

Riesz* homomorphisms on pre-Riesz spaces consisting of continuous functions

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Abstract In the theory of operators on a Riesz space (vector lattice), an important result states that the Riesz homomorphisms (lattice homomorphisms) on $C(X)$ are exactly the weighted composition operators. We extend this result to Riesz* homomorphisms on order dense subspaces of $C(X)$. On those subspace we consider and compare various classes of operators that extend the notion of a Riesz homomorphism. Furthermore, using the weighted composition structure of Riesz* homomorphisms we obtain several results concerning bijective Riesz* homomorphisms. In particular, we characterize the automorphism group for order dense subspaces of $C(X)$. Lastly, we develop a similar theory for Riesz* homomorphisms on subspace of $C_0(X)$, for a locally compact Hausdorff space X , and apply it to smooth manifolds and Sobolev spaces.

Keywords Partially ordered vector space · Order dense subspace · Pre-Riesz space · Riesz* homomorphism · Weighted composition operator · Automorphism group · Smooth manifold · Sobolev space

Mathematics Subject Classification Primary 46A40; Secondary 06F20

1 Introduction

In the theory of Riesz spaces (vector lattices) different classes of operators and their properties have been studied extensively. In particular, Riesz homomorphisms (lattice homomorphisms) are a main focus in this study. We cite an important characterization

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theorem of Riesz homomorphisms between $C(X)$ -spaces, namely Theorem 4.25 in [1] adapted to notation and terminology that will be used in this article.

Theorem 1.1 *Let X and Y be compact Hausdorff spaces. A positive operator $T : C(X) \rightarrow C(Y)$ is a Riesz homomorphism if and only if there exist a map $\pi : Y \rightarrow X$ and a weight function $\eta \in C(Y)^+$ such that we have*

$$(Tf)(y) = \eta(y)f(\pi(y)), \quad f \in C(X), \quad y \in Y. \quad (1)$$

Moreover, in this case, $\eta = T\mathbb{1}_X$ and the map π is uniquely determined and continuous on the set $\{\eta > 0\}$.

Here an operator $T : C(X) \rightarrow C(Y)$ satisfying Eq. (1) is called a *weighted composition operator*. Our aim is to extend the above theorem to partially ordered vector subspaces of $C(X)$. Our approach is to investigate operators defined on a subspace of $C(X)$ that extend to Riesz homomorphisms on $C(X)$. We use a theory developed by Van Haandel [11] on extension results in Riesz space theory. He introduced the notion of *pre-Riesz spaces* that turn out to be exactly the partially ordered vector space that can be embedded as an order dense subspace of a Riesz space called the *Riesz completion*. He characterizes the class of operators between pre-Riesz spaces that extend to Riesz homomorphisms between the corresponding Riesz completions. Precise definitions and relevant results are discussed in Sect. 2. Phrased in those words, our main goal is to determine a suitable class of pre-Riesz subspaces of $C(X)$ on which any Riesz* homomorphism is a weighted composition operator. Our main result entails that Riesz* homomorphisms defined on a subspace E of $C(X)$ that is order dense and separates the points of X are weighted composition operators. This result allows us to further investigate Riesz* homomorphisms on subspaces E of $C(X)$. In particular, we exhibit conditions, imposed on η and π , under which an operator of the form (1) is a complete Riesz homomorphism. Moreover, the weighted composition structure of Riesz* homomorphisms allows us to prove various results on bijective Riesz* homomorphism. One of which allows us to extend order isomorphisms on E to $C(X)$ yielding tools to characterize the automorphism group of E .

Lastly, we generalize our results to Riesz* homomorphisms on $C_0(X)$ where X need only be locally compact and apply the theory to spaces of differentiable functions upto arbitrary order on a locally compact smooth manifold and Sobolev spaces on domains in \mathbb{R}^d satisfying some regularity conditions.

2 Preliminaries

Let (E, \leq) be a partially ordered vector space. Often we will only write E for the pair (E, \leq) . The space E is called *Archimedean* if for every $f, g \in E$ with $nf \leq g$ for all $n \in \mathbb{N}$ one has $f \leq 0$. A set $G \subseteq E$ is called *directed* if for every $f, g \in G$ there is an element $e \in G$ such that $e \geq f$ and $e \geq g$. Suppose E is a subspace of a Riesz space F , then E is said to be *order dense* in F if for every $f \in F$ one has $f = \inf\{g \in E : g \geq f\}$. Partially ordered vector spaces that can be embedded as an order dense subspace in a Riesz space are characterized by Van Haandel in

[11] as follows. E is called a *pre-Riesz space* if for every $f, g, h \in E$ such that $\{f + g, f + h\}^u \subseteq \{g, h\}^u$ one has $f \geq 0$. E is a pre-Riesz space if and only if there exists a Riesz space F and a bipositive linear map $i : E \rightarrow F$ such that $i(E)$ is order dense in F and generates F as a Riesz space. Moreover, in this case all spaces F satisfying this property are isomorphic as Riesz spaces. We adopt the terminology to call the Riesz space F the *Riesz completion* of E and denote it by E^ρ . Van Haandel proceeds by showing also that every pre-Riesz space is directed and every directed Archimedean partially ordered vector space is a pre-Riesz space. A summary of Van Haandel's results can also be found in [10].

A natural class of operators to study between pre-Riesz spaces are the operators that extend to a Riesz homomorphism between the Riesz completions. A linear operator $T : E \rightarrow F$ between pre-Riesz space is called a *Riesz* homomorphism* if for every $f, g \in E$ one has $T(\{f, g\}^{ul}) \subseteq \{Tf, Tg\}^{ul}$ in F .

Theorem 2.1 (Van Haandel, p. 26 in [11]) *A linear operator $T : E \rightarrow F$ between pre-Riesz spaces is a Riesz* homomorphism if and only if it extends to a Riesz homomorphism $T^\rho : E^\rho \rightarrow F^\rho$.*

Let X be a compact Hausdorff space and $C(X)$ the space of all real-valued continuous functions on X . Endowed with the partial order defined by $f \geq g$ if and only if $f(x) \geq g(x)$ holds for all $x \in X$, $C(X)$ is an Archimedean Riesz space. In this article we are interested in characterizing the above defined Riesz* homomorphisms on a suitable class of subspaces in $C(X)$. Our strategy is to extend such a Riesz* homomorphism on a pre-Riesz subspace of $C(X)$ to a Riesz homomorphism on the Riesz completion, then use results in operator theory of Riesz spaces to extend this Riesz homomorphism to $C(X)$ and apply Theorem 1.1. Therefore, we are interested in subspaces E of $C(X)$ that are pre-Riesz spaces and, moreover, that their Riesz completion E^ρ is a subspace of $C(X)$. We argue that E being order dense in $C(X)$ is sufficient to satisfy these two properties. Suppose $E \subseteq C(X)$ is an order dense subspace. In particular, E is then a majorizing subspace of the Archimedean space $C(X)$ and hence a pre-Riesz space. Moreover, the Riesz subspace G of $C(X)$ generated by E is an order dense Riesz subspace of $C(X)$. Due to the uniqueness of a Riesz completion we conclude that E^ρ equals G and, hence, can be viewed as a Riesz subspace of $C(X)$. In the case that E is a Riesz subspace of $C(X)$, order denseness in $C(X)$ is equivalent with the property that for all $f \in C(X)$ with $f > 0$, there exists a $g \in E$ satisfying $0 < g \leq f$. Here the notation $f > 0$ is used whenever f satisfies both $f \geq 0$ and $f \neq 0$. In our case of general pre-Riesz spaces this equivalence does not always hold. However, the latter condition is of interest to us and we take it as the definition of E being *pervasive* in $C(X)$. This corresponds to the definition given in [8].

3 Weighted composition operators on subspaces of $C(X)$

Let E and F be order dense subspaces of $C(X)$ and $C(Y)$, respectively, where X and Y are compact Hausdorff spaces. Recall from Sect. 2 that E and F are pre-Riesz spaces and that E^ρ and F^ρ can be viewed as order dense Riesz subspaces of $C(X)$ and

$C(Y)$, respectively. Let $T : E \rightarrow F$ be a Riesz* homomorphism and $T^\rho : E^\rho \rightarrow F^\rho$ the Riesz homomorphism that extends T . An approach to show that T is a weighted composition operator is to extend T^ρ further to a Riesz homomorphism between $C(X)$ and $C(Y)$ and apply the general theory. However, generally not every Riesz* homomorphism on E is the restriction of a Riesz homomorphism on $C(X)$, which is illustrated in the following example.

Example 3.1 Let X be the unit interval and E the space consisting of continuous functions on X that satisfy $f(0) = f(1)$. E is an order dense Riesz subspace of $C(X)$. Let $T : E \rightarrow E$ be the *inside-out operator* defined by $(Tf)(x) = f(\pi(x))$ for all $f \in E$, where

$$\pi(x) = \frac{1}{2} - x + \mathbb{1}_{\left[x > \frac{1}{2}\right]},$$

holds for all $x \in X$. Clearly, π is discontinuous at $x = \frac{1}{2}$. However, for all $f \in E$ we have $f(0) = f(1)$ and, hence, that Tf is continuous on X . Moreover, we obtain the equalities $(Tf)(0) = f(\frac{1}{2}) = (Tf)(1)$ for all $f \in E$, which shows that T is a well-defined operator from E into E . T is a Riesz homomorphism on the order dense Riesz subspace E of $C(X)$. From the fact that weighted composition operators on $C(X)$ automatically have unique continuous composition maps we conclude that T does not extend to a Riesz homomorphism on $C(X)$.

This motivates us to impose an extra condition on E in order to obtain automatic continuity of weight and composition maps for every weighted composition operator on E . E is said to *separate the points of X* , or simply to be *separating* when no confusion can arise what underlying topological space is considered, if for any $x, y \in X$, $x \neq y$, there exists an $f \in E$ with $f(x) = 0$ and $f(y) = 1$.

