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Unitarily invariant strictly positive definite kernels on spheres

J. C. Guella[1] · V. A. Menegatto[1]

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Abstract We present a Fourier characterization for the continuous and unitarily invariant strictly positive definite kernels on the unit sphere in $\mathbb{C}^q$, thus adding to a celebrated work of I. J. Schoenberg on positive definite functions on real spheres.

Keywords Positive definite · Spheres · Disk polynomials · Zernike polynomials · Unitary group

Mathematics Subject Classification 42A82 · 42C10 · 43A35

1 Introduction

Let S^q be the unit sphere in the Euclidean space $\mathbb{R}^{q+1}$ and $\cdot$ the usual inner product in $\mathbb{R}^{q+1}$. Positive definite kernels of the form

$$K(x, y) = K'(x \cdot y), \quad x, y \in S^q,$$

in which $K' : [-1, 1] \to \mathbb{R}$ is continuous, were studied and characterized by Schoenberg [21] a long time ago. A kernel as above is positive definite on S^q if, and only if, the function K' has the form

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✉ V. A. Menegatto
menegatt@icmc.usp.br

J. C. Guella
jcguella@gmail.com

1 ICMC-USP - São Carlos, Caixa Postal 668, São Carlos, SP 13560-970, Brazil

$$K'(t) = \sum_{k=0}^{\infty} a_k^q P_k^q(t), \quad t \in [0, 1],$$

in which all coefficients a_k^q are nonnegative, P_k^q is the Gegenbauer or ultra-spherical polynomial of degree k associated with the real number $(q-1)/2$, as described in [22], and $\sum_{k=0}^{\infty} a_k P_k^q(1) < \infty$. Since a kernel K as above is real and symmetric, it is meaningful to recall that its positive definiteness demands that

$$\sum_{\mu,\nu=1}^{n} c_\mu c_\nu K(x_\mu \cdot x_\nu) \geq 0, \tag{1.1}$$

for all $n \geq 1$, any choice of distinct points $x_1, x_2, \ldots, x_n$ on S^q and real numbers $c_1, c_2, \ldots, c_n$.

The kernels in Schoenberg's class are usually called either *zonal* or *isotropic* on S^q, since they are invariant with respect to the orthogonal group $\mathscr{O}_q$ in $\mathbb{R}^{q+1}$, in the sense that

$$K(Ax, Ay) = K(x, y), \quad x, y \in S^q, \quad A \in \mathscr{O}_q.$$

The function K' is usually called the *isotropic part* of K.

Schoenberg's result was complemented many decades later with a characterization for the strictly positive definite kernels from his class. The term *strict* is employed if the inequalities in (1.1) are strict for nonzero scalars $c_1, c_2, \ldots, c_n$. According to [5,19] (see also [2]), a kernel K from Schoenberg's class is strictly positive definite if, and only if, in the series representation for the function K', one has:

- ($q \geq 2$): $a_k^q > 0$ for infinitely many even k and infinitely many odd k.
- ($q = 1$): $a_{|k|}^1 > 0$ for k belonging to a set that intersects every full arithmetic progression in $\mathbb{Z}$.

The strict positive definiteness of a positive definite kernel is usually required when interpolation procedures based on the kernel need to be solved. It implies that no matter how many points the interpolation procedure uses, the matrices are always positive definite, in particular, invertible. In statistics language, the strict positive definiteness of the covariance functions (positive definite kernels) provides invertible kriging coefficient matrices and, therefore, the existence of a unique solution for the associated kriging system.

Extensions of the results we have described so far, can be found in [8–11] and references therein.

In this paper, we will consider the analogous problem in Ω_{2q}, the unit sphere in $\mathbb{C}^q$. In this complex setting, the kernels have the form

$$K(z, w) = K'(z \cdot w), \quad z, w \in \Omega_{2q},$$

with a continuous generating function $K' : \Delta[0, 1] \to \mathbb{C}$. Here, we also employ the same dot notation to denote the usual inner product in $\mathbb{C}^q$, $\Delta[0, 1] := \{z \in \mathbb{C} : |z| \leq 1\}$

in the case $q \geq 2$ while $\Delta[0,1] = \Omega_2$ otherwise. The notion of positive definiteness now requires that

$$\sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu} K(z_\mu \cdot z_\nu) \geq 0, \tag{1.2}$$

