Positivity (2019) 23:291–301
https://doi.org/10.1007/s11117-018-0606-1

Positivity

Phase transitions for a model with uncountable spin space on the Cayley tree: the general case

Golibjon Botirov[2] · **Benedikt Jahnel**[1]

Received: 24 March 2018 / Accepted: 9 August 2018 / Published online: 17 August 2018

Abstract

In this paper we complete the analysis of a statistical mechanics model on Cayley trees of any degree, started in Botirov (Positivity 21(3):955–961, 2017), Eshkabilov et al. (J Stat Phys 147(4):779–794, 2012), Eshkabilov and Rozikov (Math Phys Anal Geom 13:275–286, 2010), Botirov et al. (Lobachevskii J Math 34(3):256–263 2013) and Jahnel et al. (Math Phys Anal Geom 17:323–331 2014). The potential is of nearest-neighbor type and the local state space is compact but uncountable. Based on the system parameters we prove existence of a critical value θ_c such that for $\theta \leq \theta_c$ there is a unique translation-invariant splitting Gibbs measure. For $\theta_c < \theta$ there is a phase transition with exactly three translation-invariant splitting Gibbs measures. The proof rests on an analysis of fixed points of an associated non-linear Hammerstein integral operator for the boundary laws.

Keywords Cayley trees · Hammerstein operators · Splitting Gibbs measures · Phase transitions

Mathematics Subject Classification 82B05 · 82B20 (primary); 60K35 (secondary)

1 Introduction

In the present note we complete a line of research about the phase-transition behavior of a nearest-neighbor model on Cayley trees with arbitrary degree $k \geq 2$. As first described in [4], for a given consistent family of finite-volume Gibbs measures, the

✉ Golibjon Botirov
botirovg@yandex.ru

Benedikt Jahnel
Benedikt.Jahnel@wias-berlin.de
https://www.wias-berlin.de/people/jahnel/

1 Weierstrass Institute Berlin, Mohrenstr. 39, 10117 Berlin, Germany

2 National University of Uzbekistan, University Street 4, Tashkent, Uzbekistan 100174

existence and multiplicity of a certain class of infinite-volume measures which are consistent with the prescribed finite-volume Gibbs measures, can be reduced to the analysis of fixed points of some non-linear integral equation of Hammerstein type. Every positive solution of the fixed point equation here corresponds to a measures which is called a splitting Gibbs measure. Every splitting Gibbs measure is also a Gibbs measure in the sense of the DLR formalism; see [1]. This approach has been successfully applied in the analysis of a variety of different models on Cayley trees with respect to their phase-transition properties; see [12] for a comprehensive overview. In particular, starting with [3], a phase-transition of multiple splitting Gibbs measures has been detected in a model with uncountable local state space [0, 1] and nearest-neighbor interactions. This has motivated the subsequent analysis in [2,5,9], to further understand critical behavior of this model for all degrees of the underlying tree, where also new parameters are introduced. In [6,7] the Potts model with a countable set of spin values is studied. It is the purpose of this note to complete the analysis of this model.

For nearest neighbors x, y on the *Cayley tree* Γ^k with degree $k \geq 2$ with local states $\sigma(x), \sigma(y) \in [0, 1]$, we consider the potential

$$\xi_{\sigma(x),\sigma(y)} = \log\left(1 + \theta \sqrt[2n+1]{4\left(\sigma(x) - \frac{1}{2}\right)\left(\sigma(y) - \frac{1}{2}\right)}\right) \tag{1.1}$$

where $n \in \mathbb{N} \cup \{0\}$ and $0 \leq \theta < 1$ are the system parameters. It can be interpreted as a certain symmetric pair-interaction with values in $[\log(1-\theta), \log(1+\theta)]$, admitting two distinct ground states given by the all-0 and the all-1 configuration. The main result is the existence of a sharp threshold

$$\theta_\mathrm{c} = \frac{2n+3}{k(2n+1)}$$

such that if $\theta_\mathrm{c} < \theta < 1$, there are exactly three translation-invariant splitting Gibbs measures and otherwise there is only one.

