $F(x_n) \leq^{s} F(\bar{x}) + \varepsilon_n e$. So, $\{x_n\}$ is a *LP*-minimizing sequence for (P_s) at $\bar{x}$. Conversely, it is clear that if $\{x_n\} \subset X$ is a *LP*-minimizing sequence for (P_s) at $\bar{x} \in S_{s\text{-Min}F}$, then the assertion is satisfied by setting $d_n = \varepsilon_n e$. □

Definition 3.2 The problem (P_α) is said to be

(i) *LP* well-posed at $\bar{x} \in S_{\alpha\text{-Min}F}$ if and only if any *LP*-minimizing sequence for (P_α) at $\bar{x}$ converges to $\bar{x}$;
(ii) generalized *LP* well-posed at $\bar{x} \in S_{\alpha\text{-Min}F}$ if and only if any *LP*-minimizing sequence for (P_α) at $\bar{x}$ has a subsequence converging to an element $\hat{x} \in S_{\alpha\text{-Min}F}(\bar{x}, 0)$.

Remark 3.1 When $\alpha = l$, the concept of pointwise well-posedness becomes the corresponding concepts studied in [22] (Definitions 5.1 and 5.2, respectively), even for this special case, the concept of generalized well-posedness is a new one.

The following examples illustrate the above-introduced concepts.

Example 3.1 Let $X = Y = \mathbb{R}$, $M = \mathbb{R}$, $K = \mathbb{R}_+$. Let $F : X \rightrightarrows Y$ be defined by $F(x) = [0, 1]$ for all $x \in X$. Let $e = 1 \in \text{int}K$ and $\bar{x} = 0$. We have $S_{\alpha\text{-Min}F}(\bar{x}, 0) = S_{\alpha\text{-Min}F} = \mathbb{R}$. Setting $x_n = n$, $\{x_n\}$ is a *LP*-minimizing sequence for (P_α) at $\bar{x} = 0$. Since $\{x_n\}$ admits no convergent subsequence, (P_α) is not both *LP* well-posed and generalized *LP* well-posed at 0.

Example 3.2 Let $X = Y = \mathbb{R}$, $M = \mathbb{R}$, $K = \mathbb{R}_+$. Let $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [0, 1], & \text{if} \quad -1 \leq x \leq 1, \\ [1, 1 + x^2], & \text{otherwise.} \end{cases}$$

Let $e = 1 \in \text{int}K$ and $\bar{x} = 0$. By direct computations, we have $S_{\alpha\text{-Min}F}(\bar{x}, 0) = [-1, 1]$. Setting $x_n = 1 - \frac{1}{n}$, $\{x_n\}$ is a *LP*-minimizing sequence for (P_α) at $\bar{x} = 0$ and converges to 1. Hence, (P_α) is not *LP* well-posed at 0, but it is generalized *LP* well-posed at 0. Indeed, if $\{\hat{x}_n\}$ is a *LP*-minimizing sequence for (P_α) at $\bar{x} = 0$, then there is a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \backslash \{0\}$ converging to 0 such that $d(\hat{x}_n, M) \le \varepsilon_n$ and $F(\hat{x}_n) \le^\alpha F(0) + \varepsilon_n e = [\varepsilon_n, 1 + \varepsilon_n]$. This implies that $-1 \le \hat{x}_n \le 1$ for n sufficiently large, and hence there exists a subsequence of $\{\hat{x}_n\}$ converging to some point of $S_{\alpha\text{-Min}F}(\bar{x}, 0)$.

Example 3.3 Let $X = \mathbb{R}$, $M = \mathbb{R}$, $K = \mathbb{R}_+$. Let $F : X \rightrightarrows Y$ be defined by $F(x) = [x^2, 2x^2]$ for all $x \in X$. Let $e = 1 \in \text{int}K$ and $\bar{x} = 0$. Direct cacullations give us $S_{\alpha\text{-Min}F}(\bar{x}, 0) = \{0\}$. Let $\{x_n\}$ be a *LP*-minimizing sequence for (P_α) at $\bar{x} = 0$. Then, there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \backslash \{0\}$ converging to 0 such that $d(x_n, M) \le \varepsilon_n$ and $F(x_n) \le^\alpha F(0) + \varepsilon_n e = \{\varepsilon_n\}$. It leads to $x_n^2 \le \varepsilon_n$, so $\{x_n\}$ converges to 0. Therefore, (P_α) is *LP* well-posed at 0.

Lemma 3.1 *If (P_α) is generalized LP well-posed at $\bar{x} \in S_{\alpha\text{-}MinF}$, then $S_{\alpha\text{-}MinF}(\bar{x}, 0)$ is compact.*

Proof For every sequence $\{x_n\} \subset S_{\alpha\text{-Min}F}(\bar{x}, 0)$, we always have $d(x_n, M) = 0$ and

$$F(x_n) \le^\alpha F(\bar{x}) + \varepsilon_n e$$

for any $\{\varepsilon_n\} \subset \mathbb{R}_+ \backslash \{0\}$ converging to 0. This means that $\{x_n\}$ is a LP-minimizing sequence for (P_α) at $\bar{x}$. By the generalized LP well-posedness of (P_α) at $\bar{x}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to an element $\hat{x} \in S_{\alpha\text{-Min}F}(\bar{x}, 0)$. This leads to the compactness of $S_{\alpha\text{-Min}F}(\bar{x}, 0)$. □

The next results give some properties of the mapping $S_{\alpha\text{-Min}F}(\bar{x}, \cdot)$ which are useful in the sequel.

