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Norm inequalities related to the arithmetic–geometric mean inequalities for positive semidefinite matrices

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Abstract In this paper, we propose three new matrix versions of the arithmetic–geometric mean inequality for unitarily invariant norms, which stem from the fact that the Heinz mean of two positive real numbers interpolates between the geometric and arithmetic means of these numbers. Related trace inequalities are also presented.

Keywords Unitarily invariant norm · Hilbert–Schmidt norm · Singular value · Trace · Positive semidefinite matrix · Inequality

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1 Introduction

Let a, b be positive real numbers and let $0 \leq t \leq 1$. The arithmetic–geometric mean inequality can be refined by inserting the Heinz mean between the geometric mean and the arithmetic mean:

$$\sqrt{ab} \leq \frac{a^t b^{1-t} + a^{1-t} b^t}{2} \leq \frac{a+b}{2}. \tag{1.1}$$

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The inequalities (1.1) are equivalent to the inequalities

$$\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)^2 \leq \left(a^t+b^t\right)\left(a^{1-t}+b^{1-t}\right) \leq 2\,(a+b)\,. \tag{1.2}$$

The inequalities (1.1) and (1.2) have matrix versions. We introduce some notations regarding their matrix versions: Let A, B be positive semidefinite matrices, $0 \leq t \leq 1$, and $|||.|||$ any unitarily invariant norm. Let

$$\begin{aligned} h_t &= A^t B^{1-t} + A^{1-t} B^t, \\ b_t &= A^t B^{1-t} + B^t A^{1-t}, \\ k_t &= \left(A^t + B^t\right)\left(A^{1-t} + B^{1-t}\right), \end{aligned}$$

and

$$m_t = \left(A^{1-t}+B^{1-t}\right)^{\frac{1}{2}}\left(A^t+B^t\right)\left(A^{1-t}+B^{1-t}\right)^{\frac{1}{2}}.$$

Then the matrix versions of the inequalities (1.1) are

$$2\left|\left|\left|A^{\frac{1}{2}}B^{\frac{1}{2}}\right|\right|\right| \leq \left|\left|\left|A^tB^{1-t}+A^{1-t}B^t\right|\right|\right| \leq |||A+B|||$$

which can be stated as

$$\left|\left|\left|h_{\frac{1}{2}}\right|\right|\right| \leq |||h_t||| \leq |||h_1|||. \tag{1.3}$$

For the usual operator norm, this was proved by Heinz [16] and the generalization to all unitarily invariant norms was obtained by Bhatia and Davis [7].

Because of the noncommutativity of matrix multiplication, h_t is not the same as b_t, and this gives us the ability to see the arithmetic–geometric mean inequality in a different new way in terms of b_t, which is equivalent to the following question.

Question 1.1 *Given $t \in [0, 1]$ and any unitarily invariant norm $|||.|||$, is it true that*

$$\left|\left|\left|b_{\frac{1}{2}}\right|\right|\right| \leq |||b_t||| \leq |||b_1|||? \tag{1.4}$$

The second inequality in (1.4) is a question raised by Bourin [10]. It has been proved for the Hilbert–Schmidt norm under the condition $\frac{1}{4} \leq t \leq \frac{3}{4}$, see [6,11].

Recently, an affirmative answer to Bourin's question in the trace norm has been given by the authors [15], where it has been proved that

$$\|b_t\|_1 \leq \|b_1\|_1\,.$$

Moreover, a partial answer to this question in the wider class of unitarily invariant norms has been given in [15] by proving that

$$|||\mathrm{Re}\, b_t||| \leq |||b_1|||$$

and

$$|||\mathrm{Im}\, b_t||| \leq |||b_1|||.$$

The inequalities (1.2) have the following matrix versions, which enable us to see the arithmetic–geometric mean inequality in a different new way in terms of k_t.