for $n \geq 1$, any choice of distinct points $z_1, z_2, \ldots, z_n$ on Ω_{2q} and complex numbers $c_1, c_2, \ldots, c_n$. Strict positive definiteness now demands strict inequalities in (1.2) whenever the complex numbers c_μ are nonzero. These kernels are invariant with respect to the unitary group $\mathscr{U}_q$ in $\mathbb{C}^q$ in the sense that

$$K(Az, Aw) = K(z, w), \quad z, w \in \Omega_{2q}, \quad A \in \mathscr{U}_q.$$

According to [17], a kernel K as in the previous paragraph is positive definite on Ω_{2q}, $q \geq 2$, if, and only if,

$$K'(z) = \sum_{m,n=0}^{\infty} a_{m,n}^q R_{m,n}^{q-2}(z), \quad z \in \Delta[0,1], \tag{1.3}$$

in which all the coefficients $a_{m,n}^q$ are nonnegative, $R_{m,n}^{q-2}$ is the disk polynomial of bi-degree (m,n) associated to the integer $q-2$ and normalized so that $R_{m,n}^q(1) = 1$ and $\sum_{m,n=0}^\infty a_{m,n}^q < \infty$. In the case $q = 1$, the representation becomes

$$K'(z) = \sum_{m \in \mathbb{Z}} a_m z^m, \quad z \in \Omega_2, \tag{1.4}$$

in which all coefficients a_m are nonnegative and $\sum_{m \in \mathbb{Z}} a_m < \infty$. At this point it is worth mentioning references [14,15,20] for additional information on the harmonic analysis on Ω_{2q}.

For $\alpha > -1$, the *disk polynomial* $R_{m,n}^\alpha$ of bi-degree (m,n) is given by the formula

$$R_{m,n}^\alpha(z) := r^{|m-n|} e^{i(m-n)\theta} R_{m\wedge n}^{(\alpha,|m-n|)}(2r^2 - 1), \quad z = re^{i\theta} = x + iy,$$

in which $R_{m\wedge n}^{(\alpha,|m-n|)}$ is the Jacobi polynomial of degree $m \wedge n := \min\{m,n\}$ associated to the numbers α and $|m-n|$, and normalized by $R_{m\wedge n}^{(\alpha,|m-n|)}(1) = 1$. Obviously, $R_{m,n}^\alpha$ is a polynomial of degree m in the variable z and of degree n in the variable $\overline{z}$. Due to the orthogonality relations for Jacobi polynomials, the set $\left\{R_{m,n}^\alpha : 0 \leq m, n < \infty\right\}$ is a complete orthogonal system in $L^2(\Delta[0,1], dw_\alpha)$, where dw_α is the positive measure of total mass one on $\Delta[0,1]$ defined by

$$dw_\alpha(z) = \frac{\alpha+1}{\pi}\left(1 - x^2 - y^2\right)^\alpha dxdy, \quad z = x + iy.$$

Earlier studies on disk polynomials are [4,6,14]. Disk polynomials are also known as generalized Zernike polynomials, since they are natural extensions of the standard radial Zernike polynomials used in the characterization of circular optical imaging systems with non-uniform pupil functions in Optics [13,16]. Recent references on disk polynomials are [1,24] and references therein while [23] is a source for applications.

For a function K' as in (1.4), it is shown in [19] that the kernel $K(z,w) = K'(z\cdot w)$, $z, w \in \Omega_2$, is strictly positive definite if, and only if, the set $\{m : a_m > 0\}$ from (1.4) intersects every full arithmetic progression in $\mathbb{Z}$. In view of the previous comments, our intention here is to prove the following complement:

Theorem 1.1 *Let K' be a function as in (1.3). The kernel K given by $K(z,w) = K'(z\cdot w)$, $z, w \in \Omega_{2q}$, is strictly positive definite if, and only if, the set $\left\{m-n : a_{m,n}^{q-2} > 0\right\}$ intersects every full arithmetic progression in $\mathbb{Z}$.*

The proof of the theorem will appear in Sect. 3. In Sect. 4, we will point how to extend the characterization for positive definite kernels of the same nature on the unit sphere in the complex ℓ_2.

2 Technical results

In this section, we describe an asymptotic formula for disk polynomials to be required in the closing arguments in the proof of the main result of the paper to be presented in Sect. 3.