It turns out that this condition is sufficient to guarantee that the Riesz* homomorphisms on E are exactly the weighted composition operators that extend to $C(X)$. In the proof of this assertion, we aim to apply the Riesz space theory to a Riesz homomorphism on $C(X)$ that extends T . Therefore, a crucial step in our proof is to extend T^ρ further to a Riesz homomorphism on $C(X)$. Notice that this cannot be done generally by the Lipecki–Luxemburg–Schep Theorem (see 4.36 in [1]), since $C(Y)$ fails to be Dedekind complete. Keeping in mind that our desired structure for T , namely Eq. (2), is determined pointwise, our strategy is to compose T^ρ with a point evaluation on $C(Y)$ to obtain a Riesz homomorphism from E^ρ to the Dedekind complete space \mathbb{R} .

Theorem 3.2 *Let E and F be order dense subspaces of $C(X)$ and $C(Y)$, respectively.*

- (i) *If $T : E \rightarrow F$ is a Riesz* homomorphism, then there exist $\eta : Y \rightarrow \mathbb{R}^+$ and $\pi : Y \rightarrow X$ such that*

$$(Tf)(y) = \eta(y)f(\pi(y)), \quad f \in E, \quad y \in Y. \quad (2)$$

Moreover, if E , in addition, separates the points of X , then η and π can be taken continuous on Y and $\{\eta > 0\}$, respectively, and π is uniquely determined on $\{\eta > 0\}$.

(ii) Any linear operator $T : E \rightarrow F$ that satisfies (2) for some $\eta \in C(Y)^+$ and $\pi : Y \rightarrow X$ continuous on $\{\eta > 0\}$ is a Riesz* homomorphism.

Proof Suppose E and F are given as in the first statement of (i) and $T : E \rightarrow F$ is a Riesz* homomorphism. Let $T^\rho : E^\rho \rightarrow F^\rho$ be the Riesz homomorphism that extends T . Since F is order dense in $C(Y)$, F^ρ is a Riesz subspace of $C(Y)$. Let us fix some $y \in Y$ and define the Riesz homomorphism $T_y : E^\rho \rightarrow \mathbb{R}$ as the composition of T^ρ with the point evaluation in y , i.e., $T_y f = (T^\rho f)(y)$ for all $f \in E^\rho$. We apply Theorem 4.36 in [1] to T_y , which can be done since E^ρ is a majorizing Riesz subspace in $C(X)$ and \mathbb{R} is Dedekind complete, and obtain a Riesz homomorphism $\hat{T}_y : C(X) \rightarrow \mathbb{R}$ that extends T_y . Lemma 4.23 in [1] characterizes functionals on $C(X)$ that are Riesz homomorphisms. Namely, there exist $\eta(y) \in \mathbb{R}^+$ and $\pi(y) \in X$ such that \hat{T}_y is given by $\hat{T}_y f = \eta(y)f(\pi(y))$, for all $f \in C(X)$. Applying the above reasoning for all $y \in Y$ and recalling that \hat{T}_y extends T_y yields the desired result that Eq. (2) holds.

In order to show the second part of (i), suppose that E , in addition, separates the points of X . We redefine $\eta(y)$ to be zero whenever $(Tf)(y) = 0$ holds for all $f \in E$. Equation (2) remains satisfied.

Suppose η is not bounded. Let $f \in E$ be greater than the constant one function, then Tf is not bounded, which contradicts that Tf is continuous. Therefore, we conclude that η is bounded.

Fix $y \in Y$. We aim to show that η is continuous at y and that π is continuous at y whenever $\eta(y)$ is non-zero. To this end, we will show that every net (y_α) in Y which converges to y has a subnet (v_β) such that $\lim_\beta \eta(v_\beta) = \eta(y)$ and $(\lim_\beta \pi(v_\beta) = \pi(y) \vee \eta(y) = 0)$ hold. Let (y_α) be a net in Y which converges to y . For any $f \in E$ the function Tf is continuous and we get

$$\eta(y)f(\pi(y)) = (Tf)(y) = \lim_\alpha (Tf)(y_\alpha) = \lim_\alpha \eta(y_\alpha)f(\pi(y_\alpha)). \tag{3}$$

Since X is compact and $\eta(Y)$ is bounded in \mathbb{R} , there exists a subnet of (y_α) , which we will denote by (v_β) , such that $(\eta(v_\beta))_\beta$ converges to say $t \in \mathbb{R}$ and such that $(\pi(v_\beta))_\beta$ converges to say $x \in X$. Therefore, for any $f \in E$ we get, due to Eq. (3) and the fact that f is bounded, $(Tf)(y) = t \lim_\beta f(\pi(v_\beta))$. Moreover, by continuity of f , we obtain $(Tf)(y) = tf(\lim_\beta \pi(v_\beta)) = tf(x)$. In particular, $\eta(y)f(\pi(y)) = tf(x)$ holds. Suppose $\eta(y) = 0$ holds, then t equals zero due to the existence of an $f \in E$ such that $f(x) \neq 0$ by the majorizing property of E . Therefore, $(y_\alpha)_\alpha$ has a subnet $(v_\beta)_\beta$ such that $\lim_\beta \eta(v_\beta) = \eta(y)$.

In the remaining case, namely if $\eta(y) > 0$ holds, the majorizing property of E similarly guarantees that $t > 0$ holds. Consequently, for any $f \in E$ the equation $f(x) = cf(\pi(y))$ is satisfied, where $c = \eta(y)/t$ is non-zero and independent of f . Since E separates the points of X , we obtain the equalities $x = \pi(y)$ and $c = 1$. We conclude that $(y_\alpha)_\alpha$ has a subnet $(v_\beta)_\beta$ such that $\lim_\beta \pi(v_\beta) = \pi(y)$ and, hence, that η is indeed continuous at y . Moreover, Eq. (3) reduces to $\eta(y)f(\pi(y)) = \lim_\beta \eta(v_\beta)f(\pi(y))$, hence, applying to a function $f \in E$ with $f(\pi(y)) \neq 0$ yields $\eta(y) = \lim_\beta \eta(v_\beta)$. We conclude that η is continuous at y . Additionally, π is uniquely determined on $\{\eta > 0\}$ due to E separating the points of X .

Lastly, suppose that $T : E \rightarrow F$ satisfies (2) for some $\eta \in C(Y)^+$ and $\pi : Y \rightarrow X$ continuous on $\{\eta > 0\}$. The weighted composition operator between the Riesz completions $E^\rho \subseteq C(X)$ and $F^\rho \subseteq C(Y)$ defined by η and π is a well-defined Riesz homomorphism that extends T , hence T is a Riesz* homomorphism. \square

Henceforth, for notational convenience let $T_{\eta,\pi} : E \rightarrow F$ denote the weighted composition operator between E and F with weight map $\eta : Y \rightarrow \mathbb{R}^+$ and composition map $\pi : Y \rightarrow X$, i.e.,

$$(T_{\eta,\pi} f)(y) = \eta(y)f(\pi(y)), \quad f \in E, \quad y \in Y.$$

A predecessor of the Riesz* homomorphisms are the Riesz homomorphisms, an alternative class of operators on a pre-Riesz space E that extend to a Riesz homomorphism on E^ρ . A linear operator $T : E \rightarrow F$ between pre-Riesz spaces is called a *Riesz homomorphism* if $T(\{f, g\}^u)^l \subseteq \{Tf, Tg\}^{ul}$ holds for all $f, g \in E$. Similarly to Riesz* homomorphisms these operators extend to Riesz homomorphisms, but not all Riesz homomorphisms between the completions are obtained as extensions. Also, a composition of Riesz homomorphisms between pre-Riesz spaces is generally not a Riesz homomorphism. These defects have motivated the definition of a Riesz* homomorphism.

Suppose E and F are pre-Riesz spaces and $T : E \rightarrow F$ is a positive linear operator. Clearly, for any $f, g \in E$ we obtain $T(\{f, g\}^{ul}) \subseteq T(\{f, g\}^u)^l$. Therefore, any Riesz homomorphism is a Riesz* homomorphism. A converse statement does not generally hold on pre-Riesz spaces, which will be illustrated later by a counterexample in Example 3.5. However, we show that on a wide class of subspaces of $C(X)$, which is contained in the class of separating order dense subspaces, the notions of a Riesz homomorphism and a Riesz* homomorphism coincide.

A subspace E of $C(X)$ is called *pointwise order dense* if it satisfies $f(x) = \inf\{g(x) : g \in E, g \geq f\}$ (in \mathbb{R}), for all $f \in C(X)$ and $x \in X$. Clearly, this condition is a pointwise version of being order dense in $C(X)$, which justifies its name. Moreover, any pointwise order dense subspace E of $C(X)$ is separating and order dense. It is routine to show that a norm dense subspace of $C(X)$ containing the constant functions is pointwise order dense. Before proceeding, we exhibit several examples of subspaces of $C[0, 1]$ to highlight differences between order theoretic properties defined so far.

Example 3.3 Let X be the unit interval.

- (i) For any $k \in \mathbb{N} \cup \{\infty\}$ the space $C^k([0, 1])$ of continuously differentiable functions on $[0, 1]$ upto order k is pervasive, norm dense and pointwise order dense in $C(X)$.
- (ii) The *Namioka space* defined by $\mathcal{N} = \{f \in C(X) : f(0) + f(1) = 2f(\frac{1}{2})\}$ is a pervasive, order dense and norm dense subspace of $C(X)$. However, \mathcal{N} is not pointwise order dense in $C(X)$.
- (iii) $P[0, 1]$ the space consisting of all polynomials on $[0, 1]$ is pointwise order dense and norm dense, however, not pervasive in $C(X)$.
- (iv) $P_2[0, 1]$ the space consisting of all polynomials on $[0, 1]$ of degree up to 2 is a 3-dimensional pointwise order dense subspace of $C(X)$, as follows from arguments in Example 4.4 in [9], that is not norm dense.

Suppose that E and F are pre-Riesz spaces, $T : E \rightarrow F$ is a Riesz homomorphism and $f, g \in E$ are given. In the Riesz space F^ρ we obtain the equality $\{Tf, Tg\}^{ul} = \{Tf \vee Tg\}^l$. Since T is necessarily positive we obtain

$$\inf\{Te : e \in E, e \geq f, g\} = Tf \vee Tg \text{ in } F^\rho, \quad f, g \in E. \tag{4}$$

Moreover, a linear operator $T : E \rightarrow F$ is a Riesz homomorphism if and only if it satisfies (4). Using this characterization and Theorem 3.2 we prove that any Riesz* homomorphism on a pointwise order dense subspace of $C(X)$ is automatically a Riesz homomorphism.

