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Fatou closedness under model uncertainty

Marco Maggis[1] · **Thilo Meyer-Brandis**[2] ·
Gregor Svindland[2]

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Abstract We provide a characterization in terms of Fatou closedness for weakly closed monotone convex sets in the space of $\mathcal{P}$-quasisure bounded random variables, where $\mathcal{P}$ is a (possibly non-dominated) class of probability measures. Applications of our results lie within robust versions the Fundamental Theorem of Asset Pricing or dual representation of convex risk measures.

Keywords Capacities · Fatou closedness/property · Sequential order closedness · Convex duality under model uncertainty · Fundamental Theorem of Asset Pricing

Mathematics Subject Classification 31A15 · 46A20 · 46E30 · 60A99 · 91B30

1 Introduction

A fundamental result attributed to Grothendieck ([22, p. 321, Exercise 1]) and based on the Krein–Smulian theorem characterizes weak*-closedness of a convex subset of $L^\infty_P := L^\infty(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is a probability space, by means of a property called Fatou closedness as follows:

✉ Gregor Svindland
svindla@math.lmu.de

Marco Maggis
marco.maggis@unimi.it

Thilo Meyer-Brandis
meyerbra@math.lmu.de

1 Department of Mathematics, University of Milan, Milan, Italy

2 Mathematics Institute, LMU Munich, Munich, Germany

Birkhäuser

Theorem 1.1 *Let $\mathcal{A} \subset L^\infty_P$ be convex. Equivalent are:*

(i) $\mathcal{A}$ is weak-closed (i.e. closed in $\sigma(L^\infty_P, L^1_P)$).*
(ii) $\mathcal{A}$ is Fatou closed, i.e. if $(X_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ is a bounded sequence which converges P-almost surely to X, then $X \in \mathcal{A}$.

Note that L^∞_P is a Banach lattice (see Sect. 2) and that from this point of view property (ii) in Theorem 1.1 equals sequential order closedness of $\mathcal{A}$ which in fact implies order closedness since L^∞_P has the countable sup property, i.e. every nonempty subset possessing a supremum contains a countable subset possessing the same supremum. Theorem 1.1 is very useful and often applied in the mathematical finance literature such as in the classic proof of the Fundamental Theorem of Asset Pricing, see e.g. [12] or [13], or in the dual representation of convex risk functions, see e.g. [19]. In all cases the problem is that the norm dual of L^∞_P contains undesired singular elements, whereas in the weak*-duality $(L^\infty_P, \sigma(L^\infty_P, L^1_P))$ the elements of the dual space are identified with σ-additive measures. However, as the weak*-topology is generally not first-countable, verifying that some set is weak*-closed is typically quite challenging. This is where Theorem 1.1 proves helpful.

The aim of this paper is to study the existence of a version of Theorem 1.1 for the case when the probability measure P is replaced by a class $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$. In general this class $\mathcal{P}$ does not allow for a dominating probability. Applications of such a result lie for instance in the field of mathematical finance, where currently there is much attention paid to deriving versions of the Fundamental Theorem of Asset Pricing as well as dual representations of convex risk functions in so-called *robust* frameworks as studied in [4,6–8,26,28]. These kind of frameworks have become increasingly popular to describe a decision maker who has to deal with the uncertainty which arises from model ambiguity. Here the class of probability models $\mathcal{P}$ the decision maker takes into account represents her degree of ambiguity about the right probabilistic model. If $\mathcal{P} = \{P\}$ there is no ambiguity. In many studies which account for model ambiguity $\mathcal{P}$ in fact turns out to be a non-dominated class of probability measures, see [6–8,26] and the reference therein.

We will show that there is a version of Theorem 1.1 in a robust probabilistic framework $(\Omega, \mathcal{F}, \mathcal{P})$, see Theorem 3.9. Let

$$c(A) := \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{F},$$

denote the capacity generated by $\mathcal{P}$. Under some conditions on the convex set $\mathcal{A}$ and on L^∞_c we obtain equivalence between

(WC) $\mathcal{A} \subset L^\infty_c$ is $\sigma(L^\infty_c, ca_c)$-closed,
(FC) $\mathcal{A} \subset L^\infty_c$ is Fatou closed: for any bounded sequence $\{X_n\} \subset \mathcal{A}$ and $X \in L^\infty_c$ such that $X_n \to X$ $\mathcal{P}$-quasi surely we have that $X \in \mathcal{A}$,

where L^∞_c and ca_c are the robust analogues of L^∞_P and L^1_P given by the capacity c, respectively, and $\mathcal{P}$ quasi sure convergence means Q-almost sure convergence under each $Q \in \mathcal{P}$. The conditions we have to require on $\mathcal{A}$ are monotonicity ($\mathcal{A} = \mathcal{A} + (L^\infty_c)_+$) and a property called $\mathcal{P}$-sensitivity. Monotonicity is typically satisfied in economic applications, and we show that $\mathcal{P}$-sensitivity is indeed a necessary condition to have (WC) $\Leftrightarrow$ (FC), see Proposition 3.8. If $\mathcal{P}$ is dominated, $\mathcal{P}$-sensitivity is always fulfilled.

Another requirement which is crucial for our proof of (WC) $\Leftrightarrow$ (FC) is that the dual space of ca_c may be identified with L_c^∞. This condition is in fact equivalent to the order completeness of the Banach lattice L_c^∞, i.e. the existence of a supremum for any bounded subset of L_c^∞, see Proposition 3.10, and it thus corresponds to aggregation type results as in [11,27]. If L_c^∞ is order complete, then the property (FC) equals sequential order closedness of $\mathcal{A}$. However, order completeness does not imply that L_c^∞ possesses the countable sup property, see Examples 3.11 and 3.12, so even under this condition (FC) does in general not imply order closedness of $\mathcal{A}$.

We also provide a counter example showing that for non-dominated $\mathcal{P}$ there is no proof of (WC) $\Leftrightarrow$ (FC) without further requirements such as $\mathcal{P}$-sensitivity, see Example 3.4. Moreover, we illustrate that many conditions, in particular on $\mathcal{P}$, one would think of in the first place to ensure (WC) $\Leftrightarrow$ (FC), indeed imply that $\mathcal{P}$ is dominated, so we are back to Theorem 1.1. Hence, a further contribution of this paper is to provide a deeper insight into the fallacies one might encounter when attempting to extend Theorem 1.1 to a robust case.

The paper is structured as follows: Sect. 2 provides a list of useful notations which will be used throughout the paper. Section 3 contains the main results of the paper, and in particular Theorem 3.9 is the robust version of Theorem 1.1. Finally, applications of Theorem 3.9 in the field of mathematical finance are collected in Sect. 4. Here we do not assume that the reader is familiar with mathematical finance. However, we try to keep the presentation concise, referring to the relevant literature for more background information.

2 Notation

For the sake of clarity we propose here a list of the basic notations and definitions that we shall use throughout this paper.

Let $(\Omega, \mathcal{F})$ be any measurable space.

(i) $ba := \{\mu : \mathcal{F} \to \mathbb{R} \mid \mu \text{ is finitely additive}\}$ and $ca := \{\mu : \mathcal{F} \to \mathbb{R} \mid \mu \text{ is } \sigma\text{-additive}\}$. These are both Banach lattices once endowed with the total variation norm TV and $|\mu| = \mu^+ + \mu^-$ where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition (see [1] for further details).

(ii) ba_+ (resp. ca_+) is the set of all positive additive (resp. σ-additive) set functions on $(\Omega, \mathcal{F})$.

(iii) In absence of any reference probability measure we have the following sets of random variables

$$\begin{aligned}
\mathcal{L} &:= \{f : \Omega \to \mathbb{R} \mid f \text{ is } \mathcal{F}\text{-measurable}\},\\
\mathcal{L}_+ &:= \{f \in \mathcal{L} \mid f(\omega) \geq 0, \forall\, \omega \in \Omega\},\\
\mathcal{L}^\infty &:= \{f \in \mathcal{L} \mid f \text{ is bounded}\}.
\end{aligned}$$

In particular $\mathcal{L}^\infty$ is a Banach space under the (pointwise) supremum norm $\|\cdot\|_\infty$ with dual space ba.

(iv) $\mathcal{M}_1 \subset ca_+$ is the set of all probability measures on $(\Omega, \mathcal{F})$.

(v) Throughout this paper we fix a set of probability measures $\mathcal{P} \subset \mathcal{M}_1$.

(vi) We introduce the sublinear expectation

$$c(f) := \sup_{Q \in \mathcal{P}} E_Q[f], \quad f \in \mathcal{L}_+$$

and by some abuse of notation we define the capacity $c(A) := c(1_A)$ for $A \in \mathcal{F}$.