Proposition 3.2 *Let $\bar{x} \in S_{\alpha\text{-}MinF}$ be given. Then, the following statements are true:*

(i) $S_{\alpha\text{-}MinF} = \bigcup_{z \in S_{\alpha\text{-}MinF}} S_{\alpha\text{-}MinF}(z, 0)$;
(ii) *if* $\varepsilon_1 \le \varepsilon_2$, *then* $S_{\alpha\text{-}MinF}(\bar{x}, \varepsilon_1) \subset S_{\alpha\text{-}MinF}(\bar{x}, \varepsilon_2)$;
(iii) $\bigcap_{\varepsilon > 0} S_{\alpha\text{-}MinF}(\bar{x}, \varepsilon) = S_{\alpha\text{-}MinF}(\bar{x}, 0)$ *if F is compact-valued on M.*

Proof We only demonstrate the proof of the assertions (i)-(iii) for the case $\alpha = s$; the proofs of these assertions for the cases $\alpha = l$ and $\alpha = u$ are similar.

(i) Let $z \in S_{s\text{-Min}F}$ be given. Since $z \in S_{s\text{-Min}F}(z, 0)$, $S_{s\text{-Min}F} \subset \bigcup_{z \in S_{s\text{-Min}F}} S_{s\text{-Min}F}(z, 0)$. Moreover, let $x \in \bigcup_{z \in S_{s\text{-Min}F}} S_{s\text{-Min}F}(z, 0)$, there exists $z \in S_{s\text{-Min}F}$ such that $x \in S_{s\text{-Min}F}(z, 0)$. Therefore, $d(x, M) = 0$ and $F(x) \le^s F(z)$. Since $z \in S_{s\text{-Min}F}$, $x \in S_{s\text{-Min}F}$. It implies that $\bigcup_{z \in S_{s\text{-Min}F}} S_{s\text{-Min}F}(z, 0) \subset S_{s\text{-Min}F}$.

(ii) Assume $\varepsilon_1 \le \varepsilon_2$. Let $x \in S_{s\text{-Min}F}(\bar{x}, \varepsilon_1)$, then $d(x, M) \le \varepsilon_1$ and $F(x) \le^s F(\bar{x}) + \varepsilon_1 e$. It follows from the definition of set less relation $\le^s$ that

$$F(x) \le^l F(\bar{x}) + \varepsilon_1 e \quad \text{and} \quad F(x) \le^u F(\bar{x}) + \varepsilon_1 e,$$

i.e.,

$$F(\bar{x}) + \varepsilon_1 e \subset F(x) + K \quad \text{and} \quad F(x) \subset F(\bar{x}) + \varepsilon_1 e - K.$$

We observe that

$$F(\bar{x}) + \varepsilon_2 e - K = F(\bar{x}) + \varepsilon_1 e - K + (\varepsilon_2 - \varepsilon_1)e.$$

Combining the convexity of K with Proposition 2.1, we obtain that

$$F(\bar{x}) + \varepsilon_2 e \subset F(x) + K,$$

and

$$F(x) \subset F(\bar{x}) + \varepsilon_1 e - K \subset F(\bar{x}) + \varepsilon_2 e - K.$$

Thus, $F(x) \leq^l F(\bar{x}) + \varepsilon_2 e$ and $F(x) \leq^u F(\bar{x}) + \varepsilon_2 e$. Since $\bar{x} \in S_{s\text{-Min}F}$, $F(x) \leq^s F(\bar{x}) + \varepsilon_2 e$. Moreover, $d(x, M) \leq \varepsilon_2$ as $d(x, M) \leq \varepsilon_1$. Therefore, $x \in s\text{-Min}F(\bar{x}, \varepsilon_2)$. We conclude that $S_{s\text{-Min}F}(\bar{x}, \varepsilon_1) \subset S_{s\text{-Min}F}(\bar{x}, \varepsilon_2)$.

(iii) Let $x \in S_{s\text{-Min}F}(\bar{x}, 0)$. It is clear that $x \in S_{s\text{-Min}F}(\bar{x}, \varepsilon)$ for any $\varepsilon > 0$. Therefore, $x \in \bigcap_{\varepsilon>0} S_{s\text{-Min}F}(\bar{x}, \varepsilon)$. For the converse, let $x \in \bigcap_{\varepsilon>0} S_{s\text{-Min}F}(\bar{x}, \varepsilon)$, we have $x \in S_{s\text{-Min}F}(\bar{x}, \varepsilon)$ for any $\varepsilon > 0$. It follows from definition of $S_{s\text{-Min}F}(\bar{x}, \varepsilon)$ that $d(x, M) \leq \varepsilon$ and $F(x) \leq^s F(\bar{x}) + \varepsilon e$, i.e.,

$$d(x, M) \leq \varepsilon, \ F(\bar{x}) + \varepsilon e \subset F(x) + K \quad \text{and} \quad F(x) \subset F(\bar{x}) + \varepsilon e - K. \tag{2}$$

By the compact-valuedness of F and Lemma 2.3, $F(x) - K$ and $F(\bar{x}) - K$ are closed. From (2), let $\varepsilon \to 0$, we obtain that

$$d(x, M) = 0, \ F(\bar{x}) \subset F(x) + K \quad \text{and} \quad F(x) \subset F(\bar{x}) - K,$$

i.e.,

$$d(x, M) = 0, \ F(x) \leq^s F(\bar{x}).$$

Hence, $x \in S_{s\text{-Min}\,F}(\bar{x}, 0)$. We get $\bigcap_{\varepsilon>0} S_{s\text{-Min}\,F}(\bar{x}, \varepsilon) \subset S_{s\text{-Min}\,F}(\bar{x}, 0)$. □

Next, using the Kuratowski measure of noncompactness of LP approximating solution sets, we establish metric characterizations of two types of pointwise LP well-posedness for (P_α).

Theorem 3.1 (i) *If* (P_α) *is generalized LP well-posed at* $\bar{x} \in S_{\alpha\text{-}MinF}$, *then* $\mu(S_{\alpha\text{-}MinF}(\bar{x}, \varepsilon)) \to 0$ *as* $\varepsilon \to 0$.