Theorem 3.4 *Let E be pointwise order dense in $C(X)$, F be order dense in $C(Y)$ and $T : E \rightarrow F$ be a linear operator. T is a Riesz* homomorphism if and only if T is a Riesz homomorphism.*

Proof Let E and F be given as in the statement and suppose $T : E \rightarrow F$ is a Riesz* homomorphism. Due to Theorem 3.2 there exist suitable η and π such that $T = T_{\eta, \pi} : E \rightarrow F$ holds. Suppose $f, g \in E$ are given. Since T is positive, $Tf \vee Tg$ is a lower bound of $\{Te : e \in E, e \geq f, g\}$ in F^ρ . Next, let h be a lower bound of $\{Te : e \in E, e \geq f, g\}$ in F^ρ . For any $y \in Y$ we have

$$\begin{aligned} h(y) &\leq \inf \{(Te)(y) : e \in \{f, g\}^u\} \\ &= \inf \{\eta(y)e(\pi(y)) : e \in \{f, g\}^u\} \\ &= \eta(y) (\inf \{e(\pi(y)) : e \in \{f \vee g\}^u\}) \\ &= \eta(y)(f \vee g)(\pi(y)) = (Tf \vee Tg)(y). \end{aligned}$$

Therefore, $h \leq Tf \vee Tg$ holds, which shows that T satisfies condition (4). Here we used that E is pointwise order dense in $C(X)$ in the second last equality on $f \vee g \in C(X)$ and $\pi(y) \in X$. Recall that the other implication holds for general pre-Riesz spaces as discussed earlier. □

We conclude this section with an example that shows that the above theorem does not hold generally for separating order dense subspaces of $C(X)$.

Example 3.5 Let \mathcal{N} be the Namioka space as defined in Example 3.3(ii) and recall that \mathcal{N} is indeed separating and order dense, however, not pointwise order dense in $C(X)$. Let $T : \mathcal{N} \rightarrow \mathbb{R}$ be the functional that composes any $f \in \mathcal{N}$ by the point evaluation at $x = \frac{1}{2}$. T is a Riesz* homomorphism by Theorem 3.2. However, letting $f, g \in \mathcal{N}$ be defined by $f(x) = x$ and $g(x) = 1 - x$, we obtain $Tf \vee Tg = \frac{1}{2} \vee \frac{1}{2} = \frac{1}{2}$, while for any $e \in \mathcal{N}$ with $e \geq f, g$ we get $e(0) \geq g(0) \geq 1$ and $e(1) \geq f(1) \geq 1$, hence $Te = e(\frac{1}{2}) \geq 1 > Tf \vee Tg$. Therefore, T does not satisfy condition (4) and, hence, is not a Riesz homomorphism.

4 Complete Riesz homomorphisms

Recall that the Riesz* homomorphisms on a separating order dense subspace E of $C(X)$ are exactly the weighted composition operators, due to Theorem 3.2. Using

the structure of Riesz* homomorphisms on E we will investigate another type of homomorphism, namely the *complete Riesz homomorphisms*, which encompasses the Riesz* homomorphisms. First introduced and studied by Buskes and van Rooij [5], complete Riesz homomorphisms are exactly the operators between pre-Riesz spaces that extend to order continuous Riesz homomorphisms between the completions (see [11]). We aim to characterize the complete Riesz homomorphisms between order dense subspaces of $C(X)$ and at the same time characterize the order continuous Riesz homomorphisms between Riesz subspaces of $C(X)$. More specifically, our aim is to determine a necessary condition imposed on η and π that guarantees the operator $T_{\eta,\pi} : E \rightarrow F$ to be a complete Riesz homomorphism.

Definition 4.1 Let E and F be partially ordered vector spaces. A linear operator $T : E \rightarrow F$ is called a *complete Riesz homomorphism* if for any $G \subseteq E$, $\inf G = 0$ implies $\inf T(G) = 0$.

This definition is given for general partially ordered vector spaces. In this generality any complete Riesz homomorphism is a Riesz homomorphism and, hence, a Riesz* homomorphism (see [11]). A counterexample to the converse can easily be constructed as follows. Let $T = T_{\eta,\pi} : C[0, 1] \rightarrow C[0, 1]$ where η is positive and non-vanishing and π is constant. There exists a sequence in $C[0, 1]$ that descends to zero and is constantly one on the singleton $\pi([0, 1])$. Therefore, T is not a complete Riesz homomorphism. It holds generally that $\eta \geq 0$ non-vanishing and π being an open map suffice to guarantee that $T_{\eta,\pi}$ is a complete Riesz homomorphism. However, imposing openness on π is not necessary as will be shown in Theorem 4.4.

A function $\pi : Y \rightarrow X$ is called *weak open* if for all non-empty $U \subseteq Y$ open the image $\pi(U)$ is dense somewhere, i.e., there exists a non-empty $V \subseteq X$ open such that $\pi(U) \cap V$ is dense in V , and π is called *nowhere constant* if for all non-empty $U \subseteq Y$ open the image $\pi(U)$ is not a singleton. Obviously the former implies the latter and π being open implies both properties.

When investigating complete Riesz homomorphisms, it is convenient to characterize subsets of E with infimum equal to zero.

Lemma 4.2 Let E be an order dense subspace of $C(X)$ and let $G \subseteq E^+$ be given. Then $\inf G = 0$ holds in E if and only if

$$\forall \epsilon > 0, U \subseteq X \text{ non-empty and open } \exists f \in G, y \in U \text{ such that } f(y) \leq \epsilon. \quad (5)$$

Proof Let $G \subseteq E^+$ be given satisfying $\inf G = 0$ in E . Suppose $\epsilon > 0$ and $U \subseteq X$ be non-empty and open such that for all $f \in G$ and $y \in U$ we have $f(y) > \epsilon$. Clearly, $\epsilon \mathbb{1}$ is a lower bound of G in $C(X)$. E is assumed to be order dense in $C(X)$, hence there exists a $g \in E$ with $g \not\leq 0$ and $g \leq \epsilon \mathbb{1}$. However, this contradicts the assumption that $\inf G = 0$ holds in E , hence (5) holds.

Next, suppose $G \subseteq E^+$ does not satisfy $\inf G = 0$. There exists a lower bound $g \in E$ of G such that $g \not\leq 0$. The positive part g^+ of g is a non-zero positive element of $C(X)$ and a lower bound of G . Since g^+ is continuous, there exists an $\epsilon > 0$ and a $U \subseteq X$ non-empty and open such that $f(y) \geq g^+(y) > \epsilon$ holds for all $y \in U, f \in G$. \square

Another useful tool when investigating complete Riesz homomorphisms on pre-Riesz spaces is the following extension lemma.

Lemma 4.3 *Suppose E and F are pre-Riesz spaces and $T : E \rightarrow F$ is a complete Riesz homomorphism, then $T^\rho : E^\rho \rightarrow F^\rho$ is a complete Riesz homomorphism.*

Proof Let $G \subseteq (E^\rho)^+$ be given such that $\inf G = 0$ holds and define the set

$$B := \{f \in E : \exists g \in G, f \geq g\}.$$

Since E is order dense in E^ρ we obtain $\inf B = 0$. In particular, $\inf T(B) = 0$ holds as T is assumed to be a complete Riesz homomorphism. In order to show that the infimum of $T^\rho(G)$ equals zero, we note that zero is a lower bound of $T^\rho(G)$ due to the positivity of T^ρ . Suppose $h \in F^\rho$ is another lower bound of $T^\rho(G)$. For any $f \in B$ there exists a $g \in G$ by construction such that $f \geq g$ holds and, hence, we get $Tf = T^\rho f \geq T^\rho g \geq h$. Therefore, h is a lower bound of the set $T(B)$ that has infimum equal to zero and thus is negative. \square

Arriving at the main result of this section, we note that no additional conditions are imposed on the subspace F of $C(Y)$.

Theorem 4.4 *Let E be an order dense subspace of $C(X)$ and F a subspace of $C(Y)$. Let $\eta \in C(Y)^+$ and $\pi : Y \rightarrow X$ be such that $T_{\eta,\pi} : E \rightarrow C(Y)$ maps into F . $T_{\eta,\pi} : E \rightarrow F$ is a complete Riesz homomorphism if and only if π is weak open on $\{\eta > 0\}$.*

Proof Let η and π be as in the statement and denote $T = T_{\eta,\pi}$. Suppose π is weak open on $\{\eta > 0\}$ and $G \subseteq E$ is given with $\inf G = 0$. Let us put $M := \sup\{\eta(y) : y \in Y\}$ and fix $\delta > 0$ and $U \subseteq Y$ non-empty and open. Due to Lemma 4.2 it suffices to show that there exists an $f \in G$ and $y \in U$ such that $Tf(y) \leq \delta$ holds. Suppose there exists a $y \in U \cap \{\eta = 0\}$, then for all $f \in G$ we have $(Tf)(y) = 0 < \delta$ and, hence, we are done. We assume that $U \subseteq \{\eta > 0\}$ holds and let $V \subseteq X$ be non-empty and open with $\pi(U) \cap V$ dense in V . Such a V exists due to π being weak open on $\{\eta > 0\}$. Letting $\epsilon := \delta(M + 1)^{-1} > 0$, there exists a $f \in G$ and $x \in V$ with $f(x) \leq \frac{\epsilon}{2}$. Therefore, we can find an $x_0 \in \pi(U) \cap V$ with $f(x_0) \leq \epsilon$, since f is continuous. Let $y_0 \in \pi^{-1}(\{x_0\}) \cap U$ and observe that we get

$$(Tf)(y_0) = \eta(y_0)f(\pi(y_0)) \leq Mf(\pi(y_0)) = Mf(x_0) \leq M\epsilon \leq \delta.$$

Hence, due to Lemma 4.2, T is a complete Riesz homomorphism.