Question 1.2 *Given* $t \in [0, 1]$ *and any unitarily invariant norm* $|||.|||$*, is it true that*

$$\left|\left|\left|k_{\frac{1}{2}}\right|\right|\right| \leq |||k_t||| \leq |||k_1|||? \tag{1.5}$$

The first inequality in (1.5) can be stated as

$$\left|\left|\left|\left(A^{\frac{1}{2}} + B^{\frac{1}{2}}\right)^2\right|\right|\right| \leq \left|\left|\left|\left(A^t + B^t\right)\left(A^{1-t} + B^{1-t}\right)\right|\right|\right|. \tag{1.6}$$

The inequality (1.6) is a special case of the following more general form for commuting positive semidefinite matrices:

$$\left|\left|\left|\left(A_1^{\frac{1}{2}} B_1^{\frac{1}{2}} + A_2^{\frac{1}{2}} B_2^{\frac{1}{2}}\right)^2\right|\right|\right| \leq |||(A_1 + A_2)(B_1 + B_2)|||, \tag{1.7}$$

where A_1, A_2, B_1, B_2 are positive semidefinite matrices such that $A_1 B_1 = B_1 A_1$ and $A_2 B_2 = B_2 A_2$. The inequality (1.7) is a part of Theorem 3.1 in [4] for the case $k = 2$, which has been recently stated and proved by Audenaert for all unitarily invariant norms (see also [17,19,20]). Note that the inequality (1.6) follows from the inequality (1.7) by replacing A_1, B_1, A_2, and B_2 by A^t, A^{1-t}, B^t, and B^{1-t}, respectively.

The inequalities (1.2) have other matrix versions, which enable us to see the arithmetic–geometric mean inequality in a different new way in terms of m_t.

Question 1.3 *Given* $t \in [0, 1]$ *and any unitarily invariant norm* $|||.|||$*, is it true that*

$$\left|\left|\left|m_{\frac{1}{2}}\right|\right|\right| \leq |||m_t||| \leq |||m_1|||? \tag{1.8}$$

The second inequality in (1.8) has been proved in [21]. The first inequality in (1.8) can be stated as

$$\left|\left|\left|\left(A^{\frac{1}{2}} + B^{\frac{1}{2}}\right)^2\right|\right|\right| \leq \left|\left|\left|\left(A^t + B^t\right)^{\frac{1}{2}}\left(A^{1-t} + B^{1-t}\right)\left(A^t + B^t\right)^{\frac{1}{2}}\right|\right|\right|. \tag{1.9}$$

The inequality (1.9) is a special case of the following more general form for commuting positive semidefinite matrices:

$$\left|\left|\left|\left(A_1^{\frac{1}{2}}B_1^{\frac{1}{2}}+A_2^{\frac{1}{2}}B_2^{\frac{1}{2}}\right)^2\right|\right|\right| \leq \left|\left|\left|(A_1+A_2)^{\frac{1}{2}}(B_1+B_2)(A_1+A_2)^{\frac{1}{2}}\right|\right|\right|, \quad (1.10)$$

where A_1, A_2, B_1, B_2 are positive semidefinite matrices such that $A_1B_1 = B_1A_1$ and $A_2B_2 = B_2A_2$. The inequality (1.10) is a part of Theorem 3.3 in [13] for the case $k = 2$, which has been recently stated and proved by the authors for all unitarily invariant norms in [13] (see also [17,19,20]). Note that the inequality (1.9) follows from the inequality (1.10) by replacing A_1, B_1, A_2, and B_2 by A^t, A^{1-t}, B^t, and B^{1-t}, respectively. This gives an affirmative answer to Question 1.3.

This paper is devoted towards closing some of these remaining open questions.

In Sect. 2, we will prove the following singular value inequality and majorization relations:

$$s_j\left(\left(A^t+B^t\right)\left(A^{1-t}+B^{1-t}\right)\right) \leq 2^{\frac{3}{2}}s_j\left(\left(A^2+B^2\right)^{\frac{1}{2}}\right) \text{ for } j=1,2,\ldots,n,$$
$$s\left(\left(A^t+B^t\right)\left(A^{1-t}+B^{1-t}\right)\right) \prec_w 2s\left(A+B\right),$$

and

$$s\left(A^tB^{1-t}+B^tA^{1-t}\right) \prec_w 2^{\frac{1}{2}}s\left(\left(A^2+B^2\right)^{\frac{1}{2}}\right),$$

where A and B be positive semidefinite matrices and $0 \leq t \leq 1$.