2 Setup

2.1 Gibbs measures on Cayley trees

The *Cayley tree* Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of *vertices* and L is a symmetric subset of $V \times V$, called the *edge set*. The word "symmetric" means that $(x, y) \in L$ iff $(y, x) \in L$. Here, x and y are called the *endpoints* of the edge $\langle x, y\rangle$. Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them and we denote $l = \langle x, y\rangle$. For a fixed $x^0 \in V$, called the *root*, we defines n-spheres and n-disks in the graph distance $d(x, y)$ by

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \bigcup_{i=0}^{n} W_i$$

and denote for any $x \in W_n$ the set of *direct successors* of x by

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}.$$

For $A \subset V$ let $\Omega_A = [0, 1]^A$ denote the set of all configurations σ_A on A. In particular, a configuration σ on V is then defined as a function $V \ni x \mapsto \sigma(x) \in [0, 1]$. According to the usual setup for Gibbs measure, we consider a (formal) Hamiltonian of the form

$$H(\sigma) = -\sum_{\langle x,y\rangle \in L} \xi_{\sigma(x),\sigma(y)}, \tag{2.1}$$

where $\xi : (u, v) \in [0, 1]^2 \mapsto \xi_{u,v} \in \mathbb{R}$ is the interaction (1.1) which assigns energy only to neighboring sites. Since ξ does not depend on the locations x and y, H is invariant under tree translations. Let λ be the Lebesgue measure on [0, 1] then, on the set of all configurations on A the a priori measure λ_A is introduced as the $|A|$-fold product of the measure λ. Here and in the sequel, $|A|$ denotes the cardinality of A. We equip $\Omega = \Omega_V$ with the standard sigma-algebra $\mathcal{B}$ generated by the cylindrical subsets. A probability measure μ on $(\Omega, \mathcal{B})$ is called a *Gibbs measure* (with Hamiltonian H) if it satisfies the DLR equation. That is, for any $n = 1, 2, \ldots$ and bounded measurable test function f, we have that

$$\int \mu(d\sigma) f(\sigma) = \int \mu(d\sigma) \int \gamma_{V_n}(d\tilde{\sigma}_{V_n} | \sigma_{W_{n+1}}) f(\tilde{\sigma}_{V_n} \sigma_{\Gamma^k \setminus V_n}), \tag{2.2}$$

where $\gamma_{V_n}(d\sigma_{V_n} | \sigma_{\Gamma^k \setminus V_n})$ is the Gibbsian specification

$$\gamma_{V_n}(d\tilde{\sigma}_{V_n} | \sigma_{\Gamma^k \setminus V_n}) = \frac{1}{Z_{V_n}(\sigma_{W_{n+1}})} e^{-\beta H(\tilde{\sigma}_{V_n} \sigma_{W_{n+1}})} \lambda_{V_n}(d\tilde{\sigma}_{V_n}),$$

with normalization Z_{V_n} and temperature parameter $\beta \geq 0$. Such a specification is also sometimes referred to as a Markov specification; see [8].

2.2 Representation via Hammerstein operators

A subset of the infinite-volume Gibbs measures defined via the DLR equation (2.2), called the *splitting Gibbs measures* or *Markov chains*, can be represented in terms of the fixed points of some nonlinear integral operator of Hammerstein type; see [4] for details. More precisely, for every $k \in \mathbb{N}$ consider the integral operator H_k acting on the cone $C^+[0, 1] = \{f \in C[0, 1] : f(x) \geq 0\}$ given by

$$(H_k f)(t) = \int_0^1 K(t, u) f^k(u) du. \tag{2.3}$$

Then, the translation-invariant splitting Gibbs measures for the Hamiltonian (2.1) correspond to positive fixed points of H_k with $K(t,u) = \exp(\beta\xi_{t,u})$, often called *boundary laws*. Note that H_k in general might generate ill-posed problems; see [10,11].

3 Main results

The main result of this note is the following characterization of phase-transition regimes of the model (1.1) with $\beta = 1$.