(ii) *If* (P_α) *is LP well-posed at* $\bar{x} \in S_{\alpha\text{-}MinF}$, *then* $\mathrm{diam}(S_{\alpha\text{-}MinF}(\bar{x}, \varepsilon)) \to 0$ *as* $\varepsilon \to 0$.

Proof (i) Suppose that (P_α) is generalized LP well-posed at $\bar{x} \in S_{\alpha\text{-Min}F}$. First of all, we show that $H(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon), S_{\alpha\text{-Min}F}(\bar{x}, 0)) \to 0$ as $\varepsilon \to 0$. Indeed, we observe that, for each $\varepsilon > 0$, $S_{\alpha\text{-Min}F}(\bar{x}, 0) \subset S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)$, and hence

$$H^*(S_{\alpha\text{-Min}F}(\bar{x}, 0), S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) = 0.$$

It is sufficient to show that $H^*(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon), S_{\alpha\text{-Min}F}(\bar{x}, 0)) \to 0$ as $\varepsilon \to 0$. Suppose by contrary that there exist a real number $r > 0$ and a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0, and for each $n \in \mathbb{N}$ there exists $x_n \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)$ such that $d(x_n, S_{\alpha\text{-Min}F}(\bar{x}, 0)) \geq r$. We have $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$. This means that $\{x_n\}$ is a LP-minimizing sequence for (P_α) at $\bar{x}$, and hence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some point $\hat{x} \in S_{\alpha\text{-Min}F}(\bar{x}, 0)$. Therefore, for n_k sufficiently large, we have $\left\| x_{n_k} - \hat{x} \right\| < r$ which is a contradiction.

Next, we prove that $\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \to 0$ as $\varepsilon \to 0$. By Lemma 3.1, $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is compact. Now, for any $\varepsilon > 0$, there are sets $M_1, M_2, \ldots M_n$ for some $n \in \mathbb{N}$ such that $S_{\alpha\text{-Min}F}(\bar{x}, 0) \subset \cup_{i=1}^n M_i$ with $\operatorname{diam} M_i \leq \varepsilon$ for all $i = 1, \ldots, n$. For each $i \in \{1, \ldots, n\}$, denote

$$N_i := \{x \in X \mid d(x, M_i) \leq H(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon), S_{\alpha\text{-Min}F}(\bar{x}, 0))\}.$$

We claim that $S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon) \subset \cup_{i=1}^n N_i$. Indeed, let $x \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)$ be arbitrary, we have

$$d(x, S_{\alpha\text{-Min}F}(\bar{x}, 0)) \leq H(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon), S_{\alpha\text{-Min}F}(\bar{x}, 0)).$$

Since $S_{\alpha\text{-Min}F}(\bar{x}, 0) \subset \cup_{i=1}^n M_i$, we conclude that

$$d(x, \cup_{i=1}^n M_i) \leq H(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon), S_{\alpha\text{-Min}F}(\bar{x}, 0)).$$

So, there is $k_0 \in \{1, 2, \ldots, n\}$ such that

$$d(x, M_{k_0}) \leq H(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon), S_{\alpha\text{-Min}F}(\bar{x}, 0)).$$

It means that $x \in N_{k_0}$. Therefore, $S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon) \subset \cup_{i=1}^n N_i$. Notice further that

$$\begin{aligned} \operatorname{diam} N_i &= \operatorname{diam} M_i + 2H(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon), S_{\alpha\text{-Min}F}(\bar{x}, 0)) \\ &\leq \varepsilon + 2H(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon), S_{\alpha\text{-Min}F}(\bar{x}, 0)). \end{aligned}$$

Hence, we get

$$\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \leq \mu(S_{\alpha\text{-Min}F}(\bar{x}, 0)) + 2H(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon), S_{\alpha\text{-Min}F}(\bar{x}, 0)).$$

Since $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is compact, we have $\mu(S_{\alpha\text{-Min}F}(\bar{x}, 0)) = 0$. Therefore,

$$\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \leq 2H(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon), S_{\alpha\text{-Min}F}(\bar{x}, 0)).$$

It follows that $\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \to 0$ as $\varepsilon \to 0$.

(ii) Suppose, to the contrary, that there exist a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \backslash \{0\}$ and a positive real number r such that $\operatorname{diam}(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)) > r$. Because $\bar{x} \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)$, for each n, there exists $x_n \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)$ such that $\|x_n - \bar{x}\| > \frac{r}{2}$. However, since $\{x_n\}$ is a LP-minimizing sequence for (P_α) at $\bar{x}$, $\{x_n\}$ converges to $\bar{x}$ which is a contradiction. □

The next result gives sufficient conditions for the closedness of approximating solution set.

Proposition 3.3 *$S_{\alpha\text{-}MinF}(\bar{x}, \varepsilon)$ is closed for each $\varepsilon \geq 0$ if F is continuous and compact-valued on M.*

Proof We only prove the assertion for the case $\alpha = s$. Taking $\varepsilon \geq 0$, let $\{x_n\} \subset S_{s\text{-Min}F}(\bar{x}, \varepsilon)$ converge to x, we need to prove that $x \in S_{s\text{-Min}F}(\bar{x}, \varepsilon)$. Since $x_n \in S_{s\text{-Min}F}(\bar{x}, \varepsilon)$, $d(x_n, M) \leq \varepsilon$ and

$$F(x_n) \leq^s F(\bar{x}) + \varepsilon e. \tag{3}$$

By the continuity of the function $d(., M)$, $d(x, M) \leq \varepsilon$. Next, we show that $F(x) \leq^s F(\bar{x}) + \varepsilon e$. Indeed, from (3), we have

$$F(\bar{x}) + \varepsilon e \subset F(x_n) + K, \tag{4}$$

and

$$F(x_n) \subset F(\bar{x}) + \varepsilon e - K. \tag{5}$$

Let $y \in F(x)$ be arbitrary. Since F is lower semicontinuous and $\{x_n\}$ converges to x, there exist $y_n \in F(x_n)$ such that $\{y_n\}$ converges to y. Combining this with (5), there exist $w_n \in F(\bar{x})$ such that