Notice that the above singular value inequality and majorization relations are sharp, and as a consequence of the second majorization relation, we prove the second inequality in (1.5), i.e., the inequality

$$|||k_t||| \leq |||k_1|||.$$

This gives an affirmative answer to Question 1.2.

Remark 1.4 It should be mentioned here that Question 1.1 remains open.

It can be seen that the well-known Heinz inequality and the question of Bourin (i.e. the second inequality in (1.3) and (1.4), respectively) are equivalent to the following inequalities:

$$\left|\left|\left|A^pB^q+A^qB^p\right|\right|\right| \leq \left|\left|\left|A^{p+q}+B^{p+q}\right|\right|\right|$$

and

$$\left|\left|\left|A^pB^q+B^pA^q\right|\right|\right| \leq \left|\left|\left|A^{p+q}+B^{p+q}\right|\right|\right|,$$

where A, B are positive semidefinite matrices and p, q are positive real numbers.

In a recent paper [11], and in their investigations of the Lieb–Thirring trace inequalities, and in their attempt to answer Bourin's question, Hayajneh and Kittaneh proposed the following conjecture for commuting positive semidefinite matrices.

Conjecture 1.5 *Let A_1, A_2, B_1, B_2 be positive semidefinite matrices such that $A_1B_1 = B_1A_1$ and $A_2B_2 = B_2A_2$. Then, for every unitarily invariant norm,*

$$|||A_1B_2 + A_2B_1||| \leq |||A_1B_2 + B_1A_2|||. \tag{1.11}$$

An important special case of the inequality (1.11) is the inequality

$$\left|\left|\left|A^sB^p + B^qA^t\right|\right|\right| \leq \left|\left|\left|A^sB^p + A^tB^q\right|\right|\right|, \tag{1.12}$$

where A, B are positive semidefinite matrices and s, t, p, q are positive real numbers.

The Hilbert–Schmidt norm version of (1.12) is the inequality

$$\left\|A^sB^p + B^qA^t\right\|_2 \leq \left\|A^sB^p + A^tB^q\right\|_2. \tag{1.13}$$

Recently, the authors [13] proved the inequality (1.13) under the condition that

$$\left|\frac{s}{s+t} - \frac{1}{2}\right| + \left|\frac{p}{p+q} - \frac{1}{2}\right| \leq \frac{1}{2}. \tag{1.14}$$

Replacing A and B by $A^{\frac{1}{s+t}}$ and $B^{\frac{1}{p+q}}$, we see that the inequality (1.13) is equivalent to saying

$$\left\|A^\mu B^\nu + B^{1-\nu}A^{1-\mu}\right\|_2 \leq \left\|A^\mu B^\nu + A^{1-\mu}B^{1-\nu}\right\|_2 \tag{1.15}$$

under the condition

$$\left|\mu - \frac{1}{2}\right| + \left|\nu - \frac{1}{2}\right| \leq \frac{1}{2}. \tag{1.16}$$

Consequently, one can infer from the inequality (1.15) and the condition (1.16) that

$$\|b_t\|_2 \leq \|h_t\|_2 \text{ for } \frac{1}{4} \leq t \leq \frac{3}{4}.$$

In [14], the authors also generalized the inequality (1.15) to complex values. In fact, they proved that the inequality

$$\left\|A^wB^z + B^{1-\bar{z}}A^{1-\overline{w}}\right\|_2 \leq \left\|A^wB^z + A^{1-\overline{w}}B^{1-\bar{z}}\right\|_2 \tag{1.17}$$

holds for the complex numbers w, z under the condition

$$\left|\mathrm{Re}\, w - \frac{1}{2}\right| + \left|\mathrm{Re}\, z - \frac{1}{2}\right| \leq \frac{1}{2}. \tag{1.18}$$

Section 3 is devoted to proving the following reverse-type inequality of (1.17):

$$\left\| A^w B^z - B^{1-\overline{z}} A^{1-\overline{w}} \right\|_2 \geq \left\| A^w B^z - A^{1-\overline{w}} B^{1-\overline{z}} \right\|_2 \tag{1.19}$$

under the condition (1.18).