Let us formalize the normalization for the Jacobi polynomials we are using here:

$$R_k^{(\alpha,\beta)} = \frac{P_k^{(\alpha,\beta)}}{P_k^{(\alpha,\beta)}(1)}, \quad k = 0, 1, \ldots,$$

in which $P_k^{(\alpha,\beta)}$ is the standard Jacobi polynomial as explored in [22]. Since the Jacobi polynomials satisfy the recurrence formula [22, p. 71]

$$(1-t)P_k^{(\alpha+1,\beta)}(t) = \frac{2}{2k+\alpha+\beta+2}\left[(k+\alpha+1)P_k^{(\alpha,\beta)}(t) - (k+1)P_{k+1}^{(\alpha,\beta)}(t)\right],$$

we have that

$$(1-t)R_k^{(\alpha+1,\beta)}(t) = \frac{2}{2k+\alpha+\beta+2}\left[(k+\alpha+1)\frac{P_k^{(\alpha,\beta)}(1)}{P_k^{\alpha+1,\beta}(1)}R_k^{(\alpha,\beta)}(t)\right.$$
$$\left. - (k+1)\frac{P_{k+1}^{(\alpha,\beta)}(1)}{P_k^{\alpha+1,\beta}(1)}R_{k+1}^{(\alpha,\beta)}(t)\right].$$

Recalling that

$$P_k^{(\alpha,\beta)}(1) = \binom{k+\alpha}{k}, \quad k = 0, 1, \ldots,$$

the previous equality reduces itself to the following recurrence relation for normalized Jacobi polynomials

$$(1-t)R_k^{(\alpha+1,\beta)}(t) = \frac{2(\alpha+1)}{2k+\alpha+\beta+2}\left[R_k^{(\alpha,\beta)}(t) - R_{k+1}^{(\alpha,\beta)}(t)\right].$$

In particular, for $r \in [0, 1)$, we deduce that

$$(1-r^2)R_k^{(\alpha+1,\beta)}(2r^2-1) = \frac{\alpha+1}{2k+\alpha+\beta+2}\left[R_k^{(\alpha,\beta)}(2r^2-1) - R_{k+1}^{(\alpha,\beta)}(2r^2-1)\right].$$

If $m, n \in \mathbb{Z}_+$, $\beta := |m-n|$ and $k := m \wedge n = \min\{m, n\}$, then the previous relation takes the form

$$\begin{aligned}(1-r^2)R_{m\wedge n}^{(\alpha+1,|m-n|)}(2r^2-1) = \frac{\alpha+1}{m+n+\alpha+2}&\left[R_{m\wedge n}^{(\alpha,|m-n|)}(2r^2-1)\right.\\ &\left.-R_{(m+1)\wedge(n+1)}^{(\alpha,|m-n|)}(2r^2-1)\right],\end{aligned}$$

where we have used the relation $2m \wedge n + |m-n| = m+n$ in order to simplify the equality. Another adjustment leads to

$$\begin{aligned}&e^{i\theta(m-n)}r^{|m-n|}(1-r^2)R_{m\wedge n}^{(\alpha+1,|m-n|)}(2r^2-1)\\ &= \frac{\alpha+1}{m+n+\alpha+2}\left[e^{i\theta(m-n)}r^{|m-n|}R_{m\wedge n}^{(\alpha,|m-n|)}(2r^2-1)\right.\\ &\quad\left.-e^{i\theta((m+1)-(n+1))}r^{|(m+1)-(n+1)|}R_{(m+1)\wedge(n+1)}^{(\alpha,|(m+1)-(n+1)|)}(2r^2-1)\right]\end{aligned}$$

for $r \in [0, 1)$ and $\theta \in [0, 2\pi)$. We are ready to prove the following limit formula for disk polynomials.