Theorem 3.1 *For all $n \in \mathbb{N} \cup \{0\}$ and $k \geq 2$ let $\theta_c = (2n+3)/(k(2n+1))$, then the model (1.1) has*

(1) a unique translation-invariant splitting Gibbs measure if $0 \leq \theta \leq \theta_c$ and
(2) exactly three translation-invariant splitting Gibbs measures if $\theta_c < \theta < 1$.

The proof is based on a characterization of solutions to the fixed point equation for the associated Hammerstein integral operator (2.3) as given in Proposition 3.2 below. In case of the model at hand, then the analysis can be reduced to finding the fixed points of the following 2-dimensional operator $V_k : \mathbb{R}^2 \to \mathbb{R}^2$

$$V_{k,n}(x,y) = \begin{cases} \sum\limits_{i=0,2,\ldots,\lfloor k \rfloor_{\text{even}}} \binom{k}{i} \frac{2n+1}{2n+1+i} 2^{\frac{i}{2n+1}} x^{k-i} (\theta y)^i \\ \\ \sum\limits_{i=1,3,\ldots,\lfloor k \rfloor_{\text{odd}}} \binom{k}{i} \frac{2n+1}{2n+2+i} 2^{\frac{i-1}{2n+1}} x^{k-i} (\theta y)^i \end{cases} \tag{3.1}$$

with $k \geq 2$, which is then the content of Proposition 3.3. Here we use the notation

$$\lfloor k \rfloor_{\text{even}} = \begin{cases} k, & \text{if } k \text{ is even} \\ k-1, & \text{if } k \text{ is odd} \end{cases} \qquad \text{and} \qquad \lfloor k \rfloor_{\text{odd}} = \begin{cases} k, & \text{if } k \text{ is odd} \\ k-1, & \text{if } k \text{ is even.} \end{cases}$$

Proposition 3.2 *A function $\varphi \in C[0,1]$ is a solution of the Hammerstein equation*

$$H_k f = f \tag{3.2}$$

with H_k defined in (2.3) for our model (1.1), iff φ has the following form

$$\varphi(t) = c_1 + c_2\theta \sqrt[2n+1]{4(t - \frac{1}{2})},$$

where $(c_1, c_2) \in \mathbb{R}^2$ is a fixed point of the operator $V_{k,n}$ as defined in (3.1).

In the following proposition we characterize the fixed points of $V_{k,n}$ which readily implies Theorem 3.1 using Proposition 3.2.

Proposition 3.3 *Let $\theta_c = (2n+3)/(k(2n+1))$, then there exist uniquely defined points $x_o, y_o \in (0,\infty)$ such that the number and form of the fixed points of the operator $V_{k,n}$ are as presented in the following Table 1.*

Table 1 Set of 2-dimensional fixed points of $V_{k,n}$

	Fixed points if $0 \leq \theta < 1$			Additional fixed points if $\theta_c < \theta < 1$			
k even	$(0, 0)$	$(1, 0)$		(x_o, y_o)		$(x_o, -y_o)$	
k odd	$(0, 0)$	$(1, 0)$	$(-1, 0)$	(x_o, y_o)	$(-x_o, -y_o)$	$(x_o, -y_o)$	$(-x_o, y_o)$

Further, only the fixed points $(1, 0)$, (x_o, y_o) and $(x_o, -y_o)$ give rise to positive solutions for the Hammerstein equation (3.2).

Let us finally give the references to the special cases considered prior to this work. [5, Theorem 4.2 and Theorem 5.2] proves the cases $k = 2, 3$ with $n = 1$ of (1.1) whereas in [9, Theorem 3.2.] the cases $k \geq 2$ with $n = 1$ are given. Finally, in [2, Theorem 2.3] the cases $k = 2$ with general $n \geq 1$ is provided.