$$y_n \in w_n + \varepsilon e - K. \tag{6}$$

Since $F(\bar{x})$ is compact, we can assume that $\{w_n\}$ converges to some $w \in F(\bar{x})$. By (6), there exist $k_n \in K$ such that $y_n = w_n + \varepsilon e - k_n$. This leads to $\lim_{n\to\infty} k_n = w + \varepsilon e - y$. Moreover, we get $w + \varepsilon e - y \in K$ as K is closed. Therefore, there exists $k \in K$ such that $w + \varepsilon e - y = k$. We have $y = w + \varepsilon e - k \in w + \varepsilon e - K$. It yields that $y \in F(\bar{x}) + \varepsilon e - K$ as $w \in F(\bar{x})$. We arrive at the fact that $F(x) \subset F(\bar{x}) + \varepsilon e - K$, i.e., $F(x) \leq^u F(\bar{x}) + \varepsilon e$.

Similarly, let $t \in F(\bar{x})$ be arbitrary, it follows from (4) that, for each $n \in \mathbb{N}$, there exists $v_n \in F(x_n)$ such that

$$t \in v_n - \varepsilon e + K. \tag{7}$$

Since F is upper semicontinuous and compact-valued at x, we can assume that $\{v_n\}$ converges to some element $v \in F(x)$. It implies from (7) that there exist $k_n \in K$ such that $t = v_n - \varepsilon e + k_n$. Hence, $k_n = t + \varepsilon e - v_n$. This leads to $\lim_{n\to\infty} k_n = t + \varepsilon e - v$. Since K is closed, there exists $k \in K$ such that $t + \varepsilon e - v = k$. We get $t = v - \varepsilon e + k \in v - \varepsilon e + K$. It yields that $t \in F(x) - \varepsilon e + K$ as $v \in F(x)$. We have $F(\bar{x}) \subset F(x) - \varepsilon e + K$. It means that $F(x) \leq^l F(\bar{x}) + \varepsilon e$. So, we obtain $F(x) \leq^s F(\bar{x}) + \varepsilon e$. The proof is complete. □

Theorem 3.2 *Suppose that F is continuous and compact-valued on M. Then,*

(i) (P_α) *is generalized LP well-posed at $\bar{x} \in S_{\alpha\text{-}MinF}$ if $\mu(S_{\alpha\text{-}MinF}(\bar{x}, \varepsilon)) \to 0$ as $\varepsilon \to 0$.*

(ii) (P_α) *is LP well-posed at $\bar{x} \in S_{\alpha\text{-}MinF}$ if $\operatorname{diam}(S_{\alpha\text{-}MinF}(\bar{x}, \varepsilon)) \to 0$ as $\varepsilon \to 0$.*

Proof (i) Suppose that $\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \to 0$ as $\varepsilon \to 0$. Let $\{x_n\}$ be a LP-minimizing sequence for (P_α) at $\bar{x}$. Therefore, there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \backslash \{0\}$ converging to 0 such that $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$. This means that $x_n \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)$. It is clear that $\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)) \to 0$ as $n \to \infty$, and hence by Lemma 2.2 (iii), we have $\cap_{n\in\mathbb{N}} S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)$ is a nonempty compact set and

$$H(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n), \cap_{n\in\mathbb{N}} S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)) \to 0$$

as $n \to \infty$. Note further from Proposition 3.2 (iii) that

$$S_{\alpha\text{-Min}F}(\bar{x}, 0) = \cap_{n\in\mathbb{N}} S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n).$$

Hence, we conclude that $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is compact and

$$H(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n), S_{\alpha\text{-Min}F}(\bar{x}, 0)) \to 0$$

as $n \to \infty$. Thus,

$$d(x_n, S_{\alpha\text{-Min}F}(\bar{x}, 0)) \to 0. \tag{8}$$

Therefore, there exists a sequence $\{\hat{x}_n\} \subset S_{\alpha\text{-Min}F}(\bar{x}, 0)$ such that $d(x_n, \hat{x}_n) \to 0$ as $n \to \infty$. Since $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is compact, there is a subsequence $\{\hat{x}_{n_k}\}$ of $\{\hat{x}_n\}$ converging to some $\hat{x} \in S_{\alpha\text{-Min}F}$. This implies that $\{x_n\}$ has a corresponding subsequence $\{x_{n_k}\}$ converging to $\hat{x}$. Hence, (P_α) is generalized LP well-posed at $\bar{x}$.

(ii) Assume that $\operatorname{diam}(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \to 0$ as $\varepsilon \to 0$. Then, $\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \to 0$ as $\varepsilon \to 0$, and hence (P_α) is generalized LP well-posed at $\bar{x}$. By Proposition 3.2, $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is a singleton. By Lemma 2.1 (ii), (P_α) is LP well-posed at $\bar{x}$. □

The below example shows that Theorem 3.2 is applicable.

Example 3.4 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $M = [0, 1]$, $K = \mathbb{R}^2_+$. Let $F : X \rightrightarrows Y$ be defined by

$$F(x) = [x, x + 1] \times [x, x + 1], \forall x \in X.$$

Let $e = (1, 1) \in \operatorname{int}K$, $\bar{x} = 0$. Clearly, all assumptions of Theroem 3.2 hold. By direct cacullations, we get $S_{\alpha\text{-Min}F}(0, \varepsilon) = [0, \varepsilon]$ and $S_{\alpha\text{-Min}F} = \{0\}$. So, $\operatorname{diam}(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \to 0$ as $\varepsilon \to 0$. Applying Theorem 3.2, the problem (P_α) is *LP* well-posed at $\bar{x} = 0$. In fact, if $\{x_n\}$ is a *LP* minimizing sequence for (P_α) at $\bar{x}$, then there is a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \backslash \{0\}$ converging to 0 such that $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(0) + \varepsilon_n e = [\varepsilon_n, \varepsilon_n + 1] \times [\varepsilon_n, \varepsilon_n + 1]$. We get $0 \leq x_n \leq \varepsilon_n$, and hence $\{x_n\}$ converges to 0. So, (P_α) is *LP* well-posed at 0.