Conversely, suppose that π is not weak open on $\{\eta > 0\}$. In other words, there exist $\delta > 0$ and non-empty and open $U \subseteq Y$ with $U \subseteq \{\eta \geq \delta\}$ and $\pi(U)$ is nowhere dense in X . Our aim is to show that $\hat{T} = T_{\eta,\pi} : E^\rho \rightarrow F^\rho$ is not a complete Riesz homomorphism. In that case Lemma 4.3 yields the desired contradiction.

Let us define

$$G := \{f \in E^\rho : f \geq 0 \text{ and } f \geq 1 \text{ on } \pi(U)\}.$$

We argue that $\inf G = 0$ holds. Suppose it does not hold, then there exists a lower bound of G , $g \in E^\rho$ with $g > 0$, since E^ρ is a Riesz space. In particular, there exist $\epsilon > 0$ and $W \subseteq X$ non-empty and open such that $g \geq \epsilon$ holds on W . Recall that $\pi(U)$ is nowhere dense, so $\pi(U) \cap W$ is not dense in W . Therefore, the closure of $\pi(U) \cap W$ taken in W , $V := \overline{\pi(U) \cap W}^W$, is a closed strict subset of W . So there exists a $W_0 \subseteq W$ non-empty and open with $\overline{W_0} \cap V = \emptyset$. Thus there is a $f_0 \in C(X)^+$ with $f_0 = 1$ on $V \supseteq \pi(U)$ and $f_0 = 0$ on W_0 . As E^ρ is order dense in $C(X)$, there exists an $f \in G$ with $f(x) < \epsilon$ for some $x \in W_0 \subseteq W$, which contradicts that g is a lower bound of G . However, any $g \in T(G)$ satisfies $g \geq \delta$ on the non-empty open set U . Since F^ρ is a Riesz subspace of $C(Y)$, F^ρ is pervasive and there exists an $f \in (F^\rho)^+$ with $f > 0$ on U that is a lower bound of $T(G)$. \square

Let us remark that Theorem 4.4 shows, in particular, that the order continuous Riesz homomorphisms between order dense Riesz subspaces of $C(X)$ and $C(Y)$ are exactly the composition multiplication operators where the composition map is weak open on the set where the multiplication map is non-zero.

Concluding this section we remark that, in the special case that X and Y are bounded and closed intervals of \mathbb{R} , the weak openness of π on $\{\eta > 0\}$ clause in the above theorem can be replaced by π being nowhere constant on $\{\eta > 0\}$.

Proposition 4.5 *Let X and Y be bounded and closed intervals in \mathbb{R} , E an order dense subspace of $C(X)$, F a subspace of $C(Y)$, $\eta \in C(Y)^+$ and $\pi : Y \rightarrow X$ a map that is continuous on $\{\eta > 0\}$. The linear operator $T_{\eta, \pi} : E \rightarrow F$ is a complete Riesz homomorphism if and only if π is nowhere constant on $\{\eta > 0\}$.*

Proof Due to Theorem 4.4 it suffices to verify the equivalence of π being weak open and being nowhere constant on some open subset of Y . Clearly the former implies the latter. Suppose that $U \subseteq Y$ is non-empty and open and π is nowhere constant on U . In particular, there exist distinct points $x, y \in X$ contained in $\pi(U)$. We can find $a, b \in U$ with $a < b$, $\pi(a) = x$ and $\pi(b) = y$ or the other way around. Without loss of generality we assume that $x < y$ holds. Restrict π to the continuous map $\hat{\pi} : [a, b] \rightarrow X$. For any $x < z < y$ we can find, by the Intermediate Value Theorem, a $c \in (a, b)$ with $\pi(c) = z$. Therefore, (x, y) is contained in $\pi(U)$ and we conclude that π is weak open on U .

5 Bijective homomorphisms

In the previous sections we have considered three types of homomorphisms defined on pre-Riesz spaces, namely, in order from weak to strong, Riesz* homomorphisms, Riesz homomorphisms and complete Riesz homomorphisms. Moreover, we characterized them on separating order dense subspaces of $C(X)$. We turn our attention in this section to bijective homomorphisms. More precisely, we address problems concerning their inverses. Under what conditions on E and F is the inverse of a Riesz* homomorphism $T : E \rightarrow F$ necessarily of the same type? What are necessary and sufficient conditions imposed on η and π under which $T_{\eta, \pi} : E \rightarrow F$ is an order isomorphism whenever E and F are order dense separating subspaces of $C(X)$? We start by introducing order isomorphisms and making elementary observations in a general setting.

Let E and F be partially ordered vector spaces and $T : E \rightarrow F$ a linear operator. T is called *bipositive* if for any $f \in E$ one has $f \geq 0$ if and only if $Tf \geq 0$ holds. An *order isomorphism* is a bipositive linear bijection. Recall that a bipositive linear operator is injective. The following observation on order isomorphisms holds in a general setting where E and F need not even be Archimedian or directed.

Lemma 5.1 *Suppose E and F are partially ordered vector spaces and $T : E \rightarrow F$ is a linear operator. Then T is an order isomorphism if and only if T is bijective and both T and T^{-1} are complete Riesz homomorphisms.*

Proof Suppose T is an order isomorphism. It suffices to show that T is a complete Riesz homomorphism, as T^{-1} is also an order isomorphism. To this end, let $G \subset E^+$ satisfy $\inf G = 0$. Suppose $f \in F$ is a lower bound of $T(G)$, then $T^{-1}f$ is a lower bound of G . Therefore, $T^{-1}f \leq 0$ holds as the infimum of G equals zero. Applying the positivity of T again yields $TT^{-1}f = f \leq 0$, which proves that T is a complete Riesz homomorphism.

For the converse, it suffices to observe that complete Riesz homomorphisms T are positive. Suppose f is positive and let $G := \{0, f\}$ then, in particular, $\inf G = 0$ and hence $\inf T(G) = 0$ and $Tf \geq 0$ holds. □

Lemma 5.2 *Suppose E and F are pre-Riesz spaces, $T : E \rightarrow F$ is a Riesz* homomorphism and $T^\rho : E^\rho \rightarrow F^\rho$ is the Riesz homomorphism that extends T , then the following statements hold:*

- (i) *If T is surjective, then T^ρ is surjective.*
- (ii) *If E is pervasive and T is injective, then T^ρ is injective.*

Proof (i): Let $g \in F^\rho$ be given and let $f_1, \dots, f_n, g_1, \dots, g_m \in F$ be such that $g = \bigwedge_{i=1}^n f_i + \bigvee_{j=1}^m g_j$ holds in F^ρ . As T is surjective there are $a_1, \dots, a_n, b_1, \dots, b_m \in E$ with $f_i = T(a_i)$ and $g_j = T(b_j)$, $i = 1, \dots, n$ and $j = 1, \dots, m$. We define $f := \bigwedge_{i=1}^n a_i + \bigvee_{j=1}^m b_j$ and observe that $f \in E^\rho$. The image of this f under T^ρ is computed as follows

$$\begin{aligned} T^\rho f &= T^\rho \left(\bigwedge_{i=1}^n a_i + \bigvee_{j=1}^m b_j \right) \\ &= \bigwedge_{i=1}^n T^\rho a_i + \bigvee_{j=1}^m T^\rho b_j \\ &= \bigwedge_{i=1}^n f_i + \bigwedge_{j=1}^m g_j = g, \end{aligned}$$

hence T is surjective.

(ii): Suppose E is pervasive and let $f \in (E^\rho)^+$ be non-zero. Due to the pervasiveness of E there exists a $g \in E^+$ with $0 < g \leq f$. Since T^ρ is positive this yields $0 \leq Tg = T^\rho g \leq T^\rho f$. As $Tg \neq 0$ holds we obtain $T^\rho f > 0$. Next, suppose $f \in E^\rho$ is non-zero such that $T^\rho f = 0$ holds. Then we obtain $T^\rho f^+ = (T^\rho f)^+ = 0$, hence

by the above argument we obtain $f^+ = 0$. Similarly, we get $T^\rho f^- = (T^\rho f)^- = 0$ and $f^- = 0$ follows. We conclude that T^ρ is injective. \square

In particular, the above lemma shows that bijective Riesz* homomorphisms between pervasive pre-Riesz spaces extend to bijective Riesz homomorphisms between the Riesz completions. This fact allows us to obtain a result on the inverse of a bijective Riesz* homomorphism on a pervasive pre-Riesz space.

Theorem 5.3 *Suppose E and F are pre-Riesz spaces, E is pervasive and that $T : E \rightarrow F$ is a bijective Riesz* homomorphism. Then T^{-1} is a Riesz* homomorphism and, hence, T is an order isomorphism.*

Proof Suppose T is a bijective Riesz* homomorphism. Lemma 5.2 yields that T extends to a bijective Riesz homomorphism $T^\rho : E^\rho \rightarrow F^\rho$. The inverse of a bijective Riesz homomorphism between Riesz spaces is again a Riesz homomorphism, see for example Theorem 2.15 in [3]. Therefore, $(T^\rho)^{-1} : F^\rho \rightarrow E^\rho$ is a Riesz homomorphism that extends $T^{-1} : F \rightarrow E$ and, hence, T^{-1} is a Riesz* homomorphism. \square

Let us remark that any complete Riesz homomorphism T between pre-Riesz spaces E and F is a Riesz homomorphism and, in particular, a Riesz* homomorphism (see [5]). Therefore, if E is pervasive and $T : E \rightarrow F$ is a bijective operator of any of these three types, then T is, in particular, a bijective Riesz* homomorphism and by Theorem 5.3 an order isomorphism. Thus, on pervasive pre-Riesz spaces the notions of order isomorphism, complete Riesz homomorphism, Riesz homomorphism and Riesz* homomorphisms coincide for bijective operators T and their inverses. We next exhibit an example showing that this statement fails without assuming E to be pervasive.