Lemma 2.1 *If* $\alpha > -1$, $z \in \Delta[0, 1]$ *and* $|z| \neq 1$, *then*

$$\lim_{m+n\to\infty} R_{m,n}^{\alpha+1}(z) = 0.$$

Proof Writing $z = re^{i\theta}$, with $r \in [-1, 1]$ and $\theta \in [0, 2\pi)$ and applying the equality preceding the lemma in the definition of disk polynomials leads to the following recurrence formula

$$(1-|z|^2)R_{m,n}^{\alpha+1}(z) = \frac{\alpha+1}{m+n+\alpha+2}\left[R_{m,n}^{\alpha}(z) - R_{m+1,n+1}^{\alpha}(z)\right], \quad |z| \leq 1, \quad \alpha > -1.$$

Due to the normalization adopted for the disk polynomials, we know that $|R_{m,n}^{\alpha}(z)| \leq 1$, $m, n \in \mathbb{Z}_+$. Hence, if $|z| < 1$, then

$$\left|R_{m,n}^{\alpha+1}(z)\right| \leq \frac{2}{1-|z|^2}\frac{\alpha+1}{m+n+\alpha+2},$$

which implies the limit formula in the statement of the lemma. □

Since the definition for the Jacobi polynomials $P_k^{(\alpha,\beta)}$ demands $\alpha > -1$, the previous lemma does not hold for the disk polynomials $R_{m,n}^0$. Indeed, since

$$R_{m,n}^0(0) = (-1)^m \delta_{m,n}, \quad m, n \in \mathbb{Z}_+,$$

the limit $\lim_{m+n\to\infty} R_{m,n}^0(0)$ may not exist while

$$\lim_{\substack{m+n\to\infty \\ m\neq n}} R_{m,n}^0(0) = 0.$$

However, the point $z = 0$ is the only exception, as we now show.

Lemma 2.2 *If* $z \in \Delta[0,1]$ *and* $0 < |z| < 1$*, then*

$$\lim_{m+n\to\infty} R_{m,n}^0(z) = 0.$$

Proof Here, we will employ the Bernstein inequality for Jacobi polynomials recently proved by Haagerup and Schlichtkrull [12]. For $\alpha = 0$, it reads

$$\left|(1-t^2)^{1/4}\left(\frac{1+t}{2}\right)^{\beta/2} R_k^{(0,\beta)}(t)\right| \leq \frac{C}{(2k+\beta+1)^{1/4}}, \quad k = 0, 1, \ldots, \quad t \in [-1, 1],$$

in which C is a constant at most 12 and not depending upon k. Replacing t with $2r^2-1$, leads to

$$\left[4r^2(1-r^2)\right]^{1/4} r^\beta \left|R_k^{(0,\beta)}(2r^2-1)\right| \leq \frac{C}{(2k+\beta+1)^{1/4}}, \quad k = 0, 1, \ldots, \quad r \in [0, 1].$$

It is now clear that

$$\begin{aligned} \left|R_{m,n}^0(z)\right| &\leq \frac{2^{-1/2}C}{r^{1/2}(1-r^2)^{1/4}(2m\wedge n + |m-n| + 1)^{1/4}} \\ &= \frac{2^{-1/2}C}{r^{1/2}(1-r^2)^{1/4}(m+n+1)^{1/4}}, \quad m, n \in \mathbb{Z}_+, \quad 0 < |z| < 1. \end{aligned}$$

This implies the limit formula in the statement of the lemma. □

3 The Proof of Theorem 1.1

In this section, we will assume that $q \geq 2$. We begin recalling the notion of antipodal points on Ω_{2q}: two distinct points z and w over Ω_{2q} are *antipodal* if $|z \cdot w| = 1$. In particular, z and w are antipodal if, and only if, there exists $\theta \in (0, 2\pi)$ so that $z = e^{i\theta}w$. Thus, for $z \in \Omega_{2q}$ fixed, there is a whole Ω_2 of points in Ω_{2q} that are antipodal to z.

For a finite subset $\{z_1, z_2, \ldots, z_k\}$ of Ω_{2q}, not containing any pairs of antipodal points, and a subset $\{\theta_1, \theta_2, \ldots, \theta_l\}$ of $[0, 2\pi)$, the *enhanced subset* of Ω_{2q} generated by them is the set

$$\left\{e^{i\theta_1} z_1, e^{i\theta_2} z_1, \ldots, e^{i\theta_l} z_1, e^{i\theta_1} z_2, e^{i\theta_2} z_2, \ldots, e^{i\theta_l} z_2, \ldots, e^{i\theta_1} z_k, e^{i\theta_2} z_k, \ldots, e^{i\theta_l} z_k\right\}.$$