The following example shows that the continuity of F in Theorem 3.2 is crucial.

Example 3.5 Let $X = \mathbb{R}, Y = \mathbb{R}^2, M = [-1, 1], K = \mathbb{R}^2_+$. Let $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [0, 1] \times [0, 1], & \text{if } \ x < 0, \\ [1, 2] \times [1, 2], & \text{if } \ x \geq 0. \end{cases}$$

Let $e = (1, 1) \in \text{int}K, \bar{x} = -\frac{1}{2}$. Then, F is compact-valued on M. Direct computations give us that $S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon) = [-1, 0)$, and hence $\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \to 0$ as $\varepsilon \to 0$. However, the problem (P_α) is not generalized LP well-posed at $\bar{x}$. Indeed, setting $x_n = -\frac{1}{n}$, we have $\{x_n\}$ is a LP minimizing sequence for (P_α) at $\bar{x}$ but $\{x_n\}$ converges to $0 \notin S_{\alpha\text{-Min}F}(\bar{x}, 0)$. The reason here is that F is not continuous.

Next, employing properties of the approximating solution mapping of (P_α), the connection between LP well-posedness of (P_α) and stability of approximating problem is established.

Theorem 3.3 *Let* $\bar{x} \in S_{\alpha\text{-}MinF}$.

(i) *Problem* (P_α) *is generalized* LP *well-posed at* $\bar{x}$ *if and only if* $S_{\alpha\text{-}MinF}(\bar{x}, \cdot)$ *is upper semicontinuous and compact-valued at* 0.
(ii) *Problem* (P_α) *is* LP *well-posed at* $\bar{x}$ *if and only if* $S_{\alpha\text{-}MinF}(\bar{x}, \cdot)$ *is upper semicontinuous at* 0 *and* $S_{\alpha\text{-}MinF}(\bar{x}, 0) = \{\bar{x}\}$.

Proof (i) Suppose that (P_α) is generalized LP well-posed at $\bar{x}$. By Lemma 3.1, $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is compact. Suppose by contrary that $S_{\alpha\text{-Min}F}(\bar{x}, \cdot)$ is not upper semicontinuous at 0. Then, there exists an open set $N \supset S_{\alpha\text{-Min}F}(\bar{x}, 0)$ such that for any $\delta > 0$, there exists $\varepsilon \in [0, \delta)$, $S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon) \not\subset N$. It means that there exists a sequence $\{\varepsilon_n\}$ converging to 0 such that for each $n \in \mathbb{N}$, we have $S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n) \not\subset N$. Thus, for each $n \in \mathbb{N}$, there is $x_n \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)$, $x_n \notin N$. Then, $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$, which imply that $\{x_n\}$ is a LP-minimizing sequence for (P_α) at $\bar{x}$. Because (P_α) is generalized LP well-posed at $\bar{x}$, there is a subsequence of $\{x_n\}$, denoted by $\{x_{n_k}\}$, converging to an element $\hat{x} \in S_{\alpha\text{-Min}F}(\bar{x}, 0) \subset N$. This is impossible as $x_{n_k} \notin N$ for all k.

Conversely, let $\{x_n\} \subset X$ be a LP-minimizing sequence for (P_α) at $\bar{x}$. Then, there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \backslash \{0\}$ converging to 0 such that

$$d(x_n, M) \leq \varepsilon_n, \quad F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e.$$

So, $x_n \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)$. It follows from the upper semicontinuity and compact-valuedness of $S_{\alpha\text{-Min}F}(\bar{x}, \cdot)$ at 0, Lemma 2.1 (ii) implies that there exist an element $\hat{x} \in S_{\alpha\text{-Min}F}(\bar{x}, 0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to $\hat{x}$. So, (P_α) is generalized LP well-posed at $\bar{x}$.

(ii) Let $\{x_n\} \subset X$ be a LP-minimizing sequence for (P_α) at $\bar{x}$, then there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \backslash \{0\}$ converging to 0 such that $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$. This means that, for each $n \in \mathbb{N}$,

$$x_n \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n). \tag{9}$$

Since $S_{\alpha\text{-Min}F}(\bar{x}, \cdot)$ is upper semicontinuous at 0, for any open set N, $S_{\alpha\text{-Min}F}(\bar{x}, 0) \subset N$, there is a neighborhood U of 0 such that for all $t \in U, t \geq 0$, we have

$S_{\alpha\text{-Min}F}(\bar{x}, t) \subset N$. Since $\{\varepsilon_n\}$ converges to 0, there exists $n_0 \in \mathbb{N}$ such that $\varepsilon_n \in B(0, \frac{1}{n_0})$ for all $n \geq n_0$. Combining this with (9), we obtain $x_n \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n) \subset N$ for all $n \geq n_0$. Therefore, for every neighborhood W of 0, $x_n \in S_{\alpha\text{-Min}F}(\bar{x}, 0) + W$ for all $n \geq n_0$. Since $S_{\alpha\text{-Min}F}(\bar{x}, 0) = \{\bar{x}\}$, $\{x_n\}$ converges to $\bar{x}$. So, (P$_\alpha$) is LP well-posed at $\bar{x}$.