Example 5.4 Let $X = [0, 1]$, $\eta = \mathbb{K}$, $\pi(x) = \frac{1}{2}x$, $x \in [0, 1]$, and E be the set of polynomials on X . Then E is a pre-Riesz space and its Riesz completion E^ρ is the Riesz subspace of $C([0, 1])$ consisting of all piecewise polynomial functions. Since non-constant polynomials can only be zero in finitely many points, one easily verifies that E is not pervasive. Let $T : E \rightarrow E$ be the linear operator defined by $(Tp)(x) = \eta(x)p(\pi(x)) = p(\frac{1}{2}x)$, for all $x \in [0, 1]$ and $p \in E$. Theorem 3.2 yields that T is a Riesz* homomorphism. Moreover, since π is a weak open map, Theorem 4.4 yields that T is even a complete Riesz homomorphism. Clearly, T is an injective operator. Let $g \in E$ be of the form $g(x) = \alpha_n x^n + \dots + \alpha_1 x + \alpha_0$, $x \in [0, 1]$ with $\alpha_0, \dots, \alpha_n \in \mathbb{R}$. The pre-image f of g under T is given by $f(x) = \sum_{i=0}^n \beta_i x^i$ where $\beta_i := 2^i \alpha_i$ for all $0 \leq i \leq n$. We conclude that T is a bijective Riesz* homomorphism. Suppose there exist a $\theta : X \rightarrow \mathbb{R}$ continuous and $\tau : X \rightarrow X$ continuous on $\{\theta > 0\}$ such that $T^{-1} = T_{\theta, \tau}$ on E . T has the property that $T\mathbb{K}_X = \mathbb{K}_Y$ holds so $T^{-1}\mathbb{K}_Y = 1_X$ holds. For any $x \in X$ we thus have $\theta(x) = 1$. Denote by $I_X \in E$ the identity function on X . For any $x \in X$ we have

$$x = (T^{-1}TI_X)(x) = (TI_X)(\tau(x)) = I_X \left(\frac{1}{2}(\tau(x)) \right) = \frac{1}{2}\tau(x).$$

However, the equality $\tau(x) = 2x$ can not be satisfied on all of $[0, 1]$, as τ has to map into $[0, 1]$. Therefore, T^{-1} is not a weighted composition operator. Theorem 3.2 shows that T^{-1} is not a Riesz* homomorphism.

We turn our attention to the following result which answers the second problem posed in the beginning of this section.

Theorem 5.5 *Let E and F be separating order dense subspaces of $C(X)$ and $C(Y)$, respectively, and $T : E \rightarrow F$ a linear operator. T is an order isomorphism if and only if $T = T_{\eta, \pi}$ where $\eta \in C(Y)^+$ is non-vanishing and $\pi : Y \rightarrow X$ is a homeomorphism. Moreover, in this case there exists a $\delta > 0$ such that $\eta \geq \delta \mathbb{1}_Y$ and the homeomorphism π is uniquely determined by T .*

Proof Suppose T is an order isomorphism. Due to Lemma 5.1, T is a complete Riesz homomorphism and, in particular, a Riesz* homomorphism. Therefore, Theorem 3.2 says that there exist $\eta \in C(Y)^+$ and $\pi : Y \rightarrow X$ continuous on $\{\eta > 0\}$ such that $T = T_{\eta, \pi}$. Suppose that $\eta(y) = 0$ holds for some $y \in Y$. For any $g \in T(E)$ we then obtain $g(y) = 0$, which contradicts the surjectivity of T since F is majorizing in $C(Y)$. Therefore, η is indeed non-vanishing and by compactness of Y there exists a $\delta > 0$ such that $\eta(y) \geq \delta$ for all $y \in Y$. Remark that, since $\{\eta > 0\} = Y$ holds, π is continuous and uniquely determined everywhere. Suppose that π is not injective and let $y_1, y_2 \in Y$ be such that $y_1 \neq y_2$ and $\pi(y_1) = \pi(y_2)$ hold. For any $f \in E$ we have,

$$(Tf)(y_1) = \eta(y_1)f(\pi(y_1)) = \eta(y_2)\frac{\eta(y_1)}{\eta(y_2)}f(\pi(y_2)) = \frac{\eta(y_1)}{\eta(y_2)}(Tf)(y_2).$$

Therefore, we obtain a $\lambda \geq 0$ such that $g(y_1) = \lambda g(y_2)$ holds for all $g \in \text{Im}(T)$. However, since F separates the points of Y this contradicts the surjectivity of T .

Next we show that π is surjective. Supposing the converse, there exists a $x_0 \in X$ with $x_0 \notin \pi(Y)$. Since π is continuous, $\pi(Y)$ is compact in X and, hence, $\pi(Y)$ is closed due to X being a Hausdorff space. In particular, there exists an open neighborhood U of x_0 disjoint from $\pi(Y)$. Due to Urysohn's lemma there exists an $f \in C(X)$ with $\text{supp}(f) \subseteq U$ and $f < 0$. Since E is order dense in $C(X)$ there exists a $g \in E$, $g \geq f$ and $g \not\geq 0$. However, for all $y \in Y$, $(Tg)(y) = \eta(y)g(\pi(y))$ is satisfied while $\eta \geq 0$ and $\pi(y) \notin U$. Therefore, $g(\pi(y)) \geq f(\pi(y)) = 0$ holds for all $y \in Y$, so $Tg \geq 0$, contradicting the bipositivity of T on E .

Conversely, suppose $T = T_{\eta, \pi}$ for some $\eta \in C(Y)^+$ non-vanishing and a homeomorphism $\pi : Y \rightarrow X$ and let $\delta > 0$ be such that $\eta \geq \delta \mathbb{1}_Y$. Let us define a weighted composition operator $R : F \rightarrow E$ by

$$Rf := \frac{1}{\eta \circ \pi^{-1}}(f \circ \pi^{-1}), \quad f \in F.$$

For all $f \in F$ and $y \in Y$ we have $(TRf)(y) = \eta(y)(Rf)(\pi(y)) = f(y)$, so $TR = \text{Id}_F$. Similarly, RT equals the identity operator on E and, hence, $R = T^{-1}$. Both T and T^{-1} are positive since their weight maps are positive and thus T is an order isomorphism. \square

Corollary 5.6 *Let E and F be separating order dense subspaces of $C(X)$ and $C(Y)$, respectively. An order isomorphism $T : E \rightarrow F$ extends to an order isomorphism $\hat{T} : C(X) \rightarrow C(Y)$.*

Proof Suppose $T : E \rightarrow F$ is an order isomorphism. Theorem 5.5 says that $T = T_{\eta,\pi}$ for some non-vanishing $\eta \in C(Y)^+$ and a homeomorphism $\pi : Y \rightarrow X$. Therefore, $\hat{T} = T_{\eta,\pi} : C(X) \rightarrow C(Y)$ is a well-defined linear operator and by again applying Theorem 5.5 we conclude that \hat{T} is an order isomorphism. \square

Suppose E is a separating order dense subspace of $C(X)$. The *automorphism group of E* , denoted by $\text{Aut}(E)$, is the set consisting of all order isomorphisms from E to itself equipped with the group action of composing operators. Due to Theorem 5.5 the automorphism group of $C(X)$ is isomorphic to the direct product $G \times \text{Hom}(X)$, where G is the group consisting of all $\eta \in C(X)$ satisfying $\eta \geq \delta \mathbb{1}^X > 0$, for some $\delta \in (0, \infty)$, equipped with pointwise multiplication and $\text{Hom}(X)$ is the group consisting of homeomorphisms from X to itself. Here the isomorphism map from $G \times \text{Hom}(X)$ to $\text{Aut}(C(X))$ is given by $(\eta, \pi) \mapsto T_{\eta,\pi}$.

Let T be an automorphism on E . Due to Theorem 5.5, $T = T_{\eta,\pi}$ with $(\eta, \pi) \in G \times \text{Hom}(X)$ and, moreover, π is uniquely determined by T . One easily verifies that η is also uniquely determined by T , since E separates the points of X . Combining these observations with Corollary 5.6 yields that the automorphism group of E embeds in $\text{Aut}(C(X))$. More precisely, $\text{Aut}(E)$ is the subgroup of $\text{Aut}(C(X))$ consisting of the operators that leave E invariant.

In general, $\text{Aut}(E)$ and $\text{Aut}(C(X))$ need not be isomorphic even if E is in addition pervasive. Namely, any automorphism $T_{\eta,\pi}$ on $C[0, 1]$ where η or π is not differentiable does not restrict to an automorphism on $C^k[0, 1]$. In Sect. 7 we characterize the automorphism group of a more general class of partially ordered vector spaces, namely, the spaces $C_0^k(M)$ of arbitrary order k and where M is a locally compact smooth manifold.

6 Locally compact spaces

In the previous sections we have investigated Riesz* homomorphisms on separating order dense subspaces of the space of continuous functions on some compact Hausdorff space. Most results developed so far only use the compactness structure on bounded subsets. In this section we aim to generalize Theorem 3.2 to Riesz* homomorphisms between spaces of functions on locally compact spaces.

For the rest of this section, let X and Y be locally compact Hausdorff spaces. Consider the subspace $C_0(X)$ of $C(X)$ consisting of all functions $f \in C(X)$ that *vanish at infinity*, i.e., for all $\epsilon > 0$ the set $\{f \geq \epsilon\}$ is compact in X . Clearly $C_0(X)$ is a Riesz space that coincides with $C(X)$ whenever X is compact. We aim to characterize the Riesz* homomorphisms on pre-Riesz subspaces of $C_0(X)$. However, before we can do so we have to show that Riesz homomorphisms on the whole space $C_0(X)$ are of the desired weighted composition form. Similar to the $C(X)$ case the proof uses the structure of positive bounded linear functionals on $C_0(X)$. In 7.3 of [6] the positive

bounded linear functionals on $C_0(X)$ are characterized to be exactly those functionals that are given by integration against a finite Radon measure μ .