For a positive definite kernel $K(z, w) = K'(z \cdot w)$, $z, w \in \Omega_{2q}$, with K' having the disk polynomial expansion (1.3), the quadratic form (1.2) associated to an enhanced set as above becomes

$$\sum_{\mu,\nu=1}^{k} \sum_{\tau,\lambda=1}^{l} c_\mu^\tau \overline{c_\nu^\lambda} K'\left(\left(e^{i\theta_\tau} z_\mu\right) \cdot \left(e^{i\theta_\lambda} z_\nu\right)\right).$$

Indeed, since an enhanced set may be thought as a double indexed set, we need to double index the complex scalars in the quadratic form accordingly. The quadratic form is zero if, and only if,

$$\sum_{\mu,\nu=1}^{k} \sum_{\tau,\lambda=1}^{l} c_\mu^\tau \overline{c_\nu^\lambda} R_{m,n}^{q-2}\left(e^{i(\theta_\tau-\theta_\lambda)}(z_\mu \cdot z_\nu)\right) = 0,$$

whenever (m, n) belongs to the set $\left\{(m, n) : a_{m,n}^{q-2} > 0\right\}$ associated to the representation (1.3) of K'. Taking into account that disk polynomials are homogeneous in the sense that

$$R_{m,n}^\alpha(e^{i\theta} z) = e^{i(m-n)\theta} R_{m,n}^\alpha(z), \quad m, n \in \mathbb{Z}_+, \quad z \in \Delta[0, 1], \quad \theta \in [0, 2\pi),$$

the following characterization for strict positive definiteness hold.

Theorem 3.1 *Let K' be a function as in* (1.3)*. The following assertions are equivalent:*

(i) *The kernel $K(z, w) = K'(z \cdot w)$, $z, w \in \Omega_{2q}$, is strictly positive definite;*
(ii) *If k and l are positive integers, $\{\theta_1, \theta_2, \ldots, \theta_l\}$ is a subset of $[0, 2\pi)$ and $\{z_1, z_2, \ldots, z_k\}$ is a subset of Ω_{2q}, not containing any pairs of antipodal points, then the only solution $\{c_\mu^\tau : \mu = 1, 2, \ldots, k; \tau = 1, 2, \ldots, l\}$ of the system of equations*

$$\sum_{\mu,\nu=1}^{k} \sum_{\tau,\lambda=1}^{l} c_\mu^\tau \overline{c_\nu^\lambda} e^{i(m-n)(\theta_\tau-\theta_\lambda)} R_{m,n}^{q-2}(z_\mu \cdot z_\nu) = 0, \quad (m, n) \in \left\{(m, n) : a_{m,n}^{q-2} > 0\right\},$$

is the trivial one, that is, all the complex numbers c_μ^τ are zero.

Proof One implication is obvious while the other one follows from the fact that the matrix appearing in the quadratic form (1.2) associated to an enhanced set contains, as a principal sub-matrix, the matrix in the quadratic form associated to the subset of Ω_{2q} that generates the enhanced set. □

Theorem 3.2 *Let K' be a function as in (1.3). If $K(z,w) = K'(z \cdot w)$, $z, w \in \Omega_{2q}$, is strictly positive definite, then the set $\left\{m-n : a_{m,n}^{q-2} > 0\right\}$ intersects every full arithmetic progression in $\mathbb{Z}$.*

Proof Assume

$$\left\{m-n : a_{m,n}^{q-2} > 0\right\} \cap (N\mathbb{Z}+j) = \emptyset,$$

for some $N \geq 1$ and some $j \in \{0,1,\ldots,N-1\}$. We will show that Assertion (ii) in Theorem 3.1 does not hold when we consider $l = N$, $k = 1$, and we take $\theta_\tau = e^{i2\pi\tau/N}$, $\tau = 1,2,\ldots,N$, while $\{z_1\}$ is an arbitrary unitary subset of Ω_{2q}. Indeed, the corresponding system in Theorem 3.1-(ii) takes the form

$$\sum_{\tau,\lambda=1}^{l} c_1^\tau \overline{c_1^\lambda} e^{i2\pi(\tau-\lambda)(m-n)/N} = 0, \quad (m,n) \in \left\{(m,n) : a_{m,n}^{q-2} > 0\right\},$$

that is,

$$\sum_{\tau=1}^{l} c_1^\tau e^{i2\pi\tau(m-n)/N} = 0, \quad (m,n) \in \left\{(m,n) : a_{m,n}^{q-2} > 0\right\}.$$