For the converse, suppose that (P$_\alpha$) is LP well-posed at $\bar{x}$. Using (i), $S_{\alpha\text{-Min}F}(\bar{x}, \cdot)$ is upper semicontinuous and compact-valued at 0. We show that $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is a singleton. Suppose by the contrary that there exist $x_1, x_2 \in S_{\alpha\text{-Min}F}(\bar{x}, 0)$ with $x_1 \neq x_2$. Putting $x_{2n+k} = x_k$ where $k = 1$ or $k = 2$. Clearly, $\{x_n\}$ is a LP-minimizing sequence for (P$_\alpha$) at $\bar{x}$. However, $\{x_n\}$ is not convergent. This is a contradiction. Therefore, $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is a singleton. Moreover, it is obvious that $\bar{x} \in S_{\alpha\text{-Min}F}(\bar{x}, 0)$. So, $S_{\alpha\text{-Min}F}(\bar{x}, 0) = \{\bar{x}\}$. □

The assumption about the upper semicontinuity of approximating solution mapping of (P$_\alpha$) is used in Theorem 3.3. Next, we give the sufficient conditions for this assumption.

Proposition 3.4 *Suppose that the following conditions hold:*

(i) *M is compact*;
(ii) *F is continuous and compact-valued on M.*

Then, $S_{\alpha\text{-}MinF}(\bar{x}, \cdot)$ is upper semicontinuous at 0.

Proof By the similarity, we only focus on the proof of the assertion for the case $\alpha = u$. By contradiction, suppose that $S_{u\text{-Min}F}(\bar{x}, \cdot)$ is not upper semicontinuous at 0. Then, there exist an open set $N \supset S_{u\text{-Min}F}(\bar{x}, 0)$ and a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+ \backslash \{0\}$ converging to 0 such that for each n, there exists x_n satisfying

$$x_n \in S_{u\text{-Min}F}(\bar{x}, \varepsilon_n) \backslash W_0. \tag{10}$$

Since $x_n \in S_{u\text{-Min}F}(\bar{x}, \varepsilon_n)$,

$$d(x_n, M) \leq \varepsilon_n \tag{11}$$

and

$$F(x_n) \subset F(\bar{x}) + \varepsilon_n e - K. \tag{12}$$

It implies from (11) that there exist $\hat{x}_n \in M$ such that $d(x_n, \hat{x}_n) \leq \varepsilon_n$. By the compactness of M, we can assume that $\{\hat{x}_n\}$ converges to an element $x_0 \in M$. Hence, $\{x_n\}$ converges to x_0. Next, we prove that

$$F(x_0) \subset F(\bar{x}) - K. \tag{13}$$

Indeed, by the compact-valuedness of F, the closedness of K and Lemma 2.3, $F(\bar{x}) - K$ is closed. From (12), taking $n \to \infty$, we obtain (13). It means that $x_0 \in S_{u\text{-Min}F}(\bar{x}, 0)$. Combining this, (10) and the convergence to x_0 of $\{x_n\}$, we get a contradiction. Therefore, $S_{u\text{-Min}F}(\bar{x}, \cdot)$ is upper semicontinuous at 0. □

Corollary 3.1 *Suppose that the following conditions hold:*

(i) *M is compact*;
(ii) *F is continuous and compact-valued on M.*

Then,

(a) (P_α) *is generalized LP well-posed at $\bar{x} \in S_{\alpha\text{-}MinF}$ if $S_{\alpha\text{-}MinF}(\bar{x}, 0)$ is closed.*
(b) (P_α) *is LP well-posed at $\bar{x} \in S_{\alpha\text{-}MinF}$ if $S_{\alpha\text{-}MinF}(\bar{x}, 0) = \{\bar{x}\}$.*

4 Metrically *LP* well-posed set optimization problems

Picking up the ideas in [22], we introduce the following new concepts of LP well-posedness related to metrically approach for the problem (P_α).

Definition 4.1 A sequence $\{x_n\} \subset X$ is said to be a

(i) metrically *LP*-minimizing sequence for problem (P_l) at $\bar{x} \in S_{l\text{-Min}F}$ if and only if $H^*(F(\bar{x}), F(x_n)) \to 0$ and $d(x_n, M) \to 0$ as $n \to \infty$.
(ii) metrically *LP*-minimizing sequence for problem (P_u) at $\bar{x} \in S_{u\text{-Min}F}$ if and only if $H^*(F(x_n), F(\bar{x})) \to 0$ and $d(x_n, M) \to 0$ as $n \to \infty$.
(iii) metrically *LP*-minimizing sequence for problem (P_s) at $\bar{x} \in S_{s\text{-Min}F}$ if and only if $H(F(\bar{x}), F(x_n)) \to 0$ and $d(x_n, M) \to 0$ as $n \to \infty$.

Definition 4.2 The problem (P_α) is said to be metrically *LP* well-posed if and only if $S_{\alpha\text{-Min}F} \neq \emptyset$ and for any metrically *LP*-minimizing sequence $\{x_n\}$ for problem (P_α) at some $\bar{x} \in S_{\alpha\text{-Min}F}$, we have $d(x_n, S_{\alpha\text{-Min}F}) \to 0$ as $n \to \infty$.

Remark 4.1 When $\alpha = l$, concepts introduced in Definitions 4.1 (i) and 4.2 are similar to the corresponding ones studied in Definitions 4.5 (ii) and 4.6 (ii) in [22].

Example 4.1 (a) Let $X = Y = \mathbb{R}$, $M = [0, 1]$, $K = \mathbb{R}_+$, and let $F : X \rightrightarrows Y$ be defined by $F(x) = [1, 2]$ for all $x \in X$. Obviously, $S_{\alpha\text{-Min}F} = [0, 1] = M$, and the problem (P_α) is metrically *LP* well-posed.
(b) Let $X = Y = \mathbb{R}$, $M = K = \mathbb{R}_+$. Let $F : X \rightrightarrows Y$ be defined by $F(x) = [x^2, 3x^2]$ for all $x \in X$. Direct cacullations give us $S_{\alpha\text{-Min}F} = \{0\}$ and the problem (P_α) is metrically *LP* well-posed. Indeed, let $\{x_n\}$ be a metrically *LP*-minimizing sequence for (P_α) at $\bar{x} = 0$, it implies from definition of metrically *LP*-minimizing sequence for (P_α) at $\bar{x} = 0$ that $\{x_n\}$ converges to $\bar{x}$. Therefore, $d(x_n, S_{\alpha\text{-Min}F}) \to 0$.