The special case of the inequality (1.19) when w, z are positive real numbers has been proved by Alakhrass in [1].

For a comprehensive account on Bourin's questions and related trace and norm inequalities, we refer to [3,4,6,8–15,17,19,21], and references therein.

2 Main results

Recall that a matrix norm $|||.|||$ on the space of all complex square matrices of a fixed order is called unitarily invariant if $|||UXV||| = |||X|||$ for all X and for all unitary matrices U, V. Among familiar examples of unitarily invariant norms are the Schatten $p-$norms, denoted by $\|.\|_p$ and defined for $1 \leq p \leq \infty$ as

$$\|X\|_p = \left(\mathrm{tr}\, |X|^p\right)^{\frac{1}{p}},$$

where $|X| = (X^*X)^{\frac{1}{2}}$. The values $p = 1$, $p = 2$, and $p = \infty$ correspond to the trace norm, the Hilbert–Schmidt norm, and the spectral (or the usual operator) norm, respectively.

The generalized Hölder inequality for the Schatten $p-$norms will be frequently used in proving our main results. This inequality says that for any matrices X, Y, Z and any real numbers $p, q, r \geq 1$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, we have

$$|\mathrm{tr}\,(XYZ)| \leq \|XYZ\|_1 \leq \|X\|_p \|Y\|_q \|Z\|_r\,. \tag{2.1}$$

For more details about the inequality (2.1), see [22, Theorem 2.8].

In this section, we denote the vectors of eigenvalues and singular values of a matrix A by $\lambda(A)$ and $s(A)$, respectively. These vectors are obtained by arranging the singular values and eigenvalues, as well whenever they are real, in a non-increasing order. In general, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we will write $x^{\downarrow}$ for the vector obtained by rearranging the coordinates of x in a non-increasing order.

Let $x, y \in \mathbb{R}^n$. We say that x is weakly majorized by y, denoted $x \prec_w y$, if and only if for $k = 1, \ldots, n$, $\sum_{i=1}^k x_i^{\downarrow} \leq \sum_{i=1}^k y_i^{\downarrow}$.

The Fan dominance principle [5, p. 93] says that for any two matrices X, Y, we have $s(X) \prec_w s(Y)$ if and only if $|||X||| \leq |||Y|||$ for all unitarily invariant norms. Other facts that will be used say that $\lambda(XY) = \lambda(YX)$ and $s(XY) \prec_w s(X)s(Y)$, see, e.g., [5, pp. 11, 94].

The following two lemmas are needed to prove our main results.The first lemma says that the function $f(x) = x^2$ is matrix convex on $\mathbb{R}$, and the second lemma says that the function $f(x) = x^t$ is matrix concave on $[0, \infty)$ for $0 \le t \le 1$. For these facts, we refer to [5, pp. 113, 115].

The matrices considered here are $n \times n$ complex matrices.

Lemma 2.1 *Let A and B be Hermitian matrices. Then*

$$(A+B)^2 \le 2\left(A^2 + B^2\right).$$

Lemma 2.2 *Let A and B be positive semidefinite matrices and let* $0 \le t \le 1$. *Then*

$$A^t + B^t \le 2^{1-t}(A+B)^t.$$

Our first main result can be stated as follows.

Theorem 2.3 *Let A and B be positive semidefinite matrices and let* $0 \le t \le 1$. *Then*

$$s\left(\left(A^t + B^t\right)\left(A^{1-t} + B^{1-t}\right)\right) \prec_w 2s(A+B). \tag{2.2}$$

Proof Since

$$s\left(\left(A^t + B^t\right)\left(A^{1-t} + B^{1-t}\right)\right) \prec_w s\left(A^t + B^t\right)s\left(A^{1-t} + B^{1-t}\right),$$

it follows that for $k = 1, 2, \dots, n$, we have

$$\begin{aligned}
&\sum_{j=1}^{k} s_j\left(\left(A^t + B^t\right)\left(A^{1-t} + B^{1-t}\right)\right)\\
&\quad \le \sum_{j=1}^{k} \lambda_j\left(A^t + B^t\right)\lambda_j\left(A^{1-t} + B^{1-t}\right)\\
&\quad \le \sum_{j=1}^{k} 2^{1-t}\lambda_j^t(A+B)\,2^t\lambda_j^{1-t}(A+B) \text{ (by Lemma 2.2)}\\
&\quad = 2\sum_{j=1}^{k} \lambda_j(A+B).
\end{aligned}$$