But, the scalars $c_\tau := e^{-i2\pi\tau j/N}$, $\tau = 1,2,\ldots,N$, provides a nonzero solution $\left\{c_1^\tau : \tau = 1,2,\ldots,N\right\}$ for the system. Indeed, for this choice of the scalars, the system reduces itself to

$$\sum_{\tau=1}^{l} e^{i2\pi\tau(m-n-j)/N} = 0, \quad (m,n) \in \left\{(m,n) : a_{m,n}^{q-2} > 0\right\}.$$

If $(m,n) \in \{(m,n) : a_{m,n}^{q-2} > 0\}$, then the integer $m-n-j$ is not divisible by N. Since $e^{i2\pi/N}$ is a primitive n-th root of unity, the sum is zero. Thus, K cannot be strictly positive definite in this case. □

Next, we demonstrate a technical result involving general exponentials sums of the same type of that used in the proof of the previous theorem.

Lemma 3.3 *Let $z_1, z_2, \ldots, z_n$ be distinct points on Ω_2. If $c_1, c_2, \ldots, c_n$ are complex numbers, not all zero, then the set*

$$\left\{p \in \mathbb{Z} : \sum_{\tau=1}^{l} c_\tau z_\tau^p \neq 0\right\}$$

contains a full arithmetic progression of $\mathbb{Z}$.

Proof Assume that at least one c_τ is nonzero and consider the complement of the set quoted in the statement of the lemma in $\mathbb{C}$, that is,

$$\left\{p \in \mathbb{Z} : \sum_{\tau=1}^{l} c_\tau z_\tau^p = 0\right\}.$$

This set is both a linear recurrence and a proper subset of $\mathbb{Z}$. According to the Skolem–Mahler–Lech theorem [7, p. 25], this set is the union of a finite subset of $\mathbb{Z}$ and a finite number of full arithmetic progressions of $\mathbb{Z}$. Therefore, at least one full arithmetic progression must be a subset of the set in the statement of the lemma. □

The next theorem settles the sufficiency part in Theorem 1.1.

Theorem 3.4 *Let K' be a function as in (1.3). If $\left\{m-n : a_{m,n}^{q-2} > 0\right\}$ intersects each full arithmetic progression in $\mathbb{Z}$, then $K(z,w) = K'(z \cdot w)$, $z, w \in \Omega_{2q}$, is strictly positive definite.*

Proof Assume $\left\{m-n : a_{m,n}^{q-2} > 0\right\}$ intersects each full arithmetic progression in $\mathbb{Z}$. We will apply Theorem 3.1 in order to conclude that K is strictly positive definite. Let k and l be positive integers, $\{\theta_1, \theta_2, \ldots, \theta_l\}$ be distinct angles in $[0, 2\pi)$ and $\{z_1, z_2, \ldots, z_k\}$ a subset of Ω_{2q} containing no pair of antipodal points. We will suppose that the system

$$\sum_{\mu,\nu=1}^{k} \sum_{\tau,\lambda=1}^{l} c_\mu^\tau \overline{c_\nu^\lambda} e^{i(m-n)(\theta_\tau-\theta_\lambda)} R_{m,n}^{q-2}(z_\mu \cdot z_\nu) = 0, \quad (m,n) \in \left\{(m,n) : a_{m,n}^{q-2} > 0\right\},$$

has a nontrivial solution and will reach a contradiction. Without loss of generality, we can assume that at least one of the scalars $c_1^1, c_1^2, \ldots, c_1^l$ is nonzero. Taking into account that the inner double sum in the previous equation is

$$\sum_{\tau,\lambda=1}^{l} c_\mu^\tau \overline{c_\nu^\lambda} e^{i(m-n)(\theta_\tau-\theta_\lambda)} = \sum_{\tau=1}^{l} c_\mu^\tau e^{i(m-n)\theta_\tau} \overline{\sum_{\lambda=1}^{l} c_\nu^\lambda e^{i(m-n)\theta_\lambda}},$$

we will consider the set

$$S := \left\{p \in \mathbb{Z} : \sum_{\tau=1}^{l} c_1^\tau e^{ip\theta_\tau} \neq 0\right\}.$$