(vii) Let $\widehat{\mathcal{P}}, \widetilde{\mathcal{P}} \subset \mathcal{M}_1$. $\widehat{\mathcal{P}}$ dominates $\widetilde{\mathcal{P}}$, denoted by $\widetilde{\mathcal{P}} \ll \widehat{\mathcal{P}}$, if for all $A \in \mathcal{F}$:

$$\sup_{P \in \widehat{\mathcal{P}}} P(A) = 0 \quad \Rightarrow \quad \sup_{P \in \widetilde{\mathcal{P}}} P(A) = 0.$$

We say that two classes $\widehat{\mathcal{P}}$ and $\widetilde{\mathcal{P}}$ are equivalent, denoted by $\widehat{\mathcal{P}} \approx \widetilde{\mathcal{P}}$, if $\widetilde{\mathcal{P}} \ll \widehat{\mathcal{P}}$ and $\widehat{\mathcal{P}} \ll \widetilde{\mathcal{P}}$.

(viii) A statement holds $\mathcal{P}$-quasi surely (q.s.) if the statement holds Q-almost surely (a.s.) for any $Q \in \mathcal{P}$.

(ix) The space of finitely additive (resp. countably additive) set functions dominated by c is given by $ba_c = \{\mu \in ba \mid \mu \ll c\}$ (resp. $ca_c = \{\mu \in ca \mid \mu \ll c\}$). Here $\mu \ll c$ means: $c(A) = 0$ for some $A \in \mathcal{F}$ implies $\mu(A) = 0$. When $\mathcal{P} = \{Q\}$ we shall write ba_Q or ca_Q for the sake of simplicity.

(x) We consider the quotient space $L_c := \mathcal{L}_{/\sim}$ where the equivalence is given by

$$f \sim g \quad \Leftrightarrow \quad \forall P \in \mathcal{P} : P(f = g) = 1.$$

We shall use capital letters to distinguish equivalence classes of random variables $X \in L_c$ from a representative $f \in X$, with $f \in \mathcal{L}$. In case $\mathcal{P} = \{Q\}$ we shall write L^1_Q instead of L_c. It is a well-known consequence of the Radon-Nikodym theorem ([1, Theorem 13.18]) that ca_Q may be identified with L^1_Q.

(xi) For any $f, g \in \mathcal{L}$ and $P \in \mathcal{M}_1$, we write $f \le g$ P-a.s. if and only if $P(f \le g) = 1$. Similarly $f \le g$ $\mathcal{P}$-q.s. if and only if $f \le g$ P-a.s. for all $P \in \mathcal{P}$. This relation is a partial order on $\mathcal{L}$ and it also induces a partial order on L_c where $X \le Y$ for $X, Y \in L_c$ if and only if $f \le g$ $\mathcal{P}$-q.s. for any $f \in X$ and $g \in Y$.

(xii) We define $L^\infty_c := \mathcal{L}^\infty_{/\sim}$ and endow this space with the norm

$$\|X\|_{c,\infty} := \inf\{m \mid \forall P \in \mathcal{P} : P(|X| \le m) = 1\}.$$

$(L^\infty_c, \|\cdot\|_{c,\infty})$ is a Banach lattice with the same partial order $\le$ as on L_c. Its norm dual is ba_c. In case $\mathcal{P} = \{Q\}$ we shall write L^∞_Q and $\|\cdot\|_{Q,\infty}$ for the sake of simplicity. Note that $\|\cdot\|_{c,\infty}$ is never order continuous for any choice of $\mathcal{P}$.

For simplicity of presentation, if there is no risk of confusion, we will follow the usual convention of identifying random variables in $\mathcal{L}$ with the equivalence classes they induce (in L_c, L^∞_c, L^1_Q or L^∞_Q) and vice versa.

3 Towards a robust version of Theorem 1.1

We start by recalling the proof of the non-trivial implication (ii) $\Rightarrow$ (i) of Theorem 1.1: the idea is to apply the Krein–Smulian theorem which implies that we only need to show that the sets

$$C_K := \mathcal{A} \cap \{X \in L^\infty_P \mid \|X\|_{P,\infty} \leq K\}$$

are weak*-closed for any constant $K > 0$. Now we could invoke the countable sup property of L^∞_P to find that (ii) implies (i), see e.g. [2, Definition 1.43 and following discussion]. But as, in the robust setting we envisage, L^∞_c typically does not possess this property (see for instance Examples 3.11 and 3.12), we present an alternative argument by means of the following inclusion:

$$i : \left(L^\infty_P, \sigma\left(L^\infty_P, L^1_P\right)\right) \to \left(L^1_P, \sigma\left(L^1_P, L^\infty_P\right)\right) \tag{3.1}$$

Note that i is continuous. Now, as $\mathcal{A}$ is Fatou closed, i.e. closed under bounded P-a.s. convergence, it follows that $i(C_K)$ is a closed subset of the Banach space $(L^1_P, E_P[|\cdot|])$, and thus $i(C_K)$ is also weakly (i.e. $\sigma(L^1_P, L^\infty_P)$) closed by convexity, so eventually C_K must be weak*-closed by continuity of i.

A natural approach to prove a robust version of Theorem 1.1 is to 'robustify' the spaces L^1_P and try to repeat the argument above. There are two natural candidates for this: Let $H_c := \{X \in L \mid c(|X|) < \infty\}$, with norm $\|X\|_c := c(|X|)$. Then it is readily verified that $(H_c, \|\cdot\|_c)$ is a Banach lattice. But in the robust case there is also another candidate, namely $M_c := \overline{L^\infty_c}^{\|\cdot\|_c}$ which is also a Banach lattice with the norm $\|\cdot\|_c$. These spaces have recently been studied in the literature, see e.g. [14,26], since they appear as natural environments to embed financial modelling under uncertainty. Clearly, $L^\infty_c \subset M_c \subset H_c \subset L_c$. Note that the trick with the inclusion (3.1) requires that the norm dual of L^1_P can be identified with L^∞_P, so in particular with a subset of L^1_P where in this latter case L^1_P is viewed as a representation of ca_P. Thus the reader may readily check that we could save the above argument if the norm duals M^*_c and H^*_c of M_c and H_c, respectively, would satisfy $M^*_c \subset ca$ or $H^*_c \subset ca$. The following Theorem 3.1 shows that this is the case only if $\mathcal{P}$ is dominated. To this end, denote by

$$\mathcal{Z} := \{(A_n)_{n\in\mathbb{N}} \subset \mathcal{F} \mid A_n \downarrow \emptyset \text{ and } c(A_n) \not\to 0\}, \tag{3.2}$$

where $A_n \downarrow \emptyset$ means that $A_n \supset A_{n+1}$, $A_n \neq \emptyset$, $n \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, the decreasing sequences of sets on which c is not continuous.

Theorem 3.1 *Consider the following conditions:*

(i) $\mathcal{Z} = \emptyset$.
(ii) $M^*_c \subset ca$.
(iii) $H^*_c \subset ca$.

Then (i) $\Longleftrightarrow$ *(ii)* $\Longleftarrow$ *(iii).*

In particular, if $\mathcal{Z} = \emptyset$, then there exists a countable subset $\widetilde{\mathcal{P}} \subset \mathcal{P}$ such that $\widetilde{\mathcal{P}} \approx \mathcal{P}$, and thus there is a probability measure $Q \in \mathcal{M}_1$ such that $\{Q\} \approx \mathcal{P}$.

Proof (i) $\Rightarrow$ (ii): By Proposition A.2 for any $l \in M_c^*$ there is $\mu \in ca$ such that $l(X) = \int X\, d\mu$ for all simple random variables X. Moreover, $\mu \in ca_c$, because $c(A) = 0$ implies $l(1_A) = 0$, $A \in \mathcal{F}$. Since for any $X \in L_c^\infty$ and any $n \in \mathbb{N}$ by the usual approximation method from integration theory there is a simple random variable X_n such that $|X - X_n| < 1/n$ $\mathcal{P}$-q.s., so $\|X - X_n\|_c < 1/n$, continuity of l and the dominated convergence theorem yield

$$l(X) = \lim_{n\to\infty} l(X_n) = \lim_{n\to\infty} \int X_n\, d\mu = \int X\, d\mu$$

for all $X \in L_c^\infty$. We recall that in [14] Proposition 18 the following relation was shown

$$M_c = \{X \in H_c \mid \lim_{n\to\infty} \|X 1_{\{|X|\geq n\}}\|_c = 0\}.$$

Hence, for $X \in (M_c)_+$ we have by monotone convergence that

$$l(X) = \lim_{n\to\infty} l(X 1_{\{|X|\leq n\}}) = \lim_{n\to\infty} \int X 1_{\{|X|\leq n\}}\, d\mu = \int X\, d\mu.$$

Finally, decomposing $X \in M_c$ into $X^+ - X^-$ with $X^+, X^- \in (M_c)_+$ and linearity of l and the integral shows (ii).

(ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) follow directly from Proposition A.2

The last statement of this theorem is Proposition A.1. □

Remark 3.2 Note that $\mathcal{Z} = \emptyset$ is equivalent to sequential order continuity of $\|\cdot\|_c$. According to Theorem 3.1, if $\mathcal{P}$ is not dominated, then $\mathcal{Z} \neq \emptyset$ and hence the norm $\|\cdot\|_c$ on M_c or H_c is not order continuous.

Also note that the converse of the last statement of Theorem 3.1 is not true, i.e. $\mathcal{Z} \neq \emptyset$ does not imply that $\mathcal{P}$ is not dominated. To see this, let $A_n \downarrow \emptyset$ and pick a sequence of probability measures P_n such that $P_n(A_n) = 1$ for all $n \in \mathbb{N}$, and let $\mathcal{P} = \{P_n \mid n \in \mathbb{N}\}$. Then, clearly $\|1_{A_n}\|_c = 1$ for each n. Hence, $\|\cdot\|_c$ is not order continuous and $\mathcal{Z} \neq \emptyset$ and thus $M_c^* \not\subset ca$. However, we have that $\{Q\} \approx \mathcal{P}$ for $Q = \sum_{n=1}^\infty \frac{1}{2^n} P_n$.

Recall the conditions

(WC) $\mathcal{A} \subset L_c^\infty$ is $\sigma(L_c^\infty, ca_c)$-closed.

(FC) $\mathcal{A} \subset L_c^\infty$ is Fatou closed: for any bounded sequence $X_n \subset \mathcal{A}$ and $X \in L_c^\infty$ such that $X_n \to X$ $\mathcal{P}$-q.s. we have that $X \in \mathcal{A}$.

It is easily verified that always (WC) $\Longrightarrow$ (FC) since any bounded $\mathcal{P}$-q.s. converging sequence also converges in $\sigma(L_c^\infty, ca_c)$ to the same limit. However, there is in general no proof of (FC) $\Longrightarrow$ (WC) even if $\mathcal{A}$ is convex, and also requiring monotonicity of $\mathcal{A}$, i.e. $\mathcal{A} + (L_c^\infty)_+ = \mathcal{A}$, in addition is not sufficient:

Theorem 3.3 *Let $\mathcal{A} \subset L_c^\infty$ be convex and monotone. Without further assumptions on $\mathcal{P}$ or $\mathcal{A}$, there exists no proof of (FC) $\Rightarrow$ (WC).*

The proof of Theorem 3.3 is given by the following Example 3.4 where we give a counter-example of (FC) $\Longrightarrow$ (WC) assuming the continuum hypothesis. So under the continuum hypothesis (FC) $\Longrightarrow$ (WC) is indeed wrong. Note that as the continuum hypothesis does not conflict with what one perceives as standard mathematical axioms, there is of course no way to prove (FC) $\Longrightarrow$ (WC) even if we do not believe in the continuum hypothesis.

Example 3.4 Consider the measure space $(\Omega, \mathcal{F}) = ([0,1], \mathfrak{P}([0,1]))$, where $\mathfrak{P}([0,1])$ denotes the power set of $[0,1]$. Assume the continuum hypothesis. Banach and Kuratowski have shown that for any set I with the same cardinality as $\mathbb{R}$ there is no measure μ on $(I, \mathfrak{P}(I))$ such that $\mu(I) = 1$ and $\mu(\{\omega\}) = 0$ for all $\omega \in I$; see for instance [16, Theorem C.1]. It follows that any probability measure μ over $(\Omega, \mathcal{F})$ must be a countable sum of weighted Dirac-measures, i.e. $\mu = \sum_{i=1}^\infty a_i \delta_{\omega_i}$ where $a_i \geq 0$, $\sum_{i=1}^n a_i = 1$, $\omega_i \in \Omega$, $i \in \mathbb{N}$. (Recall that for $\omega \in \Omega$ and $A \in \mathcal{F}$: $\delta_\omega(A) = 1$ if and only if $\omega \in A$ and $\delta_\omega(A) = 0$ otherwise.) Indeed, let $\mu \in \mathcal{M}_1$, and let

$$S := \{\omega \in \Omega \mid \mu(\{\omega\}) > 0\}.$$

Then S can at most be countable (consider the sets $S_n := \{\omega \in \Omega \mid \mu(\{\omega\}) > 1/n\}$, $n \in \mathbb{N}$, and note that $S = \bigcup_{n \in \mathbb{N}} S_n$). Now suppose that $\mu([0,1] \setminus S) > 0$, then as $[0,1] \setminus S$ has the same cardinality as $[0,1]$, this implies the existence of an atom for the measure μ restricted to $[0,1] \setminus S$, i.e. there exists $\hat{\omega} \in [0,1] \setminus S$ such that

$$\frac{1}{\mu([0,1] \setminus S)} \mu(\{\hat{\omega}\}) > 0.$$

This clearly contradicts the definition of S.

Let $\mathcal{P} := \{\delta_\omega \mid \omega \in [0,1]\}$ be the set of all Dirac measures. Then

$$c(|X|) = \sup_{\omega \in [0,1]} |X(\omega)|,$$

so it turns out that $L_c^\infty = M_c = H_c = \mathcal{L}^\infty$. Hence, $(L_c^\infty)^* = M_c^* = H_c^* = ba$, and, as $c(A) = 0$ is equivalent to $A = \emptyset$, we also have that $ca_c = ca$. Consider the set

$$C := \{1_A \mid \emptyset \neq A \subset [0,1] \text{ is countable}\},$$

and let $\mathcal{A}$ be the convex closure of C under bounded $\mathcal{P}$-q.s. convergence of sequences. Then $1 \notin \mathcal{A}$: Indeed, any $X = \sum_{i=1}^n a_i 1_{A_i}$, $a_i \geq 0$, $\sum_{i=1}^n a_i = 1$, $1_{A_i} \in C$, in the convex hull of C satisfies $0 \leq X \leq 1_{A_X}$ where $A_X := \bigcup_{i=1}^n A_i$ is countable. Let X_k be any sequence in the convex hull of C, then $0 \leq X_k \leq 1_B$, $k \in \mathbb{N}$, where $B := \bigcup_{k \in \mathbb{N}} A_{X_k}$ is countable. Hence, $X_k(\omega) = 0$ for all $\omega \in [0,1] \setminus B$, so $1 \notin \mathcal{A}$. Now consider the family $\mathcal{G}$ of all countable subsets of $[0,1]$ directed by $A \leq B$ if and only if $A \subset B$. Consider the net $\{1_A \mid A \in \mathcal{G}\} \subset C$. Then for any probability measure μ there is $A \in \mathcal{G}$ (namely $A = S$) such that for all $B \in \mathcal{G}$ with $B \geq A$ we have $\int 1_B \, d\mu = 1 = \int 1 \, d\mu$. Thus 1 lies in the $\sigma(L_c^\infty, ca_c)$-closure of $\mathcal{A}$.

In order to make the presentation simpler, we did not require monotonicity of $\mathcal{A}$ so far, but the same arguments as above show that if $\mathcal{A}$ is the convex closure of $-C + (L_c^\infty)_+$ under bounded $\mathcal{P}$-q.s. convergence of sequences, which is convex and monotone, then $-1 \notin \mathcal{A}$ but -1 is an element of the $\sigma(L_c^\infty, ca_c)$-closure of $\mathcal{A}$.

A consequence of Theorem 3.3 is that we need to ask for additional properties on $\mathcal{A}$ in order to have (FC) $\Longleftrightarrow$ (WC).

3.1 $\mathcal{P}$-sensitivity, $ca_c^* = L_c^\infty$, and (FC) $\Longleftrightarrow$ (WC)

A simple property on $\mathcal{A}$ which allows to prove (FC) $\Longleftrightarrow$ (WC) is to require that the convex set $\mathcal{A} \subset L_c^\infty$ behaves as in the dominated case, i.e. there is a reference probability $P \in \mathcal{P}$ such that $\mathcal{A}$ is closed under bounded P-a.s. convergence. Under this assumption the whole issue can be reduced to Theorem 1.1. Clearly, this assumption is too strong. However, it gives the idea of the $\mathcal{P}$-sensitivity property we will introduce in the following.

Given a probability $Q \in \mathcal{M}_1$ such that $\{Q\} \ll \mathcal{P}$ we define the linear map $j_Q : L_c^\infty \to L_Q^\infty$ by $Q(j_Q(X) = X) = 1$, i.e. $j_Q(X)$ is the equivalence class in L_Q^∞ such that any representative of $j_Q(X)$ and any representative of X are Q-a.s. identical. As ca_Q (which can be identified with L_Q^1) is a subset of ca_c, we deduce that $j_Q : (L_c^\infty, \sigma(L_c^\infty, ca_c)) \to (L_Q^\infty, \sigma(L_Q^\infty, L_Q^1))$ is continuous.