4 Proofs

In order to ease notation, let us write $m = 2n + 1$ in this section. Note that for the model (1.1) with $\beta = 1$, the kernel $K(t, u)$ of the Hammerstein operator H_k is given by

$$K(t, u) = 1 + \theta \sqrt[m]{4\left(t - \frac{1}{2}\right)\left(u - \frac{1}{2}\right)}.$$

Proof of Proposition 3.2 Let us start with necessity. Assume $\varphi \in C[0, 1]$ to be a solution of the equation (3.2). Then we have

$$\varphi(t) = c_1 + c_2\theta \sqrt[m]{4\left(t - \frac{1}{2}\right)}, \tag{4.1}$$

where

$$c_1 = \int_0^1 \varphi^k(u)du \quad \text{and} \quad c_2 = \int_0^1 \sqrt[m]{u - \frac{1}{2}}\varphi^k(u)du. \tag{4.2}$$

Substituting $\varphi(t)$ into the first equation of (4.2) we get

$$\begin{aligned} c_1 &= \int_0^1 \left(c_1 + c_2\theta \sqrt[m]{4\left(u - \frac{1}{2}\right)}\right)^k du \\ &= \int_0^1 \sum_{i=0}^k \binom{k}{i} c_1^{k-i} \left(c_2\theta \sqrt[m]{4} \sqrt[m]{u - \frac{1}{2}}\right)^i du \end{aligned}$$

$$= \sum_{i=0}^{k} \binom{k}{i} c_1^{k-i} (\theta c_2)^i 2^{\frac{2i}{m}} \int\limits_0^1 \left(u - \frac{1}{2}\right)^{\frac{i}{m}} du.$$

Now, we use the following equality

$$\int\limits_0^1 \left(u - \frac{1}{2}\right)^{\frac{i}{m}} du = \begin{cases} 0, & \text{if } i \text{ is odd and} \\ \frac{m}{m+i} 2^{-\frac{i}{m}}, & \text{if } i \text{ is even.} \end{cases} \tag{4.3}$$

Then we get

$$c_1 = \sum_{i=0,2,\dots,\lfloor k \rfloor_{\text{even}}} \binom{k}{i} \frac{m}{m+i} 2^{\frac{i}{m}} c_1^{k-i} (\theta c_2)^i$$

and substituting the function φ into the second equation of (4.2) we have

$$\begin{aligned} c_2 &= \int\limits_0^1 \left(u - \frac{1}{2}\right)^{\frac{1}{m}} \left(c_1 + \theta c_2 \sqrt[m]{4\left(u - \frac{1}{2}\right)}\right)^k du \\ &= \int\limits_0^1 \left(u - \frac{1}{2}\right)^{\frac{1}{m}} \sum_{i=0}^{k} \binom{k}{i} c_1^{k-i} \left(\theta c_2 \sqrt[m]{4} \sqrt[m]{u - \frac{1}{2}}\right)^i du \\ &= \sum_{i=0}^{k} \binom{k}{i} c_1^{k-i} (c_2 \theta)^i 2^{\frac{2i}{m}} \int\limits_0^1 \left(u - \frac{1}{2}\right)^{\frac{i+1}{m}} du. \end{aligned}$$

Now, using the following equality

$$\int\limits_0^1 \left(u - \frac{1}{2}\right)^{\frac{i+1}{m}} du = \begin{cases} 0, & \text{if } i \text{ is even and} \\ \frac{m}{m+1+i} 2^{-\frac{i+1}{m}}, & \text{if } i \text{ is odd} \end{cases} \tag{4.4}$$

we arrive at the equation

$$c_2 = \sum_{i=1,3,\dots,\lfloor k \rfloor_{\text{odd}}} \binom{k}{i} \frac{m}{m+1+i} 2^{\frac{i-1}{m}} c_1^{k-i} (\theta c_2)^i.$$

In particular, the point $(c_1, c_2) \in \mathbb{R}^2$ must be a fixed point of the operator $V_{k,n}$ from (3.1).