Thus, we have

$$s\left(\left(A^t + B^t\right)\left(A^{1-t} + B^{1-t}\right)\right) \prec_w 2s(A+B).$$

□

Corollary 2.4 *Let A and B be positive semidefinite matrices and let* $0 \le t \le 1$. *Then for all unitarily invariant norms, we have*

$$\left|\left|\left|\left(A^t + B^t\right)\left(A^{1-t} + B^{1-t}\right)\right|\right|\right| \le 2|||A + B|||. \tag{2.3}$$

In other words,

$$|||k_t||| \le |||k_1|||.$$

Remark 2.5 The inequality (2.3) is stronger than the second inequality in (1.8). To see this, recall that if X and Y are matrices such that XY is Hermitian, then $|||XY||| \le |||\mathrm{Re}\, YX||| \le |||YX|||$ (see, e.g., [18]). Accordingly, $|||m_t||| \le |||k_t|||$ for $0 \le t \le 1$. In the same vein, it is readily seen that the inequality (1.10) is stronger than the inequality (1.7).

Our second main result can be stated as follows.

Theorem 2.6 *Let A and B be positive semidefinite matrices and let* $0 \le t \le 1$. *Then*

$$s_j\left(\left(A^t + B^t\right)\left(A^{1-t} + B^{1-t}\right)\right) \le 2^{\frac{3}{2}} s_j\left(\left(A^2 + B^2\right)^{\frac{1}{2}}\right) \tag{2.4}$$

for $j = 1, 2, \ldots, n$.

Proof For $j = 1, 2, \ldots, n$, we have

$$\begin{aligned}
& s_j\left(\left(A^t + B^t\right)\left(A^{1-t} + B^{1-t}\right)\right) \\
&= \lambda_j\left(\left|\left(A^t + B^t\right)\left(A^{1-t} + B^{1-t}\right)\right|\right) \\
&= \lambda_j\left(\left(A^t + B^t\right)\left(A^{1-t} + B^{1-t}\right)^2\left(A^t + B^t\right)\right)^{\frac{1}{2}} \\
&= \lambda_j^{\frac{1}{2}}\left(\left(A^t + B^t\right)\left(A^{1-t} + B^{1-t}\right)^2\left(A^t + B^t\right)\right) \\
&\le 2^{\frac{1}{2}}\lambda_j^{\frac{1}{2}}\left(\left(A^t + B^t\right)\left(A^{2(1-t)} + B^{2(1-t)}\right)\left(A^t + B^t\right)\right) \text{ (by Lemma 2.1)} \\
&= 2^{\frac{1}{2}}\lambda_j^{\frac{1}{2}}\left(\left(A^{2(1-t)} + B^{2(1-t)}\right)^{\frac{1}{2}}\left(A^t + B^t\right)^2\left(A^{2(1-t)} + B^{2(1-t)}\right)^{\frac{1}{2}}\right) \\
&\le 2\lambda_j^{\frac{1}{2}}\left(\left(A^{2(1-t)} + B^{2(1-t)}\right)\left(A^{2t} + B^{2t}\right)\right) \text{ (by Lemma 2.1)} \\
&\le 2^{\frac{3}{2}}\lambda_j^{\frac{1}{2}}\left(A^2 + B^2\right) \text{ (by Lemma 2.2)} \\
&= 2^{\frac{3}{2}}\lambda_j\left(A^2 + B^2\right)^{\frac{1}{2}} \\
&= 2^{\frac{3}{2}} s_j\left(\left(A^2 + B^2\right)^{\frac{1}{2}}\right).
\end{aligned}$$

This proves the inequality (2.4). □

Using the fact that unitarily invariant norms are increasing functions of singular values, we have the following immediate consequence of Theorem 2.6.