Definition 3.5 A set $\mathcal{A} \subset L_c^\infty$ is called $\mathcal{P}$-*sensitive* if there exists a set $\mathcal{Q} \subset \mathcal{M}_1$ with $\mathcal{Q} \ll \mathcal{P}$ such that

$$j_Q(X) \in j_Q(\mathcal{A}) \text{ for all } Q \in \mathcal{Q} \text{ implies } X \in \mathcal{A}$$

or equivalently

$$\mathcal{A} = \bigcap_{Q \in \mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{A}).$$

The set $\mathcal{Q}$ will be called *reduction set* for $(\mathcal{A}, \mathcal{P})$.

Remark 3.6 Suppose that $\mathcal{P}$ is dominated. Then the Halmos Savage lemma (see [23], Lemma 7) guarantees the existence of a countable subclass $\{P_i\}_{i=1}^\infty$ such that $\{P_i\}_{i=1}^\infty \approx \mathcal{P}$. Let $P = \sum \frac{1}{2^i} P_i$. Then $\mathcal{P} \approx \{P\}$, so the space L_c^∞ can be identified with L_P^∞. Hence, in that case any set $\mathcal{A} \subset L_c^\infty$ is automatically $\mathcal{P}$-sensitive with reduction set $\mathcal{Q} = \{P\}$.

Example 3.7 The set $\mathcal{A}$ of Example 3.4 is not $\mathcal{P}$-sensitive. Since $c(A) = 0$ implies that $A = \emptyset$, any set of probabilities $\mathcal{Q} \subset \mathcal{P}$ satisfies $\mathcal{Q} \ll \mathcal{P}$. Let $Q \in \mathcal{M}_1$ be arbitrary and $S := \{\omega \in [0, 1] \mid Q(\{\omega\}) > 0\}$ such that $Q = \sum_{\omega \in S} a_\omega \delta_\omega$ with $a_\omega > 0$ and $\sum_{\omega \in S} a_\omega = 1$. Then $1_S \in \mathcal{A}$ by definition of $\mathcal{A}$ and thus $1 \in j_Q(\mathcal{A})$, or to be more precise, 1 and 1_S form the same equivalence class in L^∞_Q. Since $Q \in \mathcal{M}_1$ was arbitrary, we have $1 \in \bigcap_{Q \in \mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{A})$. As we know that $1 \notin \mathcal{A}$, the set $\mathcal{A}$ is not $\mathcal{P}$-sensitive.

Indeed $\mathcal{P}$-sensitivity is a necessary condition for (FC) $\Longleftrightarrow$ (WC).

Proposition 3.8 *Any convex set $\mathcal{A} \subset L^\infty_c$ which is $\sigma(L^\infty_c, ca_c)$-closed (i.e. satisfies (WC)) is $\mathcal{P}$-sensitive.*

Proof If $\mathcal{A} = \emptyset$ or $\mathcal{A} = L^\infty_c$, the assertion is trivial. Now assume that $\mathcal{A} \neq \emptyset$ and $\mathcal{A} \neq L^\infty_c$. As $\mathcal{A}$ is $\sigma(L^\infty_c, ca_c)$-closed and convex, the function

$$\rho(X) := \delta(X \mid \mathcal{A}) := \begin{cases} 0 & \text{if } X \in \mathcal{A} \\ \infty & \text{else} \end{cases}, \quad X \in L^\infty_c,$$

is convex and $\sigma(L^\infty_c, ca_c)$ lower-semicontinuous. Hence, by the Fenchel–Moreau theorem (see [18, Proposition 4.1]) there exists a dual representation of ρ, i.e.

$$\rho(X) = \sup_{\mu \in \mathcal{Q}} \left\{ \int X \, d\mu - \rho^*(\mu) \right\}$$

where $\mathcal{Q} := \{\mu \in ca_c \mid \rho^*(\mu) < \infty\}$ is a convex set and

$$\rho^*(\mu) := \sup_{X \in \mathcal{A}} \int X \, d\mu, \quad \mu \in ca_c.$$

$\mathcal{A} \neq L^\infty_c$ implies $\mathcal{Q} \supsetneqq \{0\}$ and therefore,

$$\mathcal{A} = \bigcap_{\mu \in \mathcal{Q}} \left\{ X \in L^\infty_c \mid \int X \, d\mu \le \rho^*(\mu) \right\} = \bigcap_{\mu \in \mathcal{Q} \setminus \{0\}} \left\{ X \in L^\infty_c \mid \int X \, d\mu \le \rho^*(\mu) \right\}.$$

Let $\tilde{\mathcal{Q}} := \{\frac{|\mu|}{|\mu|(\Omega)} \mid \mu \in \mathcal{Q} \setminus \{0\}\} \subset \mathcal{M}_1$ and note that $\tilde{\mathcal{Q}} \ll \mathcal{P}$ since $\mathcal{Q} \subset ca_c$. Consider

$$X \in \bigcap_{Q \in \tilde{\mathcal{Q}}} j_Q^{-1} \circ j_Q(\mathcal{A}).$$

Fix $Q \in \tilde{\mathcal{Q}}$ and $\nu \in \mathcal{Q}$ such that $Q = \frac{|\nu|}{|\nu|(\Omega)}$. Then, $j_Q(X) \in j_Q(\mathcal{A})$, i.e. there is $Y \in \mathcal{A}$ such that $j_Q(X) = j_Q(Y)$. Noting that $X = j_Q(X)$ and $Y = j_Q(Y)$ under ν, it follows that

$$\int X \, d\nu = \int j_Q(X) \, d\nu = \int j_Q(Y) \, d\nu = \int Y \, d\nu \le \rho^*(\nu),$$

where the inequality follows from $Y \in \mathcal{A}$. Since $Q \in \tilde{\mathcal{Q}}$ was arbitrary, we conclude that indeed $\int X \, d\mu \le \rho^*(\mu)$ for all $\mu \in \mathcal{Q}$, and hence that $X \in \mathcal{A}$. This shows that

$\bigcap_{Q\in\mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{A}) \subset \mathcal{A}$. The other inclusion $\bigcap_{Q\in\mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{A}) \supset \mathcal{A}$ is trivially satisfied, so we have that $\mathcal{A}$ is $\mathcal{P}$-sensitive with reduction set $\tilde{\mathcal{Q}}$. □

The following Theorem 3.9 gives conditions under which (FC) $\Longleftrightarrow$ (WC) for a convex set $\mathcal{A} \subset L_c^\infty$. Besides $\mathcal{P}$-sensitivity we have to require that the norm dual ca_c^* of (ca_c, TV), where TV denotes the total variation norm on ca_c, may be identified with L_c^∞. Clearly any $X \in L_c^\infty$ may be identified with a continuous linear functional on ca_c by

$$ca_c \ni \mu \mapsto \int X \, d\mu, \tag{3.3}$$

so we always have $L_c^\infty \subset ca_c^*$. However, $ca_c^* = L_c^\infty$ is obviously a very strong condition which we will characterize in Proposition 3.10 in terms of order closedness of L_c^∞.

Theorem 3.9 *Suppose that $ca_c^* = L_c^\infty$ and let $\mathcal{A} \subset L_c^\infty$ be convex and monotone ($\mathcal{A} + (L_c^\infty)_+ = \mathcal{A}$). Equivalent are*

(i) *$\mathcal{A}$ satisfies (WC).*
(ii) *$\mathcal{A}$ is $\mathcal{P}$-sensitive and satisfies (FC).*

Proof We already know that (WC) implies (FC) and $\mathcal{P}$-sensitivity. Now assume that $\mathcal{A}$ is $\mathcal{P}$ sensitive and satisfies (FC). Since $ca_c^* = L_c^\infty$, by the Krein–Smulian theorem it is sufficient to show that $C_K := \mathcal{A} \cap \{Z \in L^\infty \mid \|Z\|_{c,\infty} \le K\}$ is $\sigma(L_c^\infty, ca_c)$-closed for every $K > 0$. Let $\mathcal{Q}$ be a reduction set for $(\mathcal{A}, \mathcal{P})$ and fix any $K > 0$ and $Q \in \mathcal{Q}$.

