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Positivity

Maps between positive cones of operator algebras preserving a measure of the difference between arithmetic and geometric means

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Abstract

On the set of positive invertible elements in a finite von Neumann algebra carrying a faithful normalized trace τ the numerical quantity

$$d_\tau(A, B) = \tau(A + B)/2 - \tau\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}\right)$$

can be viewed as a measure of the difference of the arithmetic and the geometric mean. In this paper, we study maps between the positive definite cones of operator algebras which respect the above distance measure. We obtain the interesting fact that any such map originates from a trace-preserving Jordan *-isomorphisms (either algebra *-isomorphism or algebra *-antiisomorphism in the more restrictive case of factors) between the underlying von Neumann algebras.

Keywords Jordan *-isomorphisms · von Neumann algebra · Operator means

Mathematics Subject Classification Primary 47B49; Secondary 47A64

1 Introduction

Throughout the paper $\mathscr{A}$ denotes a finite von Neumann algebra acting on a separable Hilbert space. The cone of invertible positive elements in $\mathscr{A}$, by what we mean self-adjoint elements whose spectrum lies in $]0, +\infty[$, will be denoted by $\mathscr{A}_+^{-1}$. Since our

Dedicated to the memory of Professor Dénes Petz.

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result incorporates the theory of operator means on $\mathscr{A}_+^{-1}$ in the Kubo–Ando sense, we briefly review some basic concepts and facts in the forthcoming paragraphs. We further mention that the work of Professor Dénes Petz inspired many young researcher to deal with operator means. The author dedicates the paper to his memory with everlasting respect.

Recall that a numerical function f defined on an interval J is called $\mathscr{A}$-monotone if for all A, B with spectra in J, the operator inequality $A \leq B$ implies $f(A) \leq f(B)$. For example, the square root function is well-known to be $\mathscr{A}$-monotone on $[0, +\infty[$, see [1] for a recent proof of this assertion. In virtue of the fundamental results of the beautiful Kubo–Ando theory [5], the concept of operator means can be introduced in von Neumann algebras (or, more generally, in C^*-algebras) in the following way. For a fixed $\mathscr{A}$-monotone function f (called the generating function) which is normalized so that $f(1) = 1$, we define

$$A\sigma_f B = A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}, \quad A, B \in \mathscr{A}_+^{-1}. \tag{1}$$

Basic properties of this sort of means are

(i) If $A \leq C$ and $B \leq D$, then $A\sigma_f B \leq C\sigma_f D$ (monotonicity);
(ii) for any invertible $T \in \mathscr{A}$, we have $T(A\sigma_f B)T^* = (TAT^*)\sigma_f(TBT^*)$ (transfer identity).

As it was noted above, the square root function $x \mapsto \sqrt{x}$ is $\mathscr{A}$-monotone. The corresponding operator mean is called the geometric mean, that is,

$$A\#B = A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^{1/2} A^{1/2}.$$

Another important example is the arithmetic mean which is associated to the generating function $x \mapsto (1 + x)/2$. Taking this function in place of f in (1), we are led to

$$A \triangledown B = \frac{A + B}{2}.$$

In what follows, on the finite von Neumann algebra $\mathscr{A}$ the symbol τ denotes a *faithful normalized trace* τ, by what we mean a positive linear functional admitting the properties (i) $\tau(AB) = \tau(BA)$; (ii) $\tau(A^*A) = 0$ if and only if $A = 0$; (iii) $\tau(I) = 1$ where I is the identity operator. For example, if $\mathscr{A}$ is a factor, then such a functional exists uniquely. Further by the term *distance measure*, defined on a set M, we mean a function of two variables $d : M \times M \to [0, +\infty[$ satisfying the following sole property: $d(x, y) = 0$ holds if and only if $x = y$. Note that in general a distance measure may not be a true metric, as the symmetry or the triangle inequality might fail.

In the publications [4,9] the numerical quantity

$$d_\tau(A, B) = \tau(A \triangledown B) - \tau(A\#B), \quad A, B \in \mathscr{A}_+^{-1}$$

was introduced, independently, by Jenčová and Molnár in the context of matrix algebras. It was shown in [4] that the square root of the quantity $d_\tau(A, B)$ provides a strict upper bound (in the sense that equality holds only for commuting pairs of operators) for geodesic distances arising from monotone metrics. Molnár [9] considered the above quantity as a measure of the gap between the arithmetic and the geometric mean, and determined the structure of those transformations on the cones of positive (both positive definite and positive semidefinite) matrices which preserve the quantity $d_\tau(A, B)$. To be honest, the geometric mean was replaced by a general Kubo–Ando mean under some mild assumptions on the generating function but as it is pointed out in [9], the square root function fulfills these assumptions.