Corollary 2.7 *Let A and B be positive semidefinite matrices and let* $0 \le t \le 1$. *Then for every unitarily invariant norm, we have*

$$\left|\left|\left|\left(A^t + B^t\right)\left(A^{1-t} + B^{1-t}\right)\right|\right|\right| \le 2^{\frac{3}{2}}\left|\left|\left|\left(A^2 + B^2\right)^{\frac{1}{2}}\right|\right|\right|. \tag{2.5}$$

Remark 2.8 We remark here that the inequality (2.5) also follows form Corollary 2.4. For Hermitian matrices A, B, we have

$$|||A + B||| \le 2^{\frac{1}{2}}\left|\left|\left|\left(A^2 + B^2\right)^{\frac{1}{2}}\right|\right|\right| \le 2^{\frac{1}{2}}|||\,|A| + |B|\,|||. \tag{2.6}$$

In fact, in view of the matrix monotonicity of the function $f(x) = x^{\frac{1}{2}}$ on $[0, \infty)$, the first inequality in (2.6) follows from Lemma 2.1, and the second inequality in (2.6) can be inferred from [2].

Our third main result can be stated as follows.

Theorem 2.9 *Let A and B be positive semidefinite matrices and let* $0 \le t \le 1$. *Then*

$$s\left(A^t B^{1-t} + B^t A^{1-t}\right) \prec_w 2^{\frac{1}{2}} s\left(\left(A^2 + B^2\right)^{\frac{1}{2}}\right). \tag{2.7}$$

Proof Let □

$$X = \begin{bmatrix} A^t & B^t \\ 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} B^{1-t} & 0 \\ A^{1-t} & 0 \end{bmatrix}.$$

Then, using the fact that $s(T) = \lambda^{\frac{1}{2}}(T^*T) = \lambda^{\frac{1}{2}}(TT^*)$ for every matrix T, we have

$$\begin{aligned} & s\left(\left(A^t B^{1-t} + B^t A^{1-t}\right) \oplus 0\right) \\ & = s(XY) \\ & \prec_w s(X)\, s(Y) \\ & = \lambda^{\frac{1}{2}}\left(\left(A^{2t} + B^{2t}\right) \oplus 0\right) \lambda^{\frac{1}{2}}\left(\left(A^{2(1-t)} + B^{2(1-t)}\right) \oplus 0\right). \end{aligned}$$

Thus, for $k = 1, 2, \ldots, n$, we have

$$\begin{aligned} & \sum_{j=1}^{k} s_j\left(A^t B^{1-t} + B^t A^{1-t}\right) \\ & \le \sum_{j=1}^{k} \lambda_j^{\frac{1}{2}}\left(A^{2t} + B^{2t}\right) \lambda_j^{\frac{1}{2}}\left(A^{2(1-t)} + B^{2(1-t)}\right) \end{aligned}$$

$$\leq \sum_{j=1}^{k} 2^{\frac{1-t}{2}} \lambda_j^{\frac{t}{2}} \left(A^2 + B^2\right) 2^{\frac{1-(1-t)}{2}} \lambda_j^{\frac{1-t}{2}} \left(A^2 + B^2\right) \text{(by Lemma 2.2)}$$
$$= 2^{\frac{1}{2}} \sum_{j=1}^{k} \lambda_j^{\frac{1}{2}} \left(A^2 + B^2\right)$$
$$= 2^{\frac{1}{2}} \sum_{j=1}^{k} s_j \left(\left(A^2 + B^2\right)^{\frac{1}{2}}\right).$$

This proves the inequality (2.7). □

Corollary 2.10 *Let A and B be positive semidefinite matrices and let $0 \leq t \leq 1$. Then for every unitarily invariant norm, we have*

$$\left|\left|\left| A^t B^{1-t} + B^t A^{1-t} \right|\right|\right| \leq 2^{\frac{1}{2}} \left|\left|\left| \left(A^2 + B^2\right)^{\frac{1}{2}} \right|\right|\right|. \tag{2.8}$$

3 A reverse-type inequality

We start this section with the following lemma, which will be used in proving our main result in this section. This lemma has been proved by the authors in [14].