For the sufficiency, assume that, a point $(c_1, c_2) \in \mathbb{R}^2$ is a fixed point of the operator $V_{k,n}$ and define the function $\varphi \in C[0,1]$ by the equality

$$\varphi(t) = c_1 + c_2\theta \sqrt[m]{4\left(t - \frac{1}{2}\right)}.$$

Then, we can calculate

$$\begin{aligned}
(H_k\varphi)(t) &= \int_0^1 \left(1 + \sqrt[m]{4}\theta \sqrt[m]{(t-\frac{1}{2})(u-\frac{1}{2})}\right)\varphi^k(u)du \\
&= \int_0^1 \varphi^k(u)du + \sqrt[m]{4}\theta \sqrt[m]{t-\frac{1}{2}} \int_0^1 \sqrt[m]{u-\frac{1}{2}}\varphi^k(u)du \\
&= \int_0^1 \left(c_1 + c_2\theta \sqrt[m]{4\left(u-\frac{1}{2}\right)}\right)^k du \\
&\quad + \sqrt[m]{4}\theta \sqrt[m]{t-\frac{1}{2}} \int_0^1 \sqrt[m]{u-\frac{1}{2}} \left(c_1 + c_2\theta \sqrt[m]{4\left(u-\frac{1}{2}\right)}\right)^k du \\
&= \sum_{i=0}^{k} \binom{k}{i} c_1^{k-i} (\theta c_2)^i 2^{\frac{2i}{m}} \int_0^1 \left(u-\frac{1}{2}\right)^{\frac{i}{m}} du + \sqrt[m]{4}\theta \sqrt[m]{t-\frac{1}{2}} \\
&\quad \times \sum_{i=0}^{k} \binom{k}{i} c_1^{k-i} (c_2\theta)^i 2^{\frac{2i}{m}} \int_0^1 \left(u-\frac{1}{2}\right)^{\frac{i+1}{m}} du.
\end{aligned} \tag{4.5}$$

Now, we using (4.3) and (4.4), from (4.5) we get

$$\begin{aligned}
(H_k\varphi)(t) &= \sum_{i=0,2,\ldots,\lfloor k\rfloor_{\text{even}}} \binom{k}{i} \frac{m}{m+i} 2^{\frac{i}{m}} c_1^{k-i} (\theta c_2)^i \\
&\quad + \theta \sqrt[m]{4(t-\frac{1}{2})} \sum_{i=1,3,\ldots,\lfloor k\rfloor_{\text{odd}}} \binom{k}{i} \frac{m}{m+1+i} 2^{\frac{i-1}{m}} c_1^{k-i} (\theta c_2)^i = \varphi(t).
\end{aligned}$$

Thus, φ is a solution of the equation (3.2). □

Proof of Proposition 3.3 We determine the number and form of solutions to $V_{k,n}$ in equation (3.1). For $\theta = 0$, the fixed point equation for $V_{k,n}$ reduces to $x = x^k$ and $y = 0$ and hence for k even, the solutions are given by $(1, 0)$ and $(0, 0)$ and for k odd, the solutions are given by $(1, 0)$, $(-1, 0)$ and $(0, 0)$. Further note that in both cases, if $x = 0$, then also $y = 0$. So from now on we assume that $\theta > 0$ and $x \neq 0$.

Let us start by considering the case where k is even. Inspecting the equation for x in $V_{k,n}$, we see that $x \geq 0$. We introduce a reduction of the 2-dimensional fixed point equation to a 1-dimensional one. Writing $z = \theta y/x$, then for any fixed point (x, y) of $V_{k,n}$, z necessarily is a solution to the fixed point equation

$$z = \theta \frac{\sum_{i=1,3,\ldots,k-1} \binom{k}{i} \frac{m}{m+1+i} 2^{\frac{i-1}{m}} z^i}{\sum_{i=0,2,\ldots,k} \binom{k}{i} \frac{m}{m+i} 2^{\frac{i}{m}} z^i} = \theta \frac{F_1^{\text{even}}(z)}{F_2^{\text{even}}(z)} = f_{\text{even}}(z).$$