One can verify easily that the quantity $d_\tau(A, B)$ defines a distance measure. Indeed, introducing the auxiliary function $g(x) := (1+x)/2 - \sqrt{x}$, we can write

$$d_\tau(A, B) = \tau\left(A^{1/2} g\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}\right).$$

Then the non-negativity of g on the positive real line and the faithfulness of τ implies that for operators A, B in $\mathscr{A}_+^{-1}$ the equality $d_\tau(A, B) = 0$ holds if and only if

$$A^{1/2} g\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} = 0,$$

or, equivalently when $g(A^{-1/2} B A^{-1/2}) = 0$. Since the condition $g(x) = 0$ implies $x = 1$, the equality $d_\tau(A, B) = 0$ can hold only when $A^{-1/2} B A^{-1/2} = I$ which readily implies $A = B$. Using the above arguments, the reader (at least the one who is familiar with the theory of operator monotone functions) should be able to verify that the quantity $\tau(A \nabla B) - \tau(A \sigma_f B)$ for a general operator mean σ_f defines a distance measure, too. As it has no relevance here, we omit the details. Further we mention that the quantity $d_\tau(A, B)$ provides an example of *maximal f-divergences* which was introduced by Petz and Ruskai [11], and recently studied by Hiai and Mosonyi [3], and Matsumoto [6] in details. We remark that divergences are analogs of squared distances. For example, the metric property of the square root of the so-called symmetric Stein divergence has been recently established by Sra [12].

In the light of the above, Theorem 2 in [9] can be viewed as a sort of isometry theorem. In the present paper, we are concerned with establishing a von Neumann algebraic counterpart of the aforementioned result of Molnár. This was proposed as an open problem and suggested for further research at the end of the paper [9]. A closer look at our proof may convince the reader that for general operator means it is a rather challenging problem (if it is doable at all). The next best thing is to deal with the problem where only, the probably most important mean, the geometric mean appears. Thus, in the current paper we consider a more specific distance measure but in a much more general setting of finite von Neumann algebras carrying scalar valued traces. For further preserver problems related to means in operator algebras we refer to the nice papers [7,8].

2 The result

This section is devoted to the presentation of our result, as follows.

Theorem *Let $\mathscr{A}$ and $\mathscr{B}$ be finite von Neumann algebras equipped with faithful normalized traces τ and* Tr*, respectively. The bijective map $\phi : \mathscr{A}_+^{-1} \to \mathscr{B}_+^{-1}$ satisfies*

$$\begin{aligned} d_{\mathrm{Tr}}(\phi(A), \phi(B)) &= \mathrm{Tr}(\phi(A)\nabla\phi(B) - \phi(A)\#\phi(B)) \\ &= \tau(A\nabla B - A\#B) = d_\tau(A, B) \end{aligned}$$

for all $A, B \in \mathscr{A}_+^{-1}$ if and only if there is a Jordan *-isomorphism $J : \mathscr{A} \to \mathscr{B}$ and an element $T \in \mathscr{B}_+^{-1}$ such that

$$\phi(A) = TJ(A)T, \quad A \in \mathscr{A}_+^{-1}$$

and T, J satisfy $\mathrm{Tr}(TJ(A)T) = \tau(A)$ for all $A \in \mathscr{A}_+^{-1}$. Moreover, if $\mathscr{A}$ is a factor, then so is $\mathscr{B}$ and ϕ extends to either an algebra *-isomorphism or an algebra *-antiisomorphism between $\mathscr{A}$ and $\mathscr{B}$.

The core idea of the proof is reducing the problem to the description of certain unitary invariant norm preservers for (weighted) geometric means. This problem was handled successfully by Molnár in his very recent paper [7]. The following lemma [7, Proposition 3], which provides a useful characterization of the usual order (that is, the one which is induced by the positiveness) on $\mathscr{A}_+^{-1}$, helps a lot in reaching this aim.

Lemma *Let $\mathscr{A}$ be a von Neumann algebra and N be a unitary invariant norm on $\mathscr{A}$ with the property that $A \le B$, $N(A) = N(B)$ implies $A = B$. Then for any $A, B \in \mathscr{A}_+^{-1}$, we have $A \le B$ if and only if $N(A\#X) \le N(B\#X)$ ($X \in \mathscr{A}_+^{-1}$).*

Now, we are in a position to present the proof of our main result.