Lemma 3.1 *Let A, B be positive semidefinite matrices, and let w, z be complex numbers such that*

$$\left| Re\ w - \frac{1}{2} \right| + \left| Re\ z - \frac{1}{2} \right| \leq \frac{1}{2}.$$

Then

$$\left| \operatorname{tr}\left(A^w B^z A^{1-w} B^{1-z}\right) \right| \leq \operatorname{tr}(AB).$$

Using Lemma 3.1, it has been shown in [14] that

$$\left\| A^w B^z + B^{1-\overline{z}} A^{1-\overline{w}} \right\|_2 \leq \left\| A^w B^z + A^{1-\overline{w}} B^{1-\overline{z}} \right\|_2$$

As another application of Lemma 3.1, we obtain the following reverse-type inequality.

Theorem 3.2 *Let A, B be positive semidefinite matrices, and let w, z be complex numbers such that*

$$\left| Re\ w - \frac{1}{2} \right| + \left| Re\ z - \frac{1}{2} \right| \leq \frac{1}{2}.$$

Then

$$\left\| A^{w} B^{z} - B^{1-\overline{z}} A^{1-\overline{w}} \right\|_{2} \geq \left\| A^{w} B^{z} - A^{1-\overline{w}} B^{1-\overline{z}} \right\|_{2}.$$

Proof We can see that the square of the left-hand side of the desired norm inequality is equal to

$$\operatorname{tr}\left(A^{w+\overline{w}} B^{z+\overline{z}} - 2\mathrm{Re}\left(A^{1-w} B^{1-z} A^{w} B^{z} \right) + A^{2-(w+\overline{w})} B^{2-(z+\overline{z})} \right)$$

and the square of the right-hand side is equal to

$$\operatorname{tr}\left(A^{w+\overline{w}} B^{z+\overline{z}} - 2AB + A^{2-(w+\overline{w})} B^{2-(z+\overline{z})} \right).$$

Here, we have used the fact that for all matrices X, Y, $\|X\|_2 = (\operatorname{tr} X^* X)^{\frac{1}{2}}$ and the cyclicity of the trace, i.e., $\operatorname{tr} XY = \operatorname{tr} YX$.

Therefore, the desired norm inequality is equivalent to the statement

$$\mathrm{Re}\ \operatorname{tr}\left(A^{w} B^{z} A^{1-w} B^{1-z} \right) \leq \operatorname{tr}(AB). \tag{3.1}$$

By Lemma 3.1 and the fact that for every matrix X, $\mathrm{Re}\ \operatorname{tr} X \leq |\operatorname{tr} X|$, the inequality (3.1) holds provided

$$\left| \mathrm{Re}\, w - \frac{1}{2} \right| + \left| \mathrm{Re}\, z - \frac{1}{2} \right| \leq \frac{1}{2}.$$

This completes the proof. □

4 Related trace inequalities

In this section, we obtain trace inequalites related to the norm inequalities presented in Sect. 2.

Theorem 4.1 *Let A and B be positive semidefinite matrices and let $\mathcal{S} = \{z \in \mathbb{C} : 0 \leq \mathrm{Re}\, z \leq 1\}$. Then*

$$\left| \operatorname{tr}\left(\left(A^{z} B^{1-z} \right)^{2} + \left(B^{z} A^{1-z} \right)^{2} \right) \right| \leq \operatorname{tr}\left(A^{2} + B^{2} \right) \tag{4.1}$$

for all $z \in \mathcal{S}$.