In order to find solutions for $z = f_{\text{even}}(z)$, we have to find roots of the polynomial

$$\begin{aligned} P_{\text{even}}(z) &= \sum_{i=1,3,\ldots,k+1} \binom{k}{i-1} \frac{m}{m+i-1} 2^{\frac{i-1}{m}} z^i - \theta \sum_{i=1,3,\ldots,k-1} \binom{k}{i} \frac{m}{m+i+1} 2^{\frac{i-1}{m}} z^i \\ &= r_\theta(k, k+1) z^{k+1} + \sum_{i=1,3,\ldots,k-1} r_\theta(k, i) z^i \end{aligned} \tag{4.6}$$

where $r_\theta(k, k+1) = \frac{m}{m+k} 2^{\frac{k}{m}}$ and

$$r_\theta(k, i) = \binom{k}{i} \frac{m}{m+i+1} 2^{\frac{i-1}{m}} \Big[\frac{i}{k-i+1} \frac{m+i+1}{m+i-1} - \theta \Big].$$

Moreover,

$$r_\theta(k, i) \begin{cases} < 0 & \text{if } \theta > \frac{i}{k-i+1} \frac{m+i+1}{m+i-1} \\ = 0 & \text{if } \theta = \frac{i}{k-i+1} \frac{m+i+1}{m+i-1} \\ > 0 & \text{if } \theta < \frac{i}{k-i+1} \frac{m+i+1}{m+i-1} \end{cases}$$

and we denote the critical θ by $\theta_{k,i}$. Further note that $i \mapsto \theta_{k,i}$ is increasing. Indeed, the derivative of the continuous version (where $i \in \mathbb{N}$ is for a moment replaced by $x \in \mathbb{R}$) is given by

$$\frac{(1+k)(m-1)^2 + (3+k)i^2 + 2(1+k)(m-1)(1+i))}{(1+k-i)^2(m-1+i)^2}$$

which is non-negative. Hence, for θ below the lowest critical value, $\theta_c = \theta_{k,1} = \frac{m+2}{km}$, all coefficients are positive and hence there is no positive real root of P_{even} by Descartes' rule of sign. In particular, for $0 \leq \theta \leq \theta_c$, there can not be any solutions to the fixed point equation associated to $V_{k,n}$ of the form (x, y) with $0 < x, y$. Since $P_{\text{even}}(-z) = -P_{\text{even}}(z)$, there can also not exist any fixed point (x, y) with $y < 0 < x$. This settles the subcritical and critical case for even k.

Further, for the supercritical case $\theta > \theta_c$ with even k, again by Descartes' rule of sign, there is exactly one sign change in P_{even} and hence exactly one non-trivial positive real root of P_{even} exists which we denote $z_o > 0$. By the point symmetry of the polynomial P_{even}, with z_o also $-z_o$ is a root. In order to uniquely recover the solution (x_o, y_o) from the positive non-trivial solution z_o, note that

$$V_{k,n}(x,y) = \begin{cases} x^k \sum_{i=0,2,\dots,k} \binom{k}{i} \frac{m}{m+i} 2^{\frac{i}{m}} (\theta \frac{y}{x})^i = x^k F_2^{\text{even}}(\theta \frac{y}{x}) \\ x^k \sum_{i=1,3,\dots,k-1} \binom{k}{i} \frac{m}{m+i+1} 2^{\frac{i-1}{m}} (\theta \frac{y}{x})^i = x^k F_1^{\text{even}}(\theta \frac{y}{x}) \end{cases}$$

and hence $x_o = F_2^{\text{even}}(z_o)^{1/(1-k)} > 0$ and $y_o = F_1^{\text{even}}(z_o) F_2^{\text{even}}(z_o)^{k/(1-k)} > 0$ solve the 2-dimensional equation. Note that (x_o, y_o) is the only solution with $\theta y/x = z_o$. Indeed, any other such solution would be $x_1 = cx_o$ and $y_1 = cy_o$ for some $c \in \mathbb{R}\backslash\{0\}$, but then $cx_o = x_1 = x_1^k F_2(z_o) = c^k x_o^k F_2(z_o)$ which implies that $c = c^k$. But this is true if and only if $c = 1$ for even k.