Example 4.2 Let $X = Y = \mathbb{R}$, $M = [-1, 1]$, $K = \mathbb{R}_+$, and $F : X \rightrightarrows Y$ is defined by

$$F(x) = \begin{cases} [0, 1), & \text{if } \ x \le 0, \\ (0, 1], & \text{if } \ x > 0. \end{cases}$$

By direct computations, we get $S_{\alpha\text{-Min}F} = [-1, 0]$. Taking $x_n = 1 + \frac{1}{n}$, then $\{x_n\}$ is a metrically *LP*-minimizing sequence for the problem (P_α) at $\bar{x} = 0 \in S_{\alpha\text{-Min}F}$, but $d(x_n, S_{\alpha\text{-Min}F}) \to 1$. Therefore, the problem (P_α) is not metrically *LP* well-posed.

Next, we introduce a generalized form of the above concept.

Definition 4.3 The problem (P_α) is said to be generalized metrically *LP* well-posed if and only if $S_{\alpha\text{-Min}F} \neq \emptyset$ and for any metrically *LP*-minimizing sequence $\{x_n\}$ for (P_α) at some $\bar{x} \in S_{\alpha\text{-Min}F}$, $\{x_n\}$ has a subsequence, denoted by $\{x_{n_k}\}$, such that $d(x_{n_k}, S_{\alpha\text{-Min}F}) \to 0$ as $k \to \infty$.

It is clear that if (P_α) is metrically *LP* well-posed, then it is generalized metrically *LP* well-posed.

These following results give the relationships between these kinds of *LP* well-posedness considered in this study.

Theorem 4.1 (i) *If* (P_α) *is LP well-posed at all* $\bar{x} \in S_{\alpha\text{-}MinF}$, *then* (P_α) *is metrically LP well-posed.*

(ii) *If* (P_α) *is generalized LP well-posed at all* $\bar{x} \in S_{\alpha\text{-}MinF}$, *then* (P_α) *is generalized metrically LP well-posed.*

Proof (i) By the similarity we verify the assertions (i), (ii) for the case $\alpha = s$ as an example. Let $\{x_n\}$ be a metrically *LP*-minimizing sequence for problem (P_s) at some $\bar{x} \in S_{s\text{-Min}F}$. We need to prove that $d(x_n, S_{s\text{-Min}F}) \to 0$. In fact, since $\{x_n\}$ is a metrically *LP*-minimizing sequence for problem (P_s) at some $\bar{x} \in S_{s\text{-Min}F}$, $d(x_n, M) \to 0$ and

$$H(F(\bar{x}), F(x_n)) \to 0. \tag{14}$$

Observe that we can choose a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \backslash \{0\}$ converging to 0 satisfying $d(x_n, M) \leq \varepsilon_n$, both $-\varepsilon_n e + K$ and $\varepsilon_n e - K$ are neighborhoods of the origin in Y. By (14), there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have

$$F(\bar{x}) \subset F(x_n) - \varepsilon_n e + K \text{ and } F(x_n) \subset F(\bar{x}) + \varepsilon_n e - K.$$

This implies that $F(x_n) \leq^s F(\bar{x}) + \varepsilon_n e$. Hence, $\{x_n\}$ is a *LP*-minimizing sequence for (P_s) at $\bar{x}$. By the *LP* well-posedness of (P_s) at $\bar{x}$, $\{x_n\}$ converges to $\bar{x}$. Moreover, since $\bar{x} \in S_{s\text{-Min}F}$, $d(x_n, S_{s\text{-Min}F}) \leq \|x_n - \bar{x}\| \to 0$. So, (P_s) is metrically *LP* well-posed.

(ii) Using a similar argument with one above, we can prove that the statement (ii) is satisfied. □

Remark 4.2 When $\alpha = l$, (P_α) reduces to (P_l) studied in [22]. To obtain the metrically *LP* well-posedness for (P_l), the authors used an important assumption about the K-closed values of F on M, i.e., $F(x) + K$ is closed for all $x \in M$. Using another approach, as in Theorem 4.1, we can remove this assumption but also obtain the metrically *LP* well-posedness for (P_α).

Combining Theorem 4.1 and Corollary 3.1, we obtain the following results.

Theorem 4.2 *Suppose that the following conditions are satisfied:*

(i) M *is compact*;
(ii) F *is continuous and compact-valued on* M.

Then,

(a) (P_α) *is metrically LP well-posed if* $S_{\alpha\text{-}MinF}(\bar{x}, 0) = \{\bar{x}\}$ *for every* $\bar{x} \in S_{\alpha\text{-}MinF}$.

(b) (P_α) *is generalized metrically LP well-posed if* $S_{\alpha\text{-}MinF}(\bar{x}, 0)$ *is closed for every* $\bar{x} \in S_{\alpha\text{-}MinF}$.

Remark 4.3 Very recently, in [6], the authors studied several kinds of well-posedness for set optimization problems via the lower set less relation, including B-well-posedness, L-well-posedness, DH-well-posedness, and they obtained many interesting results related to this topic. In this paper, we consider the Levitin–Polyak well-posedness and the generalized Levitin–Polyak well-posedness for set optimization problems involving various kinds of set less relations, and hence the concepts of well-posedness investigated in this paper are different from those in [6]. Therefore, it could not compare our results with theirs.