Proof Let $z = x + iy, 0 \leq x \leq 1$. Then

$$\begin{aligned}
&\left|\mathrm{tr}\left(\left(A^z B^{1-z}\right)^2 + \left(B^z A^{1-z}\right)^2\right)\right| \\
&= \left|\mathrm{tr}\left(A^z B^{1-z}\right)^2 + \mathrm{tr}\left(B^z A^{1-z}\right)^2\right| \\
&\leq \left|\mathrm{tr}\left(A^z B^{1-z}\right)^2\right| + \left|\mathrm{tr}\left(B^z A^{1-z}\right)^2\right| \\
&\leq \left\|\left(A^z B^{1-z}\right)^2\right\|_1 + \left\|\left(B^z A^{1-z}\right)^2\right\|_1 \text{ (by the inequality (2.1))} \\
&= \left\|A^z B^{1-z} A^z B^{1-z}\right\|_1 + \left\|B^z A^{1-z} B^z A^{1-z}\right\|_1 \\
&= \left\|A^x B^{1-x} B^{-iy} A^x A^{iy} B^{1-x}\right\|_1 \\
&\quad + \left\|B^x A^{1-x} A^{-iy} B^x B^{iy} A^{1-x}\right\|_1 \\
&\leq \left\|A^x B^{1-x} B^{-iy}\right\|_2 \left\|A^{iy} A^x B^{1-x}\right\|_2 \\
&\quad + \left\|B^x A^{1-x} A^{-iy}\right\|_2 \left\|B^{iy} B^x A^{1-x}\right\|_2 \text{ (by the inequality (2.1))} \\
&= \left\|A^x B^{1-x}\right\|_2 \left\|A^x B^{1-x}\right\|_2 \\
&\quad + \left\|B^x A^{1-x}\right\|_2 \left\|B^x A^{1-x}\right\|_2 \left(\text{since } A^{iy} \text{ and } B^{iy} \text{ are unitary}\right) \\
&= \left\|A^x B^{1-x}\right\|_2^2 + \left\|B^x A^{1-x}\right\|_2^2 \\
&= \mathrm{tr}\left(B^{1-x} A^{2x} B^{1-x}\right) + \mathrm{tr}\left(A^{1-x} B^{2x} A^{1-x}\right) \\
&= \mathrm{tr}\ A^{2x} B^{2(1-x)} + \mathrm{tr}\ A^{2(1-x)} B^{2x} \\
&\leq \left(\mathrm{tr}\ A^2\right)^x \left(\mathrm{tr}\ B^2\right)^{1-x} + \left(\mathrm{tr}\ A^2\right)^{1-x} \left(\mathrm{tr}\ B^2\right)^x \\
&\leq \mathrm{tr}\ A^2 + \mathrm{tr}\ B^2 \text{ (by the inequalities (1.1))} \\
&= \mathrm{tr}\left(A^2 + B^2\right).
\end{aligned}$$

This completes the proof. □

Based on Theorem 4.1, we have the following trace inequality involving b_t, which is related to the second inequality in (1.4).

Corollary 4.2 *Let A and B be positive semidefinite matrices and let $\mathcal{S} = \{z \in \mathbb{C} : 0 \leq Re\ z \leq 1\}$. Then*

$$Re\ \mathrm{tr}\left(A^z B^{1-z} + B^z A^{1-z}\right)^2 \leq \mathrm{tr}\,(A + B)^2 \tag{4.2}$$

for all $z \in \mathcal{S}$.

In other words,

$$Re\ \text{tr}\,(b_z)^2 \leq \text{tr}\,(b_1)^2$$

for all $z \in \mathcal{S}$.

In particular, if $z = t \in [0, 1]$, *we have*

$$\text{tr}\,(b_t)^2 \leq \text{tr}\,(b_1)^2 .$$

Proof We can see that inequality (4.2) is equivalent to the statement

$$\text{Re}\ \text{tr}\left(\left(A^z B^{1-z}\right)^2 + \left(B^z A^{1-z}\right)^2\right) \leq \text{tr}\left(A^2 + B^2\right). \tag{4.3}$$

Thus, our goal is to show that the inequality (4.3) holds provided $z \in \mathcal{S}$. For $z \in \mathcal{S}$, we have

$$\begin{aligned}\text{Re}\ \text{tr}\left(\left(A^z B^{1-z}\right)^2 + \left(B^z A^{1-z}\right)^2\right) &\leq \left|\text{tr}\left(\left(A^z B^{1-z}\right)^2 + \left(B^z A^{1-z}\right)^2\right)\right| \\ &\leq \text{tr}\left(A^2 + B^2\right) \text{ (by Theorem 4.1)} .\end{aligned}$$

Hence, the inequality (4.2) is valid provided $z \in \mathcal{S}$. □

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