Consider the continuous inclusion

$$i : (L_Q^\infty, \sigma(L_Q^\infty, L_Q^1)) \to (L_Q^1, \sigma(L_Q^1, L_Q^\infty)).$$

In a first step we show that $C_{Q,K} := i \circ j_Q(C_K)$ is $\|\cdot\|_Q := E_Q[|\cdot|]$-closed in L_Q^1, because being convex it then follows that $C_{Q,K}$ is $\sigma(L_Q^1, L_Q^\infty)$-closed and therefore $j_Q(C_K)$ is $\sigma(L_Q^\infty, L_Q^1)$-closed by continuity of i. To this end let $(Y_n)_{n\in\mathbb{N}} \subset C_{Q,K}$ and $Y \in L_Q^1$ such that $\|Y_n - Y\|_Q \to 0$, and without loss of generality we may also assume that $Y_n \to Y$ Q-a.s. Note that Y is necessarily bounded by K. Choose $X_n \in C_K$ such that $Y_n = j_Q(X_n)$ for all $n \in \mathbb{N}$ and $X \in L_c^\infty$ such that $Y = j_Q(X)$. Consider now the set

$$F := \{\omega \in \Omega \mid X_n(\omega) \to X(\omega)\}$$

(by the usual abuse of notation, in the definition of F we still write X_n and X for arbitrary representatives of the equivalence classes X_n and X). By monotonicity of $\mathcal{A}$ we have that $\widetilde{X}_n := X_n 1_F + K 1_{F^c} \in C_K$ for all $n \in \mathbb{N}$, and $\widetilde{X}_n \to X 1_F + K 1_{F^c} =: \widetilde{X}$ $\mathcal{P}$-q.s. Consequently $\widetilde{X} \in C_K$ and since $Q(F) = 1$ we have $Y = j_Q(X) = j_Q(\widetilde{X}) \in C_{Q,K}$. Hence, $j_Q(C_K)$ is $\sigma(L_Q^\infty, L_Q^1)$ closed.

By continuity of j_Q, the preimage $j_Q^{-1} \circ j_Q(C_K)$ is $\sigma(L_c^\infty, ca_c)$-closed, and as also $\{X \mid \|X\|_{c,\infty} \le K\}$ is $\sigma(L_c^\infty, ca_c)$-closed, we conclude that

$$A_{Q,K} := j_Q^{-1} \circ j_Q(C_K) \cap \{X \mid \|X\|_{c,\infty} \le K\} \supset C_K$$

and finally also $\bigcap_{Q\in\mathcal{Q}} A_{Q,K}$ are $\sigma(L_c^\infty, ca_c)$-closed. Clearly, $\bigcap_{Q\in\mathcal{Q}} A_{K,Q} \supset C_K$. If we can show $\bigcap_{Q\in\mathcal{Q}} A_{Q,K} \subset C_K$, then we are done, because then $\bigcap_{Q\in\mathcal{Q}} A_{Q,K} = C_K$, and thus C_K is $\sigma(L_c^\infty, ca_c)$-closed. To this end, let $X \in \bigcap_{Q\in\mathcal{Q}} A_{Q,K}$. Then $j_Q(X) \in j_Q(\mathcal{A})$ for any $Q \in \mathcal{Q}$ and therefore $X \in \mathcal{A}$ by $\mathcal{P}$-sensitivity. Moreover by definition of $A_{K,Q}$ we also have $\|X\|_{c,\infty} \le K$. □

Note that Theorem 3.9 proves the so-called C-property introduced and discussed in [5] for convex and monotone sets.

Let $\mathcal{D} \subset L_c^\infty$. Recall that a supremum of $\mathcal{D}$ is a least upper bound of $\mathcal{D}$, that is an $X \in L_c^\infty$ such that $Y \le X$ for all $Y \in \mathcal{D}$, and any $Z \in L_c^\infty$ such $Y \le Z$ for all $Y \in \mathcal{D}$ satisfies $X \le Z$. The supremum of $\mathcal{D}$ is denoted by $\operatorname{ess\,sup}_{Y\in\mathcal{D}} Y$. This notation is commonly used in probability theory and it is inspired by the tradition of identifying random variables with the equivalence classes they induce. Indeed for a set of random variables in $\mathcal{L}^\infty$, a supremum in the $\mathcal{P}$-q.s. order is only essentially unique—thus called essential supremum (ess sup)—in the sense that the equivalence class generated by it in L_c^∞ is unique.

Proposition 3.10 *$ca_c^* = L_c^\infty$ if and only if L_c^∞ is order complete, i.e. there exists a supremum for any norm bounded set $\mathcal{D} \subset L_c^\infty$.*

Proof Suppose that L_c^∞ is order complete. Then L_c^∞ is in particular also monotonically complete in the sense of [25, Definition 2.4.18]. Thus [25, Theorem 2.4.22] applies which yields $ca_c^* = L_c^\infty$.

In order to prove that $ca_c^* = L_c^\infty$ implies the existence of a supremum for any norm bounded set $\mathcal{D} \subset L_c^\infty$, we recall that ca and thus also ca_c is an AL-space ([1, Theorem 10.56]), so ca_c^* is an AM-space ([1, Theorem 9.27]). In particular ca_c^* is order complete. Here, the order $\ge_*$ on ca_c^* is given by $l \ge_* 0$ if and only if $l(\mu) \ge 0$ for all $\mu \in (ca_c)_+$, and a set $\mathcal{S} \subset ca_c^*$ is order bounded from above if there is $h \in ca_c^*$ such that $h - l \ge_* 0$ for all $l \in \mathcal{S}$. Any norm bounded $\mathcal{D} \subset L_c^\infty$ is order bounded from above in ca_c^*, because $K\mu(\Omega) - \int X\,d\mu \ge 0$, $\mu \in (ca_c)_+$, for a constant $K > 0$ which is an upper bound of the norm on $\mathcal{D}$, so $(\mu \mapsto K\mu(\Omega)) \in ca_c^*$ is an upper bound with respect to $\ge_*$. Thus there is a least upper bound of $\mathcal{D}$ viewed as a subset of ca_c^*. Now suppose that ca_c^* can be identified with L_c^∞. Then this least upper bound of $\mathcal{D}$ may be identified with an element in $X \in L_c^\infty$, that is

$$\int X\,d\mu \ge \int Y\,d\mu \quad \text{for all } \mu \in (ca_c)_+ \text{ and all } Y \in \mathcal{D}.$$

Considering measures μ of type $1_A dP$ for $P \in \mathcal{P}$ and $A \in \mathcal{F}$ shows that $X \ge Y$ for all $Y \in \mathcal{D}$, and $\mu \mapsto \int X\,d\mu$ being the least amongst the upper bounds of $\mathcal{D}$ in the $\ge_*$-order implies that X is a supremum of $\mathcal{D}$. □

Example 3.11 In this example we fix a measure space $(\Omega, \mathcal{F})$ and an uncountable family $\mathcal{P} = \{P_\sigma\}_{\sigma\in\Sigma}$ of probability measures. Consider the enlarged sigma algebra $\mathcal{F}^\Sigma = \bigcap_{\sigma\in\Sigma} \mathcal{F}^\sigma$ where $\mathcal{F}^\sigma$ is the P^σ completion of $\mathcal{F}$, and notice that any P^σ uniquely extends to $\mathcal{F}^\Sigma$. Assume that there exists a family of sets $\{\Omega^\sigma\}_{\sigma\in\Sigma} \subset \mathcal{F}^\Sigma$ such that for any $\sigma \in \Sigma$, $P^\sigma(\Omega^\sigma) = 1$ and $P^{\tilde\sigma}(\Omega^\sigma) = 0$ for $\tilde\sigma \ne \sigma$. In this case it is easily seen that any norm bounded set $\mathcal{D} \subset L^\infty_c(\Omega, \mathcal{F}^\Sigma)$ admits a supremum given by

$$\text{ess sup}_{Y\in\mathcal{D}} Y = \sum_{\sigma\in\Sigma} j^{-1}_{P^\sigma}(\text{ess sup}_{Y\in\mathcal{D}} j_{P^\sigma}(Y))1_{\Omega^\sigma}.$$

Note that $\text{ess sup}_{Y\in\mathcal{D}} j_{P^\sigma}(Y)$ in $L^\infty_{P^\sigma}$ is well-defined for every $\sigma \in \Sigma$. Also notice that $\text{ess sup}_{Y\in\mathcal{D}} Y$ is $\mathcal{F}^\sigma$-measurable for any $\sigma \in \Sigma$ and therefore is also $\mathcal{F}^\Sigma$-measurable. Therefore $L^\infty_c(\Omega, \mathcal{F}^\Sigma) = ca^*_c(\Omega, \mathcal{F}^\Sigma)$. Notice that, as Σ is not countable, $L^\infty_c(\Omega, \mathcal{F}^\Sigma)$ does not possess the countable sup property, think for instance of the essential supremum of the set $\{1_{\Omega^\sigma} \mid \sigma \in \Sigma\}$. We refer to [11] for a deeper study of this example and applications to mathematical finance.