Theorem 6.1 *A linear operator $T : C_0(X) \rightarrow C_0(Y)$ is a Riesz homomorphism if and only if $T = T_{\eta,\pi}$ for some $\eta \in C_b(Y)^+$ and $\pi : Y \rightarrow X$ continuous on $\{\eta > 0\}$. Moreover, in this case π is proper on $\{\eta \geq \epsilon\}$ for each $\epsilon > 0$, i.e.,*

$$K \subseteq X \text{ compact, } \epsilon > 0 \Rightarrow \pi^{-1}(K) \cap \{\eta \geq \epsilon\} \text{ compact.} \tag{6}$$

Proof Fix a $y \in Y$ and define the positive linear functional $T_y : C_0(X) \rightarrow \mathbb{R}$ by $T_y f := (Tf)(y)$, for $f \in C_0(X)$. Since positive operators between Banach lattices are norm continuous, T is norm bounded. Therefore, T_y is a positive and norm bounded functional and by Folland [6, 7.3] T_y is given by integration against a finite Radon measure μ .

Suppose μ is supported in more than one point, i.e., there are distinct $s, t \in \text{supp}(\mu)$, where $\text{supp}(\mu)$ consists of all $x \in X$ such that for all open $U \subseteq X$ that contain x we have $\mu(U) > 0$. Since X is a Hausdorff space, we obtain disjoint open sets $A, B \subseteq X$ with $s \in A, t \in B$ and $\mu(A), \mu(B) > 0$. Furthermore, by inner regularity of the Radon measure μ we can assume that A and B are contained in some compact set $K \subseteq X$. Due to Urysohn’s lemma there exist $f, g : X \rightarrow [0, 1]$ continuous with $f = 0$ on $X \setminus A, g = 0$ on $X \setminus B$ and $f(s) = 1, g(t) = 1$. Observe that $f, g \in C_0(X)$ hold as they are zero outside the compact set K . By construction f and g are disjoint, hence their images under T are disjoint, i.e. $T_y f \wedge T_y g = 0$. However, for $\epsilon \in (0, 1]$, we also have

$$T_y f = \int f \, d\mu \geq \int_{\{f \geq \epsilon\}} \epsilon \, d\mu = \epsilon \cdot \mu(\{f \geq \epsilon\}) > 0,$$

since the set $\{f \geq \epsilon\}$ contains the point s which lies in the support of μ , since $\epsilon \leq 1$. Similarly, $T_y g > 0$ holds, which contradicts our earlier conclusion that $T_y f \wedge T_y g = 0$ holds. Thus, $\text{supp}(\mu)$ is a singleton.

Let $x_0 \in X$ be the unique element in the support of μ . For any $f \in C_0(X)$ we have

$$T_y f = \int f \, d\mu = \int_{\{x_0\}} f \, d\mu = f(x_0) \cdot \mu(\{x_0\}).$$

In other words, T_y is a weighted point evaluation, hence T is a composition multiplication operator $T = T_{\eta,\pi}$. In the proof of Theorem 3.2, where we considered the $C(X)$ case, we showed that η and π are automatically continuous on Y and $\{\eta > 0\}$, respectively. Our arguments there only require that any bounded net in X has a convergent subnet, which still holds for locally compact X since a bounded net is eventually contained in a compact subset of X . Therefore, we conclude that η and π are continuous on Y and $\{\eta > 0\}$, respectively. Additionally, η inherits the positivity and boundedness from T .

In order to prove the converse, we start with a composition multiplication operator $T = T_{\eta,\pi} : C_0(X) \rightarrow C_0(Y)$. To show that T is necessarily a Riesz homomorphism

we observe that the supremum of two functions $f, g \in C_0(X)$ equals the pointwise maximum of the two and a composition multiplication operator respects any pointwise structure.

Finally, suppose $K \subseteq X$ is compact and $\epsilon > 0$ is such that $\pi^{-1}(K) \cap \{\eta \geq \epsilon\}$ is not compact. Let $f \in C_0(X)$ be equal to one on K ; such an f exists by compactness of K . For any $y \in \pi^{-1}(K) \cap \{\eta \geq \epsilon\}$ we have $(Tf)(y) = \eta(y)f(\pi(y)) = \eta(y) \geq \epsilon$. However, $\pi^{-1}(K) \cap \{\eta \geq \epsilon\}$ is not compact which contradicts $Tf \in C_0(Y)$. \square

Analogously to Theorem 3.2 the previous result extends to Riesz* homomorphisms on separating order dense subspaces, which is the content of the following result.

Theorem 6.2 *Let E and F be order dense subspaces of $C_0(X)$ and $C_0(Y)$, respectively, let E be separating and $T : E \rightarrow F$ be a linear operator. T is a Riesz* homomorphism if and only if $T = T_{\eta,\pi}$ for some $\eta \in C_b(Y)^+$ and $\pi : Y \rightarrow X$ continuous on $\{\eta > 0\}$. Moreover, in this case η and π satisfy (6).*

Proof Suppose T is a Riesz* homomorphism and let $T^\rho : E^\rho \rightarrow F^\rho$ be the Riesz homomorphism extending T . Observe that E^ρ and F^ρ are Riesz subspaces of $C_0(X)$ and $C_0(Y)$ respectively. Let us fix $y \in Y$ and define the functional $T_y : E^\rho \rightarrow \mathbb{R}$ by $T_y f := (T^\rho f)(y)$ and observe that T_y is a Riesz homomorphism. Due to E^ρ being majorizing in $C_0(X)$ and \mathbb{R} being Dedekind complete, T_y extends to a Riesz homomorphism $\hat{T}_y : C_0(X) \rightarrow \mathbb{R}$. The arguments of Theorem 6.1 show that T_y is a positive scalar multiple of a point evaluation and, hence, that T^ρ is a weighted composition operator $T^\rho = T_{\eta,\pi} : E^\rho \rightarrow F^\rho$. In particular, due to E being order dense and separating in $C_0(X)$, arguments in the proof of Theorem 3.2 show that η and π are continuous on Y and $\{\eta > 0\}$, respectively. Moreover, η inherits positivity and boundedness from T .

Let $T = T_{\eta,\pi} : E \rightarrow F$ be given for some $\eta \in C_b(Y)^+$ and $\pi : Y \rightarrow X$ continuous on $\{\eta > 0\}$. Suppose $K \subseteq X$ is compact and $\epsilon > 0$ is given such that $\pi^{-1}(K) \cap \{\eta \geq \epsilon\}$ is not compact. Let us take a $g \in C_0(X)$ with g equal to one on K and use the majorizing property of E to find an $f \in E$ greater than g . For any $y \in \pi^{-1}(K) \cap \{\eta \geq \epsilon\}$ we have $(Tf)(y) = \eta(y)f(\pi(y)) \geq \eta(y) \geq \epsilon$. However, $\pi^{-1}(K) \cap \{\eta \geq \epsilon\}$ is not compact, which contradicts $Tf \in F \subseteq C_0(Y)$. Therefore, (6) is satisfied.

We conclude the proof by observing that $T^\rho : E^\rho \rightarrow F^\rho$ defined by $T^\rho = T_{\eta,\pi}$ is a well-defined Riesz homomorphism that extends T . \square

Theorem 6.3 *Suppose E and F be order dense subspaces of $C_0(X)$ and $C_0(Y)$, respectively, E separating and $T : E \rightarrow F$ a bijective linear operator. T is an order isomorphism if and only if $T = T_{\eta,\pi}$ holds, where $\eta \in C(Y)$ satisfies $0 < \delta \mathbb{1} \leq \eta \leq D \mathbb{1}$ for some $\delta, D > 0$ and $\pi : Y \rightarrow X$ is a homeomorphism. Moreover, in this case π is uniquely determined by T .*

Proof Suppose $T : E \rightarrow F$ is an order isomorphism. Due to Theorem 6.2 we obtain $T = T_{\eta,\pi}$ for some $\eta \in C_b(Y)^+$ and $\pi : Y \rightarrow X$ continuous on $\{\eta > 0\}$. Similar arguments as in the proof of Theorem 5.5 show that η is non-vanishing and π is invertible and, moreover, that T^{-1} , the inverse of T , is a weighted composition operator with a positive weight map equal to the reciprocal of $(\eta \circ \pi^{-1})$. In particular, T^{-1} is a

positive operator and, hence, T^{-1} is order bounded. E is majorizing in $C_0(X)$ so the weight map of T^{-1} is bounded. Since π is bijective, we infer that the reciprocal of η is bounded. Combining this with the fact that η is bounded, we obtain the desired $\delta, D > 0$ such that $0 < \delta\| \cdot \| \leq \eta \leq D\| \cdot \|$. In conclusion we observe, since η and its reciprocal are non-vanishing, that π and π^{-1} are continuous and hence π is a homeomorphism. \square

At this point it is convenient to characterize pervasive subspaces of $C_0(X)$ and relate the pervasive condition to order denseness. Note that, in contrast to [8], we do not restrict the notion of pervasiveness to majorizing subspaces.

Lemma 6.4 *Let $E \subseteq F$ be a subspace of a Riesz space F . The following assertions are equivalent.*

- (i) *E is pervasive in F , i.e., for every $f \in F$ with $f > 0$ there exists a $g \in E$ such that $0 < g \leq f$.*
- (ii) *For every $f \in F$, $f > 0$, one has $f = \sup\{g \in E : 0 \leq g \leq f\}$.*

Moreover, if E is pervasive and majorizing in F , then E is order dense in F . In the case that X is a locally compact Hausdorff space and $F = C_0(X)$, then (i) and (ii) are equivalent with:

- (iii) *For every non-empty open set $U \subseteq X$ there exists a positive non-zero $f \in E$ satisfying $\text{supp}(f) \subseteq \overline{U}$, where $\text{supp}(f)$ equals the closure of $\{x \in X : f(x) \neq 0\}$.*

Proof Theorem 1.34 on page 31 of [3] proves the equivalence of (i) and (ii) in the case that E is a Riesz space. However, the proof of this theorem only uses existence of a supremum of two elements in F . Hence, the result extends immediately to partially ordered subspaces of F .