Lemma 3.3 asserts that S contains a full arithmetic progression of $\mathbb{Z}$, say $N\mathbb{Z} + j$. Since $\left\{m-n : a_{m,n}^{q-2} > 0\right\}$ intersects each full arithmetic progression in $\mathbb{Z}$, it is clear that the set $\left\{m-n : a_{m,n}^{q-2} > 0\right\} \cap (N\mathbb{Z} + j)$ must be infinite. Now, we can select $\mu_0 \in \{1, 2, \ldots, k\}$ and an infinite set $Q \subset \left\{m-n : a_{m,n}^{q-2} > 0\right\} \cap (N\mathbb{Z} + j)$ so that

$$\left|\sum_{\tau=1}^{l} c_{\mu_0}^{\tau} e^{i(m-n)\theta_\tau}\right| \geq \left|\sum_{\tau=1}^{l} c_{\mu}^{\tau} e^{i(m-n)\theta_\tau}\right|, \quad \mu \in \{1, 2, \ldots, k\}, \quad m-n \in Q.$$

It is worth mentioning that

$$\left|\sum_{\tau=1}^{l} c_{\mu_0}^{\tau} e^{i(m-n)\theta_\tau}\right| \geq \left|\sum_{\tau=1}^{l} c_{1}^{\tau} e^{i(m-n)\theta_\tau}\right| > 0, \quad m-n \in Q.$$

Next, let us denote by Q' the unbounded set

$$\{(m,n) : m-n \in Q\backslash\{0\}\} \cap \left\{(m,n) : a_{m,n}^{q-2} > 0\right\}.$$

Here, we need to consider $Q\backslash\{0\}$ instead of Q in order to accommodate the unexpected limit quoted before Lemma 2.2 and, consequently, to be able to handle the case $q = 2$. Returning to the original system, but restricting ourselves to Q', we have that

$$\begin{aligned} 0 = R_{m,n}^{q-2}(z_{\mu_0} \cdot z_{\mu_0}) &+ \sum_{\mu \neq \mu_0} \frac{\left|\sum_{\tau=1}^{l} c_{\mu}^{\tau} e^{i(m-n)\theta_\tau}\right|^2}{\left|\sum_{\tau=1}^{l} c_{\mu_0}^{\tau} e^{i(m-n)\theta_\tau}\right|^2} R_{m,n}^{q-2}(z_\mu \cdot z_\mu) \\ &+ \sum_{\mu \neq \nu} \frac{\sum_{\tau=1}^{l} c_{\mu}^{\tau} e^{i(m-n)\theta_\tau} \overline{\sum_{\tau=1}^{l} c_{\nu}^{\tau} e^{i(m-n)\theta_\tau}}}{\sum_{\tau=1}^{l} c_{\mu_0}^{\tau} e^{i(m-n)\theta_\tau} \overline{\sum_{\tau=1}^{l} c_{\mu_0}^{\tau} e^{i(m-n)\theta_\tau}}} R_{m,n}^{q-2}(z_\mu \cdot z_\nu). \end{aligned}$$

Then, we can deduce the main inequality

$$0 \geq 1 + \sum_{\mu \neq \nu} \frac{\sum_{\tau=1}^{l} c_{\mu}^{\tau} e^{i(m-n)\theta_\tau} \overline{\sum_{\tau=1}^{l} c_{\nu}^{\tau} e^{i(m-n)\theta_\tau}}}{\sum_{\tau=1}^{l} c_{\mu_0}^{\tau} e^{i(m-n)\theta_\tau} \overline{\sum_{\tau=1}^{l} c_{\mu_0}^{\tau} e^{i(m-n)\theta_\tau}}} R_{m,n}^{q-2}(z_\mu \cdot z_\nu), \quad (m,n) \in Q'.$$

Since Q' is unbounded, the same is true of the set $\{m+n : (m,n) \in Q'\}$. On the other hand, since the set $\{z_1, z_2, \ldots, z_k\}$ does not contain pairs of antipodal points, we have that

$$|z_\mu \cdot z_\nu| < 1, \quad \mu, \nu = 1, 2, \ldots, k, \quad \mu \neq \nu.$$