Example 3.12 Recall Example 3.4. Clearly any norm bounded set $\mathcal{D} \subset L^\infty_c = \mathcal{L}^\infty$ admits an essential supremum which is simply given by $\omega \mapsto \sup_{Y\in\mathcal{D}} Y(\omega)$. Hence $ca^* = ca^*_c = \mathcal{L}^\infty$ by Proposition 3.10. This holds without the continuum hypothesis, but is also easily directly verified using the continuum hypothesis: Let $l \in ca^*_c$ and define $X(\omega) = l(\delta_\omega)$, $\omega \in [0, 1]$. Then by linearity, for all $\mu \in ca$ it follows that $l(\mu) = \sum_{\omega\in S} a_\omega l(\delta_\omega) = \int X\, d\mu$ where $S := \{\omega \in [0, 1] \mid \mu(\{\omega\}) > 0\}$ and $a_\omega = \mu(\{\omega\})$, $\omega \in S$. Moreover, it is also readily verified that in this case L^∞_c does not have the countable sup property.

4 Applications of Theorem 3.9

4.1 Dual representation of (quasi-) convex increasing functionals

In this section we provide a dual representation of (quasi-) convex increasing functionals. Such results are key in the study of robustness of financial risk measures. An exhaustive introduction to the dual representation of convex risk measures can be found in [19] (see also [15] for the quasiconvex case and [10] for recent developments). To the best of our knowledge, in presence of model uncertainty, the only result available in the literature is [6, Theorem 3.1] which is obtained for the closure of the space of continuous functions under the norm $\|\cdot\|_c$.

Definition 4.1 A function $f : L^\infty_c \to (-\infty, \infty]$ is

- quasiconvex (resp. convex) if for every $\lambda \in [0, 1]$ and $X, Y \in L^\infty$ we have $f(\lambda X+(1-\lambda)Y) \leq \max\{X, Y\}$ (resp. $f(\lambda X+(1-\lambda)Y) \leq \lambda f(X)+(1-\lambda)f(Y)$).
- τ-lower semicontinuous (l.s.c.) for some topology τ on L^∞_c if for every $a \in \mathbb{R}$ the lower level set $\{X \in L^\infty_c \mid f(X) \leq a\}$ is τ-closed.
- $\mathcal{P}$-sensitive if the lower level sets $\{X \in L^\infty_c \mid f(X) \leq a\}$ are $\mathcal{P}$-sensitive for every $a \in \mathbb{R}$.

The following Lemma provides a huge class of $\mathcal{P}$-sensitive functions.

Lemma 4.2 *Consider a function* $f : L^\infty_c \to [-\infty, \infty]$ *such that*

$$f(X) = \sup_{P\in\mathcal{Q}} f_P(j_P(X)), \tag{4.1}$$

for some $\mathcal{Q} \subset \mathcal{M}_1$ and $f_P : L_P^\infty \to [-\infty, \infty]$. If $\mathcal{Q} \ll \mathcal{P}$ then f is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$.

Proof From representation (4.1) we automatically have

$$\{X \in L_c^\infty \mid f(X) \le a\} = \bigcap_{P \in \mathcal{Q}} \{X \in L_c^\infty \mid f_P(j_P(X)) \le a\}.$$

As $\{X \in L_c^\infty \mid f_P(j_P(X)) \le a\} = j_P^{-1} \circ j_P\{X \in L_c^\infty \mid f_P(j_P(X)) \le a\}$, we conclude that f is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$. □

Theorem 4.3 *Assume that $ca_c^* = L_c^\infty$. Let $f : L_c^\infty \to (-\infty, \infty]$ be a quasiconvex (resp. convex), monotone non-decreasing ($X \le Y$ $\mathcal{P}$-q.s. implies $f(X) \le f(Y)$) and $\mathcal{P}$-sensitive function. The following are equivalent:*

- *(i) f is $\sigma(L_c^\infty, ca_c)$-lower semi continuous.*
- *(ii) f has the Fatou property: for any bounded sequence $(X_n)_{n\in\mathbb{N}} \subset L_c^\infty$ converging $\mathcal{P}$-q.s. to $X \in L_c^\infty$ we have $f(X) \le \liminf_{n\to\infty} f(X_n)$.*
- *(iii) For any sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ and $X \in L_c^\infty$ such that $X_n \uparrow X$ $\mathcal{P}$-q.s. we have that $f(X_n) \uparrow f(X)$.*
- *(iv) f admits a bidual representation which in the quasiconvex case is*

$$f(X) = \sup_{P \in ca_c \cap \mathcal{M}_1} R\left(E_P[X], P\right), \quad X \in L_c^\infty,$$

with dual function $R : \mathbb{R} \times ca_c \to (-\infty, \infty]$ given by

$$R(t, \mu) := \sup_{t' < t} \inf_{Y \in L_c^\infty} \left\{ f(Y) \mid \int Y \, d\mu = t' \right\};$$

and in the convex case the dual representation is

$$f(X) = \sup_{\mu \in (ca_c)_+} \left\{ \int X \, d\mu - f^*(\mu) \right\}, \quad X \in L_c^\infty,$$

where the dual function $f^* : ca_c \to (-\infty, \infty])$ is given by

$$f^*(\mu) := \sup_{Y \in L_c^\infty} \left\{ \int Y \, d\mu - f(Y) \right\}.$$

In addition, if $f(X + c) = f(X) + c$ for every $X \in L_c^\infty$ and $c \in \mathbb{R}$ then f is necessarily convex and

$$f(X) = \sup_{P \in ca_c \cap \mathcal{M}_1} \left\{ E_P[X] - f^*(P) \right\}, \quad X \in L_c^\infty.$$

Proof According to Theorem 3.9 (i) holds if and only if (ii) is satisfied.
(ii) ⇒ (iii) is due to

$$f(X) \leq \liminf_{n\to\infty} f(X_n) \leq f(X)$$

where the last ineqaulity follows from monotonicity. Conversely (iii) ⇒ (ii) follows by considering $Y_n := \text{ess}\inf_{k\geq n} X_k$ and noting that $Y_n \uparrow X$ $\mathcal{P}$-q.s. and $f(Y_n) \leq f(X_n)$; see also [19, Lemma 4.16].

In the convex case $(i) \Leftrightarrow (iv)$ is Fenchel's Theorem (see [18, Proposition 4.1]) together with monotonicity (see [21, Corollary 7]).

In the quasiconvex case showing $(i) \Rightarrow (iv)$ is a consequence of the Penot-Volle duality Theorem (see Appendix B) and together with monotonicity (see [9, Lemma 8]), and $(iv) \Rightarrow (iii)$ follows from the monotone convergence theorem and the definition of R. □

4.2 Fundamental Theorem of Asset Pricing

Pricing theory in mathematical finance is based on the Fundamental Theorem of Asset Pricing, which roughly asserts that in a market without arbitrage opportunities (the so-called no-arbitrage condition) discounted prices are expectations under some risk-neutral probability measure. This characterisation is essential to develop a pricing theory for financial instruments which are not traded in the market. In the classical dominated framework on some probability space $(\Omega, \mathcal{F}, P)$ the risk-neutral probability measures are martingale measures for the discounted price process which are equivalent to the reference probability P, see [13] for a detailed review and related literature. Also note that the no-arbitrage condition is necessary and sufficient the existence of an economic equilibrium, see e.g. [24].

It is well understood that the Fundamental Theorem of Asset Pricing in a classical dominated framework is highly related to duality arguments. There are also robust approaches applying duality, see e.g. [4] based on an extended order dual space, the so-called super order dual introduced in [3]. However, most recent studies of robust Fundamental Theorems of Asset Pricing do not use duality arguments given the difficulties we outlined in this paper, see e.g. [7]. However, under the conditions that we have derived in Sect. 3 we will see that it is possible to reconcile the Fundamental Theorem of Asset Pricing, the Superhedging Duality, and duality theory on the pair (L_c^∞, ca_c) using the well-known arguments.

Throughout this section we assume that $ca_c^* = L_c^\infty$ holds true. We consider a discrete time market model with terminal time horizont $T \in \mathbb{N}$, and trading times $I := \{0, \dots, T\}$. The price process is given by a $\mathcal{P}$-q.s. bounded $\mathbb{R}^d$-valued stochastic process $S = (S_t)_{t\in I} = (S_t^j)_{t\in I}^{j=1,\dots,d}$ on $(\Omega, \mathcal{F})$, and we also assume the existence of a numeraire asset $S_t^0 = 1$ for all $t \in I$. Moreover, we fix a filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in I}$ such that the process S is $\mathbb{F}$-adapted. Denote by $\mathcal{H}$ the class of $\mathbb{R}^d$-valued, $\mathbb{F}$-predictable stochastic processes, which is the class of all admissible trading strategies. Let

$$\mathcal{C} := \left\{X \in L_c^\infty \mid X \leq (H \bullet S)_T \ \mathcal{P}\text{-q.s. for some } H \in \mathcal{H}\right\}$$

where

$$(H \bullet S)_t := \sum_{k=1}^{t} \sum_{j=1}^{d} H_k^j (S_k^j - S_{k-1}^j)$$

is the payoff of the self-financing trading strategy at time $t \in I \setminus \{0\}$ with initial investment $(H \bullet S)_0 = 0$ given by the predictable process $H = (H_t)_{t \in I \setminus \{0\}}$. In this framework the no-arbitrage condition (NA($\mathcal{P}$)) was introduced by [7] as given by the following definition.

Definition 4.4 The described market model is called arbitrage-free, if it satisfies the no-arbitrage condition

NA($\mathcal{P}$) $(H \bullet S)_T \geq 0$ $\mathcal{P}$-q.s. implies $(H \bullet S)_T = 0$ $\mathcal{P}$-q.s..

Note that NA($\mathcal{P}$) is equivalent to $\mathcal{C} \cap (L_c^\infty)_+ = \{0\}$.

Lemma 4.5 *Under $NA(\mathcal{P})$ if $\mathcal{C}$ is $\mathcal{P}$-sensitive then $\mathcal{C}$ is $\sigma(L_c^\infty, ca_c)$-closed.*

Proof [7, Theorem 2.2] shows that under $NA(\mathcal{P})$ the cone $\mathcal{C}$ is closed under $\mathcal{P}$-q.s. convergence of sequences and therefore $\mathcal{C}$ satisfies (FC). We remark that [7, Theorem 2.2] holds in full generality without the product structure on the underlying probability space assumed in [7]. Therefore applying Theorem 3.9 we deduce that $\mathcal{C}$ is $\sigma(L_c^\infty, ca_c)$-closed. □

Suppose that $\mathcal{C}$ is $\mathcal{P}$-sensitive. As $\mathcal{C}$ is a $\sigma(L_c^\infty, ca_c)$-closed convex cone, the bipolar Theorem yields

$$\begin{aligned} \mathcal{C} = \mathcal{C}^{00} &= \left\{ Y \in L_c^\infty \mid \forall Q \in \mathcal{C}_1^0 : \ E_Q[Y] \leq 0 \right\} \\ \text{where } \mathcal{C}_1^0 &:= \mathcal{C}^0 \cap \mathcal{M}_1 = \left\{ \mu \in \mathcal{C}^0 \mid \mu(1_\Omega) = 1 \right\} \\ \text{and } \mathcal{C}^0 &:= \left\{ \mu \in ca_c \mid \forall X \in \mathcal{C} : \int X \, d\mu \leq 0 \right\}. \end{aligned} \tag{4.2}$$

Notice that since $\mathcal{C} \supset -(L_c^\infty)_+$ then $\mu \in (ca_c)_+$ for every $\mu \in \mathcal{C}^0$ which explains $\mathcal{C}_1^0$.

Lemma 4.6 *$\mathcal{C}_1^0$ is the set of all martingale measures dominated by the capacity c, that is*

$$\mathcal{C}_1^0 = \{Q \ll \mathcal{P} \mid S \textit{ is a } Q\textit{-martingale}\}$$

Proof The proof is well-known and straightforward, so we just give the basic arguments: indeed choose any $Q \in \{Q \ll \mathcal{P} \mid S$ is a Q-martingale$\}$, and let $X \in \mathcal{C}$ and $H \in \mathcal{H}$ such that $X \leq (H \bullet S)_T$ $\mathcal{P}$-q.s. Then $E_Q[X] \leq E_Q[(H \bullet S)_T] = (H \bullet S)_0 = 0$ since $((H \bullet S)_t)_{t \in I}$ is a Q-martingale (using generalized conditional expectations, see [7, Appendix]). Thus $Q \in \mathcal{C}_1^0$.

If $Q \in \mathcal{C}_1^0$ then $E_Q[(H \bullet S)_T] = 0$ for any $H \in \mathcal{H}$ and by choosing appropriate strategies in $\mathcal{H}$ such as $H_t^j = 1_A$ for $A \in \mathcal{F}_{t-1}$, $H_t^i = 0$ for $i \neq j$ and $H_s = 0$ for $s \neq t$ one verifies that Q is a martingale measure for S.

Theorem 4.7 (First Fundamental Theorem of Asset Pricing) *Suppose $\mathcal{C}$ is $\mathcal{P}$-sensitive. The following are equivalent:*

(i) $NA(\mathcal{P})$
(ii) $\mathcal{C}_1^0 \approx \mathcal{P}$

Moreover, the Superhedging Duality holds, that is for any $X \in L_c^\infty$ the minimal superhedging price

$$\pi(X) := \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ s.t. } x + (H \bullet S)_T \geq X \ \mathcal{P}\text{-q.s.}\}$$

satisfies

$$\pi(X) = \sup_{Q \in \mathcal{C}_1^0} E_Q[X]. \tag{4.3}$$

Proof (i) $\Rightarrow$ (ii): Clearly, $c(A) = 0$ implies $\sup_{Q \in \mathcal{C}_1^0} Q(A) = 0$ as $\mathcal{C}_1^0 \subset ca_c$. Let $B \in \mathcal{F}$ such that $Q(B) = 0$ for all $Q \in \mathcal{C}_1^0$. Thus $1_B \in \mathcal{C}$ by (4.2), so $1_B = 0$ in L_c^∞ by $NA(\mathcal{P})$, i.e. $c(B) = 0$.

(ii) $\Rightarrow$ (i): let $H \in \mathcal{H}$ such that $(H \bullet S)_T \geq 0$ $\mathcal{P}$-q.s. Then $Q\{(H \bullet S)_T \geq 0\} = 0$ for every $Q \in \mathcal{C}_1^0$, because $(H \bullet S)_t$ is a Q-martingale with expectation 0, and therefore $(H \bullet S)_T = 0$ $\mathcal{P}$-q.s.

As for the Superhedging Duality note that clearly $\pi(X) \leq \|X\|_{c,\infty}$ since $0 \in \mathcal{H}$, and as $\mathcal{C}_1^0 \neq \emptyset$ ($\mathcal{C} \neq L_c^\infty$) it follows that $\pi(X) > -\infty$. Moreover, by (4.2) we have for any $y \in \mathbb{R}$ that $X - y \in \mathcal{C}$ if and only if $0 \geq \sup_{Q \in \mathcal{C}_1^0} E_Q[X - y] = -y + \sup_{Q \in \mathcal{C}_1^0} E_Q[X]$ which proves (4.3). □

A Auxiliary results for Theorem 3.1

Recall the set $\mathcal{Z}$ defined in (3.2).

Proposition A.1 *If $\mathcal{Z} = \emptyset$, then there exists a countable subset $\widetilde{\mathcal{P}} \subset \mathcal{P}$ such that $\widetilde{\mathcal{P}} \approx \mathcal{P}$. The latter implies that there is a probability measure $Q \in \mathcal{M}_1$ such that $\{Q\} \approx \mathcal{P}$.*

Proof We claim that for each $\varepsilon > 0$, there exists $P_1, \dots, P_n \in \mathcal{P}$ and $\delta > 0$ such that $P_i(A) < \delta$ for all $i = 1, \dots, n$ implies that for all $P \in \mathcal{P}$ we have $P(A) < \varepsilon$. Suppose this is not the case. Then there exists $\varepsilon > 0$ such that for any $P_1 \in \mathcal{P}$ there is $A_1 \in \mathcal{F}$ and $P_2 \in \mathcal{P}$ satisfying

$$P_1(A_1) < 1/2 \quad \text{and} \quad P_2(A_1) \geq \varepsilon.$$

Then there also exists $A_2 \in \mathcal{F}$ and $P_3 \in \mathcal{P}$ such that

$$P_1(A_2) < 1/4, \ P_2(A_2) < 1/4 \quad \text{while} \quad P_3(A_2) \geq \varepsilon.$$

Continuing this procedure we find sequences $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ and $(P_n)_{n\in\mathbb{N}} \in \mathcal{P}$ such that

$$P_i(A_n) < \frac{1}{2^n},\ i = 1, \ldots, n, \quad \text{and} \quad P_{n+1}(A_n) \geq \varepsilon.$$

Consider $N := \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$. Then $P_i(N) = 0$ for each $i \in \mathbb{N}$, because for all $n > (i-1)$

$$P_i(N) \leq \sum_{k=n}^{\infty} P_i(A_k) \leq \frac{1}{2^{n-1}}.$$

Hence, replacing the above sequence A_n by $B_n := A_n \setminus N$, $n \in \mathbb{N}$, we still have

$$P_i(B_n) < \frac{1}{2^n},\ i = 1, \ldots, n, \quad \text{and} \quad P_{n+1}(B_n) \geq \varepsilon.$$

Now let $E_n := \bigcup_{k\geq n} B_k$, $n \in \mathbb{N}$. It follows that $E_n \downarrow \emptyset$. However, for each $n \in \mathbb{N}$

$$c(E_n) \geq P_{n+1}(E_n) \geq P_{n+1}(B_n) \geq \varepsilon$$

which contradicts $\mathcal{Z} = \emptyset$.