Next we suppose that E is pervasive and majorizing in F and let $f \in F$ be given. By assumption there exists a $g \in E$ with $g \geq f$. Let $h := g - f \geq 0$ and apply property (ii) to obtain $h = \sup\{e \in E : 0 \leq e \leq h\}$. From the equality $f = g - h$ we obtain

$$\begin{aligned} f &= g - \sup\{e \in E : 0 \leq e \leq h\} \\ &= \inf\{g - e : e \in E, 0 \leq e \leq h\} \\ &= \inf\{e \in E : f \leq e \leq g\}, \end{aligned}$$

where the last equality is due to $g \in E$ and the fact that $0 \leq e \leq h$ holds if and only if $g \geq g - e \geq f$. In particular, for any lower bound v of $\{e \in E : f \leq e\}$ we have $v \leq \inf\{e \in E : f \leq e \leq g\} = f$. Hence, $f = \inf\{e \in E : f \leq e\}$, which shows that E is order dense in F .

Next, suppose $F = C_0(X)$ holds, where X is a locally compact Hausdorff space. Let E be a pervasive subspace of F and let $U \subseteq X$ be open and non-empty. Due to Urysohn's lemma we know there exists a non-zero $f \in C(X)^+$ with $\text{supp}(f) \subseteq \overline{U}$. Since E is pervasive in F , there exists a $g \in E$ satisfying the same properties as f and hence E satisfies property (iii).

Conversely, suppose that (iii) holds and let $f \in C_0(X)^+$ be non-zero. Let us fix an $0 < \epsilon < \|f\|_\infty$, then the set $U := \{f > \epsilon\}$ is a non-empty open subset of X . Applying

property (iii), there exists a non-zero $g \in E^+$ with $\text{supp}(g) \subseteq \bar{U}$. Let $h = \epsilon g / \|g\|_\infty$, then $h \in E^+$, $h \neq 0$ and $h \leq f$ are all satisfied and thus E is pervasive in $C_0(X)$. \square

Regarding Theorems 6.2 and 6.3 we are interested to know whether the space $C_0^k(X)$, where $X \subseteq \mathbb{R}^d$ is an open subset and $k \in \mathbb{N} \cup \{\infty\}$, is separating and order dense in $C_0(X)$. It turns out that this in fact holds and that $C_0^k(X)$ is even pervasive. This is the content of Theorem 6.5 which is easily proved using Lemma 6.4. In the next section we show a more general version of this statement, namely Theorem 7.2, where X is replaced by any locally compact smooth manifold, therefore, we omit the proof.

Theorem 6.5 *Let $d \in \mathbb{N}$ and $X \subseteq \mathbb{R}^d$ be an open subset and $k \in \mathbb{N} \cup \{\infty\}$ be given. Then $C_0^k(X)$ is a separating, pervasive and order dense subspace of $C_0(X)$.*

7 Applications

We dedicate this section to an investigation of various spaces consisting of functions on a locally compact Hausdorff space X that can be embedded as a separating, pervasive and order dense subspace of $C_0(X)$. An immediate result in this vain is obtained by observing the following chain of embeddings

$$C_0^k(X) \subseteq C_0^{k,\alpha}(X) \subseteq UC_0(X) \subseteq C_0(X),$$

where X is an open subset of \mathbb{R}^d and $C_0^{k,\alpha}(X)$ is the space consisting of real-valued functions on X that vanish at infinity having continuous derivatives up to order k such that the k th partial derivatives are Hölder continuous with exponent $0 < \alpha \leq 1$, meaning that $|f(x) - f(y)| \leq C \|x - y\|^\alpha$ holds for all $x, y \in X$ and some constant $C > 0$. $UC_0(X)$ denotes the space of uniformly continuous functions on X that vanish at infinity. Moreover, $C_0^{k,\alpha}(X)$ equals the space $LC_0(X)$ consisting of all Lipschitz continuous functions on X whenever $k = 0$ and $\alpha = 1$ hold. Due to Theorem 6.5 the above embeddings yield that $LC_0(X)$, $C_0^{k,\alpha}(X)$ and $UC_0(X)$ are separating, pervasive and order dense subspaces of $C_0(X)$.

In the remainder of this section we exhibit two more elaborate examples of spaces consisting of functions that can be embedded as separating, pervasive and order dense subspaces of $C_0(X)$. Namely, first we investigate spaces of differentiable functions on a locally compact smooth manifold. Secondly we consider Sobolev spaces on a domain of \mathbb{R}^d satisfying certain regularity conditions.

We recall several elementary definitions concerning smooth manifolds (see [7]). Let (X, τ) be a second countable Hausdorff space. X is called a d -dimensional *topological manifold* if there exists an open cover $(U_i)_{i \in I}$ of X such that for all $i \in I$, U_i is homeomorphic to an open subset V_i of \mathbb{R}^d . In that case, the collection of triplets $\mathcal{A} = \{(U_i, h_i, V_i) : i \in I\}$ is called an *atlas* of M , where $h_i : U_i \rightarrow V_i$ are homeomorphisms. One such a triplet is called a *chart* of U . $M = (X, \mathcal{A})$ is an *m-smooth manifold* if in addition for all $i, j \in I$ the gluing map $(h_i \circ h_j^{-1})|_{h_j(U_i \cap U_j)} : h_j(U_i \cap U_j) \rightarrow h_i(U_i \cap U_j)$ is m -times differentiable as a map on \mathbb{R}^d , or simply a smooth manifold whenever $m = \infty$.

Definition 7.1 Suppose M is a m -smooth d -dimensional manifold and $f : M \rightarrow \mathbb{R}$ is a continuous map. Then f is called m -times differentiable if for all charts (U, h, V) of M the map $(f \circ h^{-1}) : V \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$ is m -times differentiable.

Let M be a m -smooth d -dimensional manifold and let $C_0^\infty(M)$ be the space of functions that vanish at infinity that are infinitely many times differentiable according to Definition 7.1. A useful tool when dealing with the space $C_0^\infty(M)$ is the notion of a partition of unity. Suppose $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ is an open cover of M . A *partition of unity subordinate to \mathcal{U}* is a collection of continuous functions $\varphi_\alpha : M \rightarrow [0, 1]$, $\alpha \in A$, such that $\text{supp}(\varphi_\alpha) \subseteq U_\alpha$, $\{\text{supp}(\varphi_\alpha) : \alpha \in A\}$ is a locally finite cover and $\sum_{\alpha \in A} \varphi_\alpha = 1$. Since the supports of the φ_α form a locally finite cover, $\sum_\alpha \varphi_\alpha$ has only finitely many non-zero terms in a neighborhood around every point and we encounter no convergence problems. A partition of unity is called *m -smooth* if every φ_α is a m -smooth function. An important result in the study of m -smooth manifolds is the existence of a m -smooth partition of unity subordinate to any given open cover (see [7, Theorem 2.25, p.54]). A useful consequence of the existence of a m -smooth partition of unity is the existence of m -smooth *bump functions* on M . Let U and V be open subsets of M such that $\overline{V} \subseteq U$ holds. Letting $U_1 = U$ and $U_2 = M \setminus \overline{V}$ we get an open cover $\{U_1, U_2\}$ of M , hence there exists a subordinated m -smooth partition of unity $\{\varphi_1, \varphi_2\}$. Observe that φ_1 is a m -smooth map on M with values in $[0, 1]$, supported in U and constantly one on \overline{V} . A map φ_1 satisfying these properties is called a m -smooth bump function of V supported in U .

For the remainder of this section let M be an n -dimensional locally compact m -smooth manifold with $m \in \mathbb{N} \cup \{\infty\}$ and $k \leq m$ an integer or $k = \infty$.

Theorem 7.2 *The space $C_0^k(M)$ is a separating, pervasive and order dense subspace of $C_0(M)$.*

Proof The existence of bump functions in $C_0^k(M)$ described above immediately yields that $C_0^k(M)$ separates the points of M and is pervasive due to Lemma 6.4. Moreover, due to the same lemma it suffices to show that $C_0^\infty(M)$ is majorizing in $C_0(M)$ to obtain order denseness. We first argue that it is sufficient to majorize positive $f \in C_0(M)$ that vanish nowhere. To this end, we show the existence of a positive function $h \in C_0(M)$ that vanishes nowhere, since then $f \vee h \in C_0(M)$ is positive, vanishes nowhere and is greater than f . Lemma 2.23 in [7] states that there exists a countable locally finite cover $(U_n)_{n=1}^\infty$ of M consisting of precompact open sets. Let $W_1 = U_1$ and observe that (U_n) covers the compact set \overline{W}_1 , hence there exist $n_1, \dots, n_k \in \mathbb{N}$ such that $\overline{W}_1 \subseteq \bigcup_{j=1}^k U_{n_j} =: W_2$. Inductively, we obtain a cover (W_n) of M consisting of precompact open sets satisfying $\overline{W}_n \subseteq W_{n+1}$, for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ let $h_n \in C_0(M)$ be a bump function of W_n supported in W_{n+1} and let $h = \sum_n 2^{-n} h_n$. Clearly, $h \in C_0(M)$ satisfies $h(x) > 0$ for all $x \in M$.