Further, note that $F_2^{\text{even}}(-z_o)^{1/(1-k)} = F_2^{\text{even}}(z_o)^{1/(1-k)} = x_o$ and $F_1^{\text{even}}(-z_o) F_2^{\text{even}}(-z_o)^{k/(1-k)} = -F_1^{\text{even}}(z_o) F_2^{\text{even}}(z_o)^{k/(1-k)} = -y_o$ and hence also $(x_o, -y_o)$ is a solution to the 2-dimensional fixed point equation. Finally, as in the case of z_o, the tuple $(x_o, -y_o)$ is the only solution with $\theta y/x = -z_o$ since any other solution would be of the form $x_1 = cx_o$ and $y_1 = -cy_o$ for some $c \in \mathbb{R} \setminus \{0\}$. But then again $cx_o = x_1 = x_1^k F_2^{\text{even}}(z_o) = c^k x_o^k F_2^{\text{even}}(z_o)$ which implies that $c = c^k$ which is true if and only if $c = 1$ for even k. So, for even k, we have now completely charaterized solutions of the fixed point equation associated with also in the super critical regime.

For odd k and $\theta > 0$, the situations is slightly more complicated, since we can not exclude certain signs from the fixed points of

$$V_{k,n}(x,y) = \begin{cases} x^k \sum_{i=0,2,\dots,k-1} \binom{k}{i} \frac{m}{m+i} 2^{\frac{i}{m}} (\theta \frac{y}{x})^i \\ x^k \sum_{i=1,3,\dots,k} \binom{k}{i} \frac{m}{m+i+1} 2^{\frac{i-1}{m}} (\theta \frac{y}{x})^i. \end{cases}$$

Writing again $z = \theta y/x$, the fixed point equation for (3.1) becomes

$$z = \theta \frac{\sum_{i=1,3,\dots,k} \binom{k}{i} \frac{m}{m+1+i} 2^{\frac{i-1}{m}} z^i}{\sum_{i=0,2,\dots,k-1} \binom{k}{i} \frac{m}{m+i} 2^{\frac{i}{m}} z^i}$$

and the corresponding point symmetric polynomial is given by

$$P_{\text{odd}}(z) = \sum_{i=1,3,\dots,k} r_\theta(k,i) z^i.$$

Hence, the exact same arguments as in the case of even k apply, yielding again no roots for $\theta \le \theta_c$ and two roots z_o and $-z_o$ for $\theta > \theta_c$. In contrast to the case for even k, for odd k, both (x_o, y_o) and $(-x_o, -y_o)$ are 2-dimensional fixed points corresponding to z_o since $c = c^k$ can be solved by ± 1. Finally, following the exact same arguments as in the case of even k, the fixed points $(x_o, -y_o)$ and $(-x_o, y_o)$ correspond to $-z_o$. The complete list of fixed points is recorded in Table 1.

For $(\pm x_o, \pm y_o)$ to give rise to a positive solution of the Hammerstein fixed point equation, by the form of solutions φ we must have that for all $t \in [0, 1]$

$$\pm x_o \pm y_o \theta \sqrt[m]{4(t - 1/2)} > 0. \tag{4.7}$$

Clearly, for $-x_o$, in $t = 1/2$, the inequality is violated and it suffices to consider the points $(x_o, \pm y_o)$. Note that for all $t \in [0, 1]$

$$-2^{1/m} = \sqrt[m]{4(0 - 1/2)} \leq \sqrt[m]{4(t - 1/2)} \leq \sqrt[m]{4(1 - 1/2)} = 2^{1/m}$$

and hence, it suffices to show that

$$2^{-1/m} > \theta y_o / x_o = z_o \tag{4.8}$$

where z_o is the unique positive root of the polynomial $P \in \{P_{\text{even}}, P_{\text{odd}}\}$. Note that the sign change of P in z_o must be from minus to plus, i.e., $P(z) < 0$ for $z < z_o$ and $P(z) > 0$ for $z > z_o$, it suffices to show that $P(2^{-1/m}) > 0$. Let us do this here only for P_{even}, the case of P_{odd} can be solved identically. Note that