Now let $\delta_n > 0$ and let $P_1^{(n)}, \ldots, P_{m(n)}^{(n)} \in \mathcal{P}$ be such that for all $P \in \mathcal{P}$ it holds $P(A) < 1/n$ whenever $P_i^{(n)}(A) < \delta_n$ for all $i = 1, \ldots, m(n)$. Define

$$\mu := \sum_{n=1}^{\infty} \sum_{i=1}^{m(n)} \frac{1}{2^n} \frac{1}{2^i} P_i^{(n)}.$$

Then $\mu \in ca_+$, and $\mu(A) = 0$ implies that $P_i^{(n)}(A) = 0$ for all $i = 1, \ldots, m(n)$ and $n \in \mathbb{N}$. Eventually this implies that for all $P \in \mathcal{P}$ we have $P(A) < 1/n$ for all $n \in \mathbb{N}$, hence $P(A) = 0$. Thus

$$\widetilde{\mathcal{P}} := \{P_i^{(n)} \mid i \in \{1, \ldots, m(n)\}, n \in \mathbb{N}\} \quad \text{and} \quad Q := \frac{1}{\mu(\Omega)}\mu$$

satisfy the assertion. □

Proposition A.2 *Let* $(B, \|\cdot\|)$ *be a Banach lattice of (equivalence classes of) random variables on* $(\Omega, \mathcal{F})$ *containing all simple random variables such that the order* $\leq$ *on* B *satisfies* $0 \leq 1_A \leq 1_{A'}$ *whenever* $A \subset A'$ *for* $A, A' \in \mathcal{F}$. *If* $B^* \subset ca$, *in the sense that every* $l \in B^*$ *is of type*

$$l(X) = \int X \, d\mu, \quad X \in B,$$

for some $\mu \in ca$, *then* $\|1_{A_n}\| \to 0$ $(n \to \infty)$ *for all* $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ *such that* $A_n \downarrow \emptyset$.

Conversely, if $\|1_{A_n}\| \to 0$ $(n \to \infty)$ *for all* $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ *such that* $A_n \downarrow \emptyset$*, then for every* $l \in B^*$ *there is a* $\mu \in ca$ *such that* $l(Y) = \int Y\, d\mu$ *for all simple random variables* Y*.*

Proof Suppose that $B^* \subset ca$ and let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $A_n \downarrow \emptyset$. Then $1_{A_n} \to 0$ with respect to $\sigma(B, B^*)$ since every element in B^* corresponds to a σ-additive measure. Hence,

$$0 \in \overline{co\{1_{A_n} \mid n \in \mathbb{N}\}}$$

where the closure is taken in the $\sigma(B, B^*)$-topology. As the closed convex set in the $\sigma(B, B^*)$-topology and in the norm topology coincide, we have that there is a sequence of convex combinations

$$c_k := \sum_{i=1}^{m(k)} a_i(k) 1_{A_{n_i(k)}}, \quad k \in \mathbb{N},$$

where $a_i(k) \in \mathbb{R}$ and $n_1(k) \leq n_2(k) \leq \ldots \leq n_{m(k)}(k)$ for all $k \in \mathbb{N}$ such that $\|c_k\| \to 0$ for $k \to \infty$. Moreover, since $0 \in \overline{co\{1_{A_n} \mid n \geq N\}}$ for any $N \in \mathbb{N}$, we may assume that $n_1(k) \leq n_1(k+1)$ for all $k \in \mathbb{N}$. However, $c_k \geq 1_{A_k}$ where $A_k = A_{n_{m(k)}(k)}$, because $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$. Thus, as $\|\cdot\|$ is a lattice norm, the subsequence 1_{A_k} converges to 0 in norm and hence also 1_{A_n} converges to 0 in the norm topology (again due to $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$).

Finally suppose that $\|1_{A_n}\| \to 0$ $(n \to \infty)$ for all $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $A_n \downarrow \emptyset$. Then for any $l \in B^*$, the set function

$$\mu(A) := l(1_A), \quad A \in \mathcal{F},$$

is σ-additive. By linearity of l we deduce that $l(X) = \int X\, d\mu$ for all simple random variables X. □

B Penot–Volle duality theorem

Theorem B.1 (see e.g. *[20, Theorem 1.1]*) *Let* L *be a locally convex topological vector space,* L' *be its dual space and* $f : L \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ *be quasiconvex and lower semicontinuous. Then*

$$f(X) = \sup_{X' \in L'} R(X'(X), X') \tag{B.1}$$

where $R : \mathbb{R} \times L' \to \overline{\mathbb{R}}$ *is defined by*

$$R(t, X') := \inf_{\xi \in L} \left\{ f(\xi) \mid X'(\xi) \geq t \right\}. \tag{B.2}$$

References

1. Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis. Springer, Berlin (2006)
2. Aliprantis, C.D., Burkinshaw, O.: Locally Solid Riesz Spaces with Applications to Economics, Mathematical Surveys and Monographs, vol. 105, 2nd edn. American Mathematical Society, Providence (2003)
3. Aliprantis, C.D., Tourky, R.: The super order dual of an ordered vector space and the Riesz–Kantorovich formula. Trans. AMS **354**(5), 2055–2077 (2001)
4. Beißner P.: Coherent price systems and uncertainty-neutral valuation. IMW working paper series (2013)
5. Biagini, S., Frittelli, M.: On the extension of the Namioka-Klee theorem and on the Fatou property for risk measures. In: Delbaen, F., et al. (eds.) Optimality and Risk-Modern Trends in Mathematical Finance, pp. 1–28. Springer, Berlin, Heidelberg (2009)
6. Bion-Nadal, J., Kervarec, M.: Risk measuring under model uncertainty. Ann. Appl. Probab. **22**(1), 213–238 (2012)
7. Bouchard, B., Nutz, M.: Arbitrage and duality in nondominated discrete-time models. Ann. Appl. Probab. **25**(2), 823–859 (2015)
8. Burzoni, M., Frittelli, M., Maggis, M.: Model-free super hedging duality. Ann. Appl. Probab. (**forthcoming**) (2016)
9. Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., Montrucchio, L.: Complete monotone quasiconcave Duality. Extended version: Working Paper 80 Carlo Alberto Notebooks
10. Cheridito, P., Kupper, M., Tangpi, L.: Representation of increasing convex functionals with countably additive measures. preprint (2015)
11. Cohen, S.N.: Quasi-sure analysis, aggregation and dual representations of sublinear expectations in general spaces. Electron. J. Probab **17**(62), 1–15 (2012)
12. Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math. Ann. **300**, 463–520 (1994)
13. Delbaen, F., Schachermayer, W.: The Mathematics of Arbitrage. Springer, New York (2006)
14. Denis, L., Hu, M., Peng, S.: Functions spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths. Potential Anal. **34**(2), 139–161 (2011)
15. Drapeau, S., Kupper, M.: Risk preferences and their robust representation. Math. Oper. Res. **28**(1), 28–62 (2013)
16. Dudley, R.M.: Real Analysis and Probability. Cambridge University Press, Cambridge (2002)
17. Dunford, N., Schwartz, J.: Linear Operators. Part I: General Theory. Interscience Publishers, New York (1958)
18. Ekeland, I., Témam, R.: Convex Analysis and Variational Problems (Part I), Classics in Applied Mathematics. SIAM, Philadelphia (1999)
19. Föllmer, H., Schied, A.: Stochastic Finance. An Introduction in Discrete Time, 3rd edn. de Gruyter Studies in Mathematics (2011)
20. Frittelli, M., Maggis, M.: Dual representation of quasiconvex conditional maps. SIAM J. Financ. Math. **2**, 357–382 (2011)
21. Frittelli, M., Rosazza, G.E.: Putting order in risk measures. J. Bank. Finance **26**(7), 1473–1486 (2002)
22. Grothendieck, A.: Espaces Vectoriels Topologiques. Sociedade de Matematica de Sao Paulo, Sao Paulo (1954)
23. Halmos, P.R., Savage, L.J.: Application of the Radon–Nikodym theorem to the theory of sufficient statistics. Ann. Math. Stat. **20**(2), 225–241 (1949)
24. Kreps, D.: Arbitrage and equilibrium in economies with infinitely many commodities. J. Math. Econ. **8**(1), 15–35 (1981)
25. Meyer-Nieberg, P.: Banach Lattices. Springer, Berlin (1991)
26. Nutz, M.: Superreplication under model uncertainty in discrete time. Financ. Stoch. **18**(4), 791–803 (2014)
27. Soner, H.M., Touzi, N., Zang, J.: Quasi-sure stochastic analysis through aggregation. Electron. J. Probab. **16**, 1844–1879 (2011)
28. Vorbrink, J.: Financial markets with volatility uncertainty. J. Math. Econ. **53**(issue C), 64–78 (2014)