Proof of Theorem First we intend to show that the transformation ϕ preserves the usual order between the positive invertible elements. To this end, we establish the following characterization of the order on the cone $\mathscr{A}_+^{-1}$. For any $A, B \in \mathscr{A}_+^{-1}$, we have $A \le B$ if and only if the set

$$\left\{ d_\tau(A, X) - d_\tau(B, X) \ : \ X \in \mathscr{A}_+^{-1} \right\}$$

is bounded from below. To see this, assume that $A \le B$. Then by [7, Proposition 3] we conclude that

$$\tau(A\#X) \le \tau(B\#X), \quad X \in \mathscr{A}_+^{-1}.$$

Hence we obtain

$$\begin{aligned} &2d_\tau(A, X) - 2d_\tau(B, X) \\ &\quad = \tau(A) - \tau(B) + 2\tau(B\#X) - 2\tau(A\#X) \ge \tau(A) - \tau(B). \end{aligned}$$

This gives us the necessity part of the statement. To see the converse, suppose that the set

$$\left\{2d_\tau(A, X) - 2d_\tau(B, X) \ : \ X \in \mathscr{A}_+^{-1}\right\}$$

is bounded from below. This can happen only when the set

$$\left\{\tau(B\#X) - \tau(A\#X) \ : \ X \in \mathscr{A}_+^{-1}\right\}$$

itself is bounded from below. It means that there is a real number c such that

$$\tau(B\#X) - \tau(A\#X) \geq c$$

is satisfied for all positive invertible elements $X \in \mathscr{A}_+^{-1}$. Now, assume for contradiction that $A \not\leq B$. Applying [7, Proposition 3], again, we conclude that there exists an $X \in \mathscr{A}_+^{-1}$ such that

$$\tau(B\#X) - \tau(A\#X) < 0$$

is satisfied. Pick a number $t > 0$ and substitute tX in place of X in the last displayed inequality. We get that

$$\tau(B\#(tX)) - \tau(A\#(tX)) = \sqrt{t}\,(\tau(B\#X) - \tau(A\#X)) < 0, \quad t > 0.$$

Thus, by taking the limit $t \to +\infty$ we conclude that the set

$$\left\{\tau(B\#X) - \tau(A\#X) \ : \ X \in \mathscr{A}_+^{-1}\right\}$$

cannot be bounded from below, a contradiction. Clearly, the same characterization of the order can be obtained on the cone $\mathscr{B}_+^{-1}$ in terms of the distance measure d_{Tr}, too. It follows that ϕ preserves the order, as asserted.

Next, we are concerned with verifying that ϕ is trace-preserving, by what we mean that $\mathrm{Tr}\,\phi(A) = \tau(A)$ holds for all $A \in \mathscr{A}_+^{-1}$. In fact, this follows directly from the above verified order-preserving property. Indeed, consider a sequence (A_n) of operators in $\mathscr{A}_+^{-1}$. Then A_n tends to the zero operator if and only if for every $X \in \mathscr{A}_+^{-1}$ we have $A_n \leq X$ for large enough n. It follows that $A_n \to 0$ in the operator norm if and only if $\phi(A_n) \to 0$. Note that for an arbitrary but fixed $B \in \mathscr{A}_+^{-1}$, the sequence of operators $A_n\#B$ tends to 0 whenever A_n converges to 0. Thus, by the continuity of the trace functional τ, we conclude that $d_\tau(A_n, B)$ tends to $\tau(B)/2$ and $d_\tau(\phi(A_n), \phi(B))$ tends to $\mathrm{Tr}(\phi(B))/2$ from which the required trace-preserving property follows.

Since ϕ is trace-preserving, we see that the transformation ϕ satisfies

$$\mathrm{Tr}(\phi(A)\#\phi(B)) = \tau(A\#B), \quad A, B \in \mathscr{A}_+^{-1}.$$

The structure of those transformations which respect any given unitary invariant norm of geometric means is described in [7]. As the trace norm is unitary invariant on a finite Neumann algebra [10], the main result of [7] applies. Namely, [7, Theorem 2] tells us that there is a Jordan *-isomorphism $J : \mathscr{A} \to \mathscr{B}$ and a positive invertible element $T \in \mathscr{A}_+^{-1}$ such that ϕ is of the form

$$\phi(A) = TJ(A)T, \qquad A \in \mathscr{A}_+^{-1}$$

and $\mathrm{Tr}(TJ(A)T) = \tau(A)$ holds for all $A \in \mathscr{A}_+^{-1}$. Further if $\mathscr{A}$ is a factor, then according to [7, Corollary 5] $\mathscr{B}$ is a factor, too, and ϕ is either an algebra isomorphism or an algebra antiisomorphism. This completes the necessity part.

As for the sufficiency, we recall the very well-known property of Jordan *-homomorphisms [2, Lemma 2] that

$$J(BAB) = J(B)J(A)J(B)$$

holds for all $A, B \in \mathscr{A}$. By repeated applications of this property to various integer powers of an operator A in $\mathscr{A}$ and then taking linear combinations of these powers, we note that $J(p(A)) = p(J(A))$ for every polynomial p. The Stone–Weierstrass theorem and the norm-continuity of J lead us to the conclusion that

$$J(f(A)) = f(J(A)), \quad A \in \mathscr{A}_+^{-1}$$

for every continuous function f defined on the spectrum of A. With this consideration in mind, the rest follows straightforwardly, referring to the mentioned transfer identity of Kubo–Ando means. □

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