$$\begin{aligned} P_{\text{even}}(2^{-1/m}) = 2^{-1/m} \Bigg[& \sum_{i=1,3,\ldots,k+1} \binom{k}{i-1} \frac{m}{m+i-1} \\ & -\theta \sum_{i=1,3,\ldots,k-1} \binom{k}{i} \frac{m}{m+1+i} \Bigg] > 0 \end{aligned}$$

is implied by

$$\sum_{i=1,3,\ldots,k+1} \binom{k}{i-1} \frac{1}{m+i-1} - \sum_{i=1,3,\ldots,k-1} \binom{k}{i} \frac{1}{m+1+i} > 0 \tag{4.9}$$

since $\theta < 1$. We can further bound the left hand side of (4.9) from below by

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{1}{m+1+i} = \frac{k!m!}{(m+1+k)!}$$

which is positive for all m, k. By direct computation we also see that $(1, 0)$ satisfies inequality (4.7) whereas $(0, 0)$ and $(-1, 0)$ do not. This completes the proof. □

Proof of Theorem 3.1 According to [4], the translation-invariant splitting Gibbs measures are in one-to-one correspondence with the positive solutions of equation (2.3) for $K(t, u) = \exp(\xi_{t,u})$. By Proposition 3.2 solutions to (2.3) are characterized by 2-dimensional fixed points of the operator $V_{k,n}$. Now, Proposition 3.3 provides a complete list of these fixed points in the two parameter regimes for θ. Proposition 3.3 further asserts that only exactly one respectively three of them give rise to positive solutions. This completes the proof. □

Acknowledgements Golibjon Botirov thanks the DAAD program for the financial support and the Weierstrass Institute Berlin for its hospitality. Benedikt Jahnel thanks the Leibniz program 'Probabilistic methods for mobile ad-hoc networks' for the support.

References

1. Baxter, R.J.: Exactly Solved Models in Statistical Mechanics. Academic, London (1982)
2. Botirov, G.I.: A model with uncountable set of spin values on a Cayley tree: phase transitions. Positivity **21**(3), 955–961 (2017)
3. Eshkabilov, Y.K., Haydarov, F.H., Rozikov, U.A.: Non-uniqueness of Gibbs measure for models with uncountable set of spin values on a Cayley tree. J. Stat. Phys. **147**(4), 779–794 (2012)
4. Eshkabilov, Y.K., Rozikov, U.A.: On models with uncountable set of spin values on a Cayley tree: integral equations. Math. Phys. Anal. Geom. **13**, 275–286 (2010)
5. Botirov, G.I., Eshkabilov, Y.K., Rozikov, U.A.: Phase transition for a model with uncountable set of spin values on Cayley tree. Lobachevskii J. Math. **34**(3), 256–263 (2013)
6. Ganikhodjaev, N.N.: Potts model on $\mathbb{Z}^d$ with countable set of spin values. J. Math. Phys. **45**, 1121–1127 (2004)
7. Ganikhodjaev, N.N., Rozikov, U.A.: The Potts model with countable set of spin values on a Cayley tree. Lett. Math. Phys. **74**, 99–109 (2006)
8. Georgii, H.-O.: Gibbs Measures and Phase Transitions. De Gruyter, New York (2011)
9. Jahnel, B., Külske, C., Botirov, G.I.: Phase transition and critical values of a nearest-neighbor system with uncountable local state space on Cayley trees. Math. Phys. Anal. Geom. **17**, 323–331 (2014)
10. Krasnosel'skii, M.A.: Topological Methods in the Theory of Nonlinear Integral Equations. Macmillan, Basingstoke (1964)
11. Krasnosel'skii, M.A., Zabreiko, P.P.: Geometrical Methods of Nonlinear Analysis. Springer, Berlin (1984)
12. Rozikov, U.A.: Gibbs Measures on Cayley Trees. World Scientific, Singapore (2013)