Suppose $f \in C_0(M)$ is positive and non-vanishing. Without loss of generality we can assume that $\|f\|_\infty = 1$ holds after rescaling. For any $n \in \mathbb{N}$ we define the open set $V_n = \{p \in M : 2^{-(n+2)} < f(p) < 2^{-n}\}$ and $V_0 = \{p \in M : f(p) > 2^{-2}\}$. The collection $(V_n)_{n=0}^\infty$ is a locally finite countable open cover of $\{f > 0\}$, which equals M . Let $(\varphi_n : M \rightarrow \mathbb{R})_{n=0}^\infty$ be a partition of unity subordinate to $(V_n)_{n=0}^\infty$ and define

$$g(p) := \sum_{j=0}^{\infty} 2^{-j} \varphi_j(p), \quad p \in M.$$

For any point $p \in M$ only finitely many terms are non-zero in a neighborhood of p , hence g is well-defined and m -smooth. Let $\epsilon > 0$ be given and let $j_0 \in \mathbb{N}$ be such that $\epsilon > \sum_{j=j_0}^{\infty} 2^{-j}$, then we get

$$\{g \geq \epsilon\} \subseteq \bigcup_{n=0}^{j_0} V_n \subseteq \{f \geq 2^{-(j_0+2)}\}. \tag{7}$$

Indeed, whenever $p \in M \setminus \bigcup_{n=0}^{j_0} V_n$ we have $g(p) = \sum_{j=j_0}^{\infty} 2^{-j} \varphi_j(p) \leq \sum_{j=j_0}^{\infty} 2^{-j} < \epsilon$, showing the first inclusion while the second inclusion follows from the construction of the set V_n . Since f vanishes at infinity, the set on the right hand side of (7) is compact. Therefore, the closed set $\{g \geq \epsilon\}$ is compact, showing that g vanishes at infinity. We are left to show that $g \geq f$ holds. Let $p \in M$ and $n \in \mathbb{N}$ the largest index such that $p \in V_n$. Then we have $g(p) = \sum_{j=0}^n 2^{-j} \varphi_j(p) \geq 2^{-n} \sum_{j=0}^n \varphi_j(p) = 2^{-n}$. On the other hand, we have $f(p) < 2^{-n} \leq g(p)$ as $p \in V_n$ holds. \square

Suppose M and N are m - and n -smooth manifolds of independent dimension and let $k \leq n, m$ be an integer or $k = \infty$. Combination of Theorems 6.2 and 7.2 yields that any Riesz* homomorphism $T : C_0^k(M) \rightarrow C_0^k(N)$ is a weighted composition operator $T = T_{\eta, \pi}$, where $\eta \in C_b(Y)^+$ and π is continuous and proper on $\{\eta > 0\}$. Moreover, Theorem 6.3 gives a characterization of the order isomorphisms $T_{\eta, \pi} : C_0^k(M) \rightarrow C_0^k(N)$. We aim to give a full description of the automorphism group $\text{Aut}(C_0^k(M))$. To this end we show that any bijective weighted composition operator on $C_0^k(M)$ has automatically k -smooth weight and composition maps, which is precisely stated and proved in Lemma 7.4. We exhibit an intermediate observation concerning the existence of k -smooth maps on M that locally behave like coordinate projections in \mathbb{R}^d .

Lemma 7.3 *Suppose M is d -dimensional, $p \in M$ and (U, h, V) is a chart of M with $p \in U$. For any index $1 \leq n \leq d$ there exists a k -smooth function $f \in C_0(M)$ and a neighborhood U_0 of p contained in U such that $f = f_n \circ h$ on U_0 , where $f_n(x_1, \dots, x_d) = x_n$, for all $(x_1, \dots, x_d) \in \mathbb{R}^d$.*

Proof Suppose $p \in M$ is given and (U, h, V) is a chart in M containing p . Let U_0 be a neighborhood of p with $\overline{U_0} \subseteq U$ and $\varphi : M \rightarrow \mathbb{R}$ a k -smooth bump function of U_0 supported in U . Define $g : M \rightarrow \mathbb{R}$ by $g(q) = f_n(h(q))$ for all $q \in U$ and $g(q) = 0$ elsewhere, where f_n is the n -th coordinate projection in \mathbb{R}^d as in the statement of the lemma. Since φ is supported in U the map f on M defined by $f = \varphi \cdot g$ is k -smooth. Moreover, since φ is constantly equal to one on U_0 , $f = f_n \circ h$ holds on U_0 . \square

Lemma 7.4 *Suppose $T = T_{\eta, \pi} : C_0^k(M) \rightarrow C_0^k(N)$ is a well-defined bijective weighted composition operator, then η and π are k -times differentiable on N .*

Proof Let $q \in N$ be given and U a compact neighborhood of q in N . Recall from Theorems 6.3 and 7.2 that there exist $\delta, D > 0$ such that $\delta \llcorner \leq \eta \leq D \llcorner$ and that

π is a homeomorphism. In particular, $\pi(U)$ is compact in M . Let V be a compact neighborhood of $\pi(U)$ and $f \in C_0^k(M)$ be a bump function of V . From $(Tf)(p) = \eta(p)(f(\pi(p)))$, for $p \in N$, we obtain that Tf equals η on $\pi^{-1}(V)$ which contains U . Since $Tf \in C_0^k(N)$ holds, we infer that η is k -times differentiable at q .

Next, let us fix $q \in N$. Let (U', h', V') be a chart of M with $\pi(q) \in U'$ and (U, h, V) a chart of N with $q \in U$ and $1 \leq n \leq d$. Due to Lemma 7.3 we find an $f \in C_0^k(M)$ and some neighborhood U_n of $\pi(q)$ contained in U' such that $f = f_n \circ h'$ holds on U_n , where f_n is the n -th coordinate projection on \mathbb{R}^d . Since the reciprocal of η is well-defined and k -times differentiable on N , we get $\eta^{-1}(Tf) = f \circ \pi$ and, hence, $(f \circ \pi)$ is k -times differentiable on N . Therefore, the map $(f_n \circ h' \circ \pi)$ is k -times differentiable on $\pi^{-1}(U_n)$ which is a neighborhood of q , since π is a homeomorphism. In particular, $(f_n \circ h' \circ \pi \circ h^{-1})$ is k -times differentiable on $h(\pi^{-1}(U_n))$. Let $W := h(\pi^{-1}(U_1)) \cap \dots \cap h(\pi^{-1}(U_d))$ and observe that the map $(h' \circ \pi \circ h^{-1})$ is k -times differentiable on this neighborhood W of q when composed with any of the coordinate projection on \mathbb{R}^d . In conclusion, $(h' \circ \pi \circ h^{-1})$ is k -times differentiable at q and, hence, π is k -times differentiable at q in accordance to Definition 7.1. \square

We obtain the following description of the automorphism group of $C_0^k(M)$.

Theorem 7.5 *Let M be an m -smooth manifold of arbitrary dimension and let $k \leq m$ be given, where $m, k \in \mathbb{N} \cup \{\infty\}$. The automorphism group of $C_0^k(M)$ can be described by*

$$\text{Aut}(C_0^k(M)) \simeq G \times \text{Diff}^k(M),$$

where G is the multiplicative subgroup of $C_0^k(M)$ consisting of all η that satisfy $\delta \mathcal{K} \leq \eta \leq D \mathcal{K}$ for some $\delta, D > 0$ and $\text{Diff}^k(M)$ consists of all k -times diffeomorphisms on M .

We turn our attention to Sobolev spaces on a domain of \mathbb{R}^d . Definitions and terminology used concerning Sobolev spaces are taken from Adams (see [2]). Let $d \in \mathbb{N}$ be given and Ω a domain in \mathbb{R}^d , i.e., $\Omega \subseteq \mathbb{R}^d$ is open. For any $m \in \mathbb{N}$ and $1 \leq p < \infty$ we define the Sobolev space $W^{m,p}(\Omega)$ as the space consisting of L^p -functions f on Ω for which all distributional partial derivatives $D^\alpha f$, with $1 \leq |\alpha| \leq m$, are in L^p . Equipped with the norm $\|\cdot\|_{m,p}$ defined by $\|f\|_{m,p} = (\sum_{0 \leq |\alpha| \leq m} \|D^\alpha f\|_p^p)^{\frac{1}{p}}$, the space $W^{m,p}(\Omega)$ is a Banach space. For smooth functions the distributional and classical partial derivatives coincide, hence, we infer $C^\infty(\Omega) \subseteq W^{m,p}(\Omega)$.

For our purpose of characterizing the Riesz* homomorphisms between Sobolev spaces, we need to be able to embed $W^{m,p}(\Omega)$ in a space of continuous functions. Suppose for now that any equivalence class $u \in W^{m,p}(\Omega)$ contains a unique continuous function. We define the space $W_0^{m,p}(\Omega)$ to be the norm-closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. Our aim is to show that $W_0^{m,p}(\Omega)$ is a pervasive, separating and order dense subspace of $C_0(\Omega)$. To this end it suffices to show that

$$C_0^\infty(\Omega) \subseteq W_0^{m,p}(\Omega) \subseteq C_0(\Omega) \tag{8}$$

holds. It turns out that this holds true if we impose a regularity condition on Ω . The following definitions can be found on page 66 in [2].

Definition 7.6 Let $d \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^d$ open be given.

- (i) Let $x \in \mathbb{R}^d$ be given and open balls B_1, B_2 in \mathbb{R}^d with $x \in B_1$ and $x \notin B_2$. The set $C_x := B_1 \cap \{x + \lambda(y - x) : y \in B_2, \lambda > 0\}$ is a *finite cone* with vertex x .
- (ii) Every domain Ω for which there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C is said to have the *cone property*.

The classical Sobolev Embedding Theorem (see for example Theorem 5.4 part III(C) in [2]) states that if Ω is a domain in \mathbb{R}^d which has the cone property and $mp > d$ holds, then $W_0^{m,p}(\Omega) \subseteq C_0(\Omega)$ holds. In particular, Eq. (8) is satisfied which yields the following corollary.

Theorem 7.7 Suppose Ω_1 and Ω_2 are domains in \mathbb{R}^d having the cone property, $1 \leq p, q < \infty$ and $m, n \in \mathbb{N}$ be such that $pm > d$ and $qn > d$ hold. If $T : W_0^{m,p}(\Omega_1) \rightarrow W_0^{n,q}(\Omega_2)$ is a Riesz* homomorphisms, then there exist $\eta : \Omega_2 \rightarrow \mathbb{R}$ bounded positive and $\pi : \Omega_1 \rightarrow \Omega_2$ such that $(Tf)(\omega) = \eta(\omega)f(\pi(\omega))$ holds for all $f \in W_0^{m,p}(\Omega_1)$ and almost every $\omega \in \Omega_2$.

Theorem 7.7 is very similar to a result by Biegert, which states that any Riesz homomorphisms on $W_0^{1,p}(\Omega)$ is a weighted composition operator, Theorem 4.4 in [4]. In his proof Biegert does not use the order structure of the space $W^{1,p}(\Omega)$ nor the Sobolev Embedding Theorem. Due to the latter he does not need to impose the cone property on Ω or any condition on p and d . However, in constrast to Biegert's result Theorem 7.7 can deal with Sobolev space upto any order as long as $pm > d$ is satisfied. Remark that $W^{m,p}(\Omega)$ is a Riesz space exactly when $m = 1$ holds. In conclusion, under extra conditions on Ω , p and d we can generalize the result of Biegert to higher order Sobolev space by considering Riesz* homomorphisms.

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