Taking into account these two pieces of information and also that

$$\left|\frac{\sum_{\tau=1}^{l} c_{\mu}^{\tau} e^{i(m-n)\theta_\tau} \overline{\sum_{\tau=1}^{l} c_{\nu}^{\tau} e^{i(m-n)\theta_\tau}}}{\sum_{\tau=1}^{l} c_{\mu_0}^{\tau} e^{i(m-n)\theta_\tau} \overline{\sum_{\tau=1}^{l} c_{\mu_0}^{\tau} e^{i(m-n)\theta_\tau}}}\right| \leq 1, \quad \mu \neq \nu, \quad (m,n) \in Q',$$

we can apply Lemmas 2.1 and 2.2, to conclude that

$$\lim_{m+n\to\infty} \frac{\sum_{\tau=1}^{l} c_\mu^\tau e^{i(m-n)\theta_\tau} \overline{\sum_{\tau=1}^{l} c_\nu^\tau e^{i(m-n)\theta_\tau}}}{\sum_{\tau=1}^{l} c_{\mu_0}^\tau e^{i(m-n)\theta_\tau} \overline{\sum_{\tau=1}^{l} c_{\mu_0}^\tau e^{i(m-n)\theta_\tau}}} R_{m,n}^{q-2}(z_\mu \cdot z_\nu) = 0, \quad \mu \neq \nu,$$

as long as $(m, n) \in Q'$. Therefore, we can return to the main inequality to deduce that $0 \geq 1 - 1/2$, a clear contradiction. □

We would like to observe that Theorem 1.1 proved here corrects a wrong argument developed in the proof of the main Theorem in [18]. There, the reader may also find some other partial results on positive definiteness and strict positive definiteness of kernels fitting in the complex setting considered here.

4 The unit sphere in the complex ℓ_2

Here, we consider kernels of the form $K(z, w) = K'(z \cdot w)$, $z, w \in \Omega_\infty$, in which Ω_∞ is the unit sphere in the complex ℓ_2, $\cdot$ is the usual inner product of ℓ_2 and K' is a complex continuous function on $\Delta[0, 1] = \{z \in \mathbb{C} : |z| \leq 1\}$. The concepts previously introduced for kernels on Ω_{2q} hold true for kernels on Ω_∞ modulus obvious modifications. The positive definiteness of the kernel corresponds to the following series representation for K' [3, p. 171]:

$$K'(z) = \sum_{m,n=0}^{\infty} a_{m,n}^\infty R_{m,n}^\infty(z), \quad z \in \Delta[0, 1], \tag{4.1}$$

in which all the coefficients $a_{m,n}^\infty$ are nonnegative,

$$R_{m,n}^\infty(z) = z^m \overline{z}^n, \quad z \in \Delta[0, 1],$$

and $\sum_{m,n=0}^{\infty} a_{m,n}^\infty < \infty$.

The characterization for strict positive definiteness follows the same pattern of that in Theorem 1.1.

Theorem 4.1 *Let K' be a function as in (4.1). The kernel $K(z, w) = K'(z \cdot w)$, $z, w \in \Omega_\infty$, is strictly positive definite if, and only if, the set $\{m - n : a_{m,n}^\infty > 0\}$ intersects every full arithmetic progression in $\mathbb{Z}$.*

Proof The necessity part of the theorem goes along the lines of the proof of Theorem 3.2. If we assume that

$$\left\{m - n : a_{m,n}^\infty > 0\right\} \cap (N\mathbb{Z} + j) = \emptyset,$$

for some $N \geq 1$ and some $j \in \{0, 1, \ldots, N-1\}$, we may consider the points $z_1, z_2, \ldots, z_N$ on Ω_∞ given by

$$z_\mu := (e^{i2\pi\mu/N}, 0, 0, \ldots), \quad \mu = 1, 2, \ldots, N,$$

and the scalars

$$c_\mu = \exp(-i2\pi\mu j/N), \quad \mu = 1, 2, \ldots, N,$$

in order to see that

$$\sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} K'(z_\mu \cdot z_\nu) = \sum_{m,n=0}^{\infty} a_{m,n}^{\infty} \left| \sum_{\mu=1}^{N} e^{i2\pi\mu(m-n-j)/N} \right|^2 = 0,$$

a contradiction with the strict positive definiteness of the kernel. Since

$$\lim_{m+n\to\infty} \left| R_{m,n}^{\infty}(z) \right| = \lim_{m+n\to\infty} |z|^{m+n} = 0, \quad z \in \Delta[0,1], \quad |z| \neq 1,$$

the proof of Theorem 3.4 can be adapted to hold in the present case, after one verifies that Theorem 3.1 can be also adapted. Thus, the sufficiency of the condition holds in this case as well. □

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