References

1. Anh, L.Q., Duy, T.Q.: Tykhonov well-posedness for lexicographic equilibrium problems. Optimization **65**, 1929–1948 (2016)
2. Anh, L.Q., Khanh, P.Q., Van, D.T.M., Yao, J.C.: Well-posedness for vector quasiequilibria. Taiwan J. Math. **13**, 713–737 (2009)
3. Aubin, J.P., Frankowska, H.: Set-valued analysis. Springer, Boston (2009)
4. Bednarczuk, E., Penot, J.P.: Metrically well-set minimization problems. Appl. Math. Optim. **26**, 273–285 (1992)
5. Chen, J., Ansari, Q.H., Yao, J.C.: Characterizations of set order relations and constrained set optimization problems via oriented distance function. Optimization **66**, 1741–1754 (2017)
6. Crespi, G.P., Dhingra, M., Lalitha, C.S.: Pointwise and global well-posedness in set optimization: a direct approach. Ann. Oper. Res. **269**, 149–166 (2018)
7. Crespi, G.P., Guerraggio, A., Rocca, M.: Well posedness in vector optimization problems and vector variational inequalities. J. Optim. Theory Appl. **132**, 213–226 (2007)
8. Crespi, G.P., Schrage, C.: Set optimization meets variational inequalities. In: Hamel, A., Heyde, F., Löhne, A., Rudloff, B., Schrage, C. (eds.) Set optimization and applications-the state of the art, pp. 213–247. Springer, Berlin (2015)
9. Dhingra, M., Lalitha, C.: Well-setness and scalarization in set optimization. Optim. Lett. **10**, 1657–1667 (2016)
10. Dontchev, A.L., Zolezzi, T.: Well-posed optimization problems. Springer, New York (1993)
11. Gaydu, M., Geoffroy, M.H., Jean-Alexis, C., Nedelcheva, D.: Stability of minimizers of set optimization problems. Positivity **21**, 127–141 (2017)
12. Gutiérrez, C., Miglierina, E., Molho, E., Novo, V.: Pointwise well-posedness in set optimization with cone proper sets. Nonlinear Anal. **75**, 1822–1833 (2012)
13. Ha, T.X.D.: Some variants of the Ekeland variational principle for a set-valued map. J. Optim. Theory Appl. **124**, 187–206 (2005)
14. Hamel, A.H., Heyde, F.: Duality for set-valued measures of risk. SIAM J. Financ. Math. **1**, 66–95 (2010)
15. Hamel, A.H., Heyde, F., Löhne, A., Rudloff, B., Schrage, C.: Set optimization and applications-the state of the art: from set relations to set-valued risk measures. Springer, Berlin (2015)
16. Han, Y., Huang, N.: Well-posedness and stability of solutions for set optimization problems. Optimization **66**, 17–33 (2017)
17. Hernández, E., Rodríguez-Marín, L.: Existence theorems for set optimization problems. Nonlinear Anal. **67**, 1726–1736 (2007)
18. Huang, X.X., Yang, X.Q.: Generalized Levitin–Polyak well-posedness in constrained optimization. SIAM J. Optim. **17**, 243–258 (2006)
19. Jahn, J., Ha, T.X.D.: New order relations in set optimization. J. Optim. Theory Appl. **148**, 209–236 (2011)

20. Karaman, E., Soyertem, M., Atasever Güvenç, I., Tozkan, D., Küçük, M., Küçük, Y.: Partial order relations on family of sets and scalarizations for set optimization. Positivity **22**, 783–802 (2017)
21. Khan, A.A., Tammer, C., Zălinescu, C.: Set-valued optimization. Springer, Berlin (2016)
22. Khoshkhabar-amiranloo, S., Khorram, E.: Scalarization of Levitin–Polyak well-posed set optimization problems. Optimization **66**, 113–127 (2017)
23. Kuratowski, K.: Topology, vol. 2. Academic Press, London (1968)
24. Kuroiwa, D.: Some duality theorems of set-valued optimization with natural criteria. In: Proceedings of the international conference on nonlinear analysis and convex analysis, pp 221–228. World Scientific River Edge, (NJ) (1999)
25. Kuroiwa, D.: On set-valued optimization. Nonlinear Anal. **47**, 1395–1400 (2001)
26. Kuroiwa, D., Tanaka, T., Ha, T.X.D.: On cone convexity of set-valued maps. Nonlinear Anal. **30**, 1487–1496 (1997)
27. Lalitha, C., Chatterjee, P.: Levitin–Polyak well-posedness for constrained quasiconvex vector optimization problems. J. Glob. Optim. **59**, 191–205 (2014)
28. Levitin, E., Polyak, B.: Convergence of minimizing sequences in conditional extremum problems. Sov. Math. Doklady **7**, 764–767 (1966)
29. Long, X.J., Peng, J.W., Peng, Z.Y.: Scalarization and pointwise well-posedness for set optimization problems. J. Glob. Optim. **62**, 763–773 (2015)
30. Loridan, P.: Well-posedness in vector optimization. In: Lucchetti, R., Revalski, J. (eds.) Recent developments in well-posed variational problems, pp. 171–192. Kluwer Academic Publishers, Dordrecht (1995)
31. Miglierina, E., Molho, E.: Well-posedness and convexity in vector optimization. Math. Meth. Oper. Res. **58**, 375–385 (2003)
32. Miglierina, E., Molho, E., Rocca, M.: Well-posedness and scalarization in vector optimization. J. Optim. Theory Appl. **126**, 391–409 (2005)
33. Milovanovic-Arandjelovic, M.M.: Measures of noncompactness on uniform spaces-the axiomatic approach. Filomat **15**, 221–225 (2001)
34. Neukel, N.: Order relations of sets and its application in socio-economics. Appl. Math. Sci. (Ruse) **7**, 5711–5739 (2013)
35. Sach, P.H.: Solution existence in bifunction-set optimization. J. Optim. Theory Appl. **176**, 1–16 (2018)
36. Tikhonov, A.N.: On the stability of the functional optimization problem. USSR Comput. Math. Math. Phys. **6**, 28–33 (1966)
37. Wangkeeree, R., Anh, L.Q., Boonman, P.: Well-posedness for general parametric quasi-variational inclusion problems. Optimization **66**, 93–111 (2017)
38. Zhang, W., Li, S., Teo, K.L.: Well-posedness for set optimization problems. Nonlinear Anal. **71**, 3769–3778 (2009)