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Ordered Banach algebras and multi-norms: some open problems

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Abstract This is a list of open problems posed at the workshop on *Ordered Banach Algebras* held at the Lorentz Center, Leiden, during the week 21–25 July, 2014.

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We call A a *Banach lattice algebra* if it is both a Banach lattice and a Banach algebra and furthermore the product of two positive elements is positive.

Problem 1 Characterize those Banach lattice algebras A for which the left regular representation $a \mapsto L_a$, where $L_a(x) = ax$, into the space of regular operators $\mathcal{L}^r(A)$ preserves finite suprema and infima, as well as being an algebra homomorphism. Even an answer for Dedekind complete Banach lattice algebras would be of interest, when we are asking about it being a lattice homomorphism. In addition, consider when such representations are either faithful or isometric. The natural norm to take on $\mathcal{L}^r(A)$ would be the regular norm, but it is conceivable that the question might be of interest for the operator norm as well.

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Problem 2 Characterize those Banach lattice algebras A for which there exists any faithful algebra homomorphism, preserving finite suprema and infima, into some space of regular operators, $\mathcal{L}^r(E)$. Note that the existence of such a representation does not imply that the left regular representation possesses these properties, see [20]. In particular it seems to be open as to whether or not every Banach lattice algebra can be represented in this manner. If such a representation exists, when can it be taken to be isometric, either for the regular or operator norm on $\mathcal{L}^r(E)$?

Problem 3 If there is such a representation of A in $\mathcal{L}^r(E)$, as asked in Problem 2, then it would be tempting to compose it with the map $T \mapsto T^{**} : \mathcal{L}^r(E) \to \mathcal{L}^r(E^{**})$ to obtain a representation of A as a space of regular operators on a Dedekind complete Banach lattice. However, the map $T \mapsto T^{**}$ need not be a lattice homomorphism, so this process will fail. Therefore it is a separate question to characterize those A which can be represented in this way in $\mathcal{L}^r(E)$, with E being Dedekind complete.

Problem 4 In the absence of complete solutions to Problems 1, 2 and 3, identify relatively simple sufficient conditions for a Banach lattice algebra to have such representations.

Problem 5 Given that Problem 3 is an analogue of the Gelfand–Naimark theorem, for which the Gelfand–Naimark–Segal construction is a key ingredient, we ask if there is a positive Gelfand–Naimark–Segal construction for lattice algebra homomorphisms from a Banach lattice algebra (in a suitable class) into the Banach lattice algebra of regular operators on a Dedekind complete Banach lattice?

Problem 6 Arendt, [2,3], has given an indirect proof using deep results of Brainerd and Edwards that if G is a locally compact group then the left regular representation of $L^1(G)$ in $\mathcal{L}^r\big(L^1(G)\big)$ is a lattice homomorphism. He also proves that if $1 < p < \infty$ and G is amenable then the action of $L^1(G)$ on $L^p(G)$ gives a lattice homomorphism of $L^1(G)$ into $\mathcal{L}^r\big(L^p(G)\big)$. Give a direct proof.
Update: This has been solved in [11], where the amenability condition on G in the case $1 < p < \infty$ is removed. This is ongoing research and will be published in due course.

Problem 7 It also follows from [2,3] that there is an isometric Banach lattice and algebra isomorphism of $M(G)$ into $\mathcal{L}^r\big(L^p(G)\big)$ for any $p \in [1, \infty)$ if either $p = 1$ or G is amenable. It does not seem to be so clear that the left regular representation of $M(G)$ in $\mathcal{L}^r\big(M(G)\big)$ possesses these properties.
Update: In view of [11], is the amenability of G necessary for the representation of $M(G)$ in $\mathcal{L}^r\big(L^p(G)\big)$ to be a lattice homomorphism in the case $p > 1$?

Problem 8 When there is a faithful or isometric representation of a Banach lattice algebra A in $\mathcal{L}^r(E)$, what extra conditions on A are needed for the representation to be chosen so that the image of A can be a band (resp. σ-ideal or ideal) in $\mathcal{L}^r(E)$? In particular, is this always possible? Of course, the definitions of *band* etc. will need to be modified when $\mathcal{L}^r(E)$ is not a lattice.

Problem 9 If a Banach lattice algebra has an approximate identity (e_γ) with all $\|e_\gamma\| \le 1$, must it have an approximate identity composed of positive elements?

Problem 10 Suppose that A is a non-unital Banach lattice algebra. Let B_1 and B_2 be unital Banach lattice algebras, containing A as a sublattice and subalgebra, such that the smallest unital sublattice and subalgebra of B_k containing A is the whole of B_k $(k = 1, 2)$. Is it true that for all $x \in A$, $\sigma_{B_1}(x) = \sigma_{B_2}(x)$? The same question is of interest if B_1 and B_2 are restricted to be Banach lattice algebras with identities of norm 1.

Problem 11 Find conditions on a unital Banach lattice algebra A such that if A is a sublattice and unital subalgebra of a larger Banach lattice algebra B then $\sigma_A(x) = \sigma_B(x)$ for all $x \in A$. It would be of particular interest if some familiar order theoretic property of A, such as Dedekind completeness, were sufficient.

Problem 12 If A is a Banach lattice algebra that is not Dedekind complete, can it be embedded in a Dedekind complete Banach lattice superalgebra B in such a way that every lattice algebra homomorphism from A into a Dedekind complete Banach lattice algebra C extends uniquely to a lattice algebra homomorphism from B into C? A candidate construction for such B might consist of the Dedekind completion of A with a judicious choice of extensions of the norm and multiplication on A, but it is not even clear whether this can always be done so as to make B a Banach lattice algebra.

Problem 13 There is a theory of free Banach lattices, [15]. Is there a sensible notion of a free Banach lattice algebra? If so, what can be said about its representations?

Problem 14 Develop a categorically satisfying theory of Banach lattice algebra tensor products. Presumably the Fremlin tensor product of Banach lattices would be a starting point, but this is not entirely clear.

Problem 15 If A is a non-unital Banach lattice algebra, can it be embedded in a unital Banach lattice superalgebra B in such a way that every lattice algebra homomorphism from A into a unital Banach lattice algebra C extends uniquely to a unital lattice and algebra homomorphism from B into C? If we restrict C to lie in the class of unital Banach lattice algebras with positive identities, can we make B have a positive identity?

Problem 16 If the Banach lattice algebra A has a positive algebra identity e and also has a faithful lattice and algebra representation in $\mathcal{L}^r(E)$, is the ideal generated by the image of e necessarily an f-algebra? If $E = A$ and the representation is the left regular representation then the image of e is the identity operator on A and this ideal is precisely $Z(A)$ which is certainly an f-algebra.

Problem 17 Is there a "positive" Cohen factorization for Banach lattice algebras?
Update: As pointed out in [10], Rudin has given a negative answer to the general question in [19], but there are some algebras for which the answer is positive. The picture as a whole is not clear yet.

Problem 18 A unital C^*-algebra is generated, in fact spanned, by its unitaries. Is there an interesting class of Banach lattice algebras that is generated (or even spanned) as a Banach lattice algebra by its positive invertible norm one elements that have a positive norm one inverse?

Problem 19 Certain specific well-known (and lesser known) Banach algebras still need to be investigated in order to determine if they are, in fact, ordered Banach algebras and, if so, what properties their algebra cones possess. This list includes $L^1(G)$ (with G a locally compact commutative group) and, in particular, $L^1(\mathbb{Z}) = \ell^1$ (the Wiener algebra); the disc algebra both as an ordered subalgebra of the corresponding algebra of continuous functions and with the positive elements being the functions such that all coefficients in their Taylor series expansion are positive; the order bounded operators on a complex Banach lattice E (note that they are the same as the regular operators on E if E is Dedekind complete) and the multi-bounded operators on a complex Banach lattice.

Problem 20 Given a two-sided (algebraic) ideal, or a multiplicative ideal, or an order ideal (how should the latter be defined?) I in an ordered Banach algebra A, and $a, b \in A$ with $0 \leq a \leq b$, under what conditions does it follow from $b \in I$ that $a^n \in I$ for some natural number n? This constitutes a more general form of the well-known domination problem (see [14], Sect. 4.2).

Problem 21 We know (see [18]) that the algebra cone πC in the quotient algebra A/I of an ordered Banach algebra A modulo a closed ideal I (with $\pi : A \to A/I$ the canonical homomorphism) is proper if and only if it follows from $a, b \in A,\ 0 \leq a \leq b$ and $b \in I$ that $a \in I$. Would it be possible to characterize some property of the algebra cone in A/I in terms of the property $[a, b \in A$ with $0 \leq a \leq b$ and $b \in I] \Rightarrow [a^n \in I$ for some $n \in \mathbb{N}]$ mentioned in Problem 20?

Problem 22 Let A be an ordered Banach algebra and let $a, b \in A$ such that $0 \leq a \leq b$. Given a spectral property (P), provide conditions which will ensure that if a satisfies (P), then b^n satisfies (P) for some natural number n. This constitutes a "converse" domination problem.

Problem 23 Can any of the existing domination results currently relying on weak monotonicity of the spectral radius in the quotient algebra (see Sect. 4.2 in [14]) be proved without this condition? Maybe if the algebra cone in the original Banach algebra is assumed to be normal and/or generating? (This could allow these results to apply to other cases than the regular operators.)

Problem 24 Can the ergodic domination theorem (see Theorem 5.5 in [13]) be extended by replacing the condition "$0 \leq a \leq b$" with the weaker condition "$\pm a \leq b$"?

Problem 25 Can any of the relevant Gelfand-Hille/Huijsmans-De Pagter results (see Sect. 4.3 in [14]) be improved if the algebra cone is assumed to be normal and/or generating? In particular, in Theorem 4.3.8 in [14] where 1 is assumed to be a pole of the resolvent of a, can the result be proved for orders higher than 2?

Problem 26 The concept of irreducible elements in ordered Banach algebras was established in [1]. Several aspects of these elements, in particular the peripheral spectrum, still have to be investigated.

Problem 27 Investigate the possibility of generalizing order continuity results from operators on Banach lattices to ordered Banach algebras.

The reader is referred to [9] for the primary definitions concerning multi-norms and to [5–7] for further material.

Problem 28 In [9] Theorem 6.33, the space of multi-bounded operators between two Banach lattices, taken with their Banach lattice multi-norm, is identified in some cases. Often they are just the space of regular operators between the Banach lattices. What are they in other cases?

Problem 29 The theory of decompositions of Banach spaces in relation to multi-norms involves *orthogonal* and *small* decompositions, using the language of §7.1 of [9]. Each small decomposition is orthogonal. It would be nice if the converse were true.

Problem 30 In [7] and [5], the authors attempted to determine when two (p, q)-multi-norms (for $1 \leq p \leq q$) based on the spaces ℓ^r, where $r \geq 1$, are equivalent. They have a full solution when $r = 1$ and when $r \geq 2$, but less than a full solution when $1 < r < 2$. There is a strong connection between this question and the calculation of absolutely summing norms. One method of showing that two points (p_1, q_1) and (p_2, q_2) do not give equivalent multi-norms would be to show that there is no constant $C > 0$ such that

$$\pi_{q_1,p_1}(I_n : \ell_n^{r'} \to \ell_n^r) \leq C\pi_{q_2,p_2}(I_n : \ell_n^{r'} \to \ell_n^r)$$

for all $n \in \mathbb{N}$, where $\pi_{q,p}$ denotes the (q, p)-summing norm. A good step towards the resolution of the remaining case would be to calculate an asymptotic formula for $\pi_{2,1}(I_n : \ell_n^{r'} \to \ell_n^r)$. This seems to be an interesting question in its own right: its is surprising that there is apparently no answer in the classical literature.

It is possible to define the notion of a "multi-Banach algebra". Indeed, let A be a Banach algebra and $(\|\cdot\|_n)$ be a multi-norm based on the Banach space A. Then $(A, \|\cdot\|_n)$ is a *multi-Banach algebra* if

$$\|(a_1b_1, \ldots, a_nb_n)\|_n \leq \|(a_1, \ldots, a_n)\|_n \|(b_1, \ldots, b_n)\|_n$$

for all $n \in \mathbb{N}$ and all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$.

Problem 31 Is there any useful general theory of multi-Banach algebras?

Problem 32 Several multi-norms on the group algebra $L^1(G)$ of a locally compact group give a multi-Banach algebra. Does this lead to any interesting information about $L^1(G)$? Which multi-norms work here?

Problem 33 The multi-bounded operators $\mathcal{M}(E)$ based on a fixed multi-normed space E give a multi-Banach algebra. This generalizes the Banach algebra $\mathcal{L}^r(E)$ of all regular operators on a (suitable) Banach lattice E. Is this interesting? What are the Banach algebras that we obtain in cases where $\mathcal{M}(E)$ is not equal to $\mathcal{L}^r(E)$?

Problem 34 Both $L^1(G)$ and $\mathcal{M}(E)$ (the multi-bounded operators on a Banach lattice) are often examples of Banach lattice algebras and of multi-Banach algebras. Is there a useful resonance between the theories of Banach lattice algebras and of multi-Banach algebras?

The notion of a p-multi-norm was first given in [17], under the name "type-p multi-norm". They are mentioned in [9], whilst [8] is a memoir on this topic. The *canonical p-multi-norm*, $(\|\cdot\|_n^{L,P})$, on a Banach lattice E, for $p \in [1, \infty)$, is defined by

$$\|\mathbf{x}\|_n^{L,p} = \left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\|,$$

for $n \in \mathbb{N}$ and $\mathbf{x} = (x_1, \ldots, x_n) \in E^n$.

Problem 35 In at least some cases, the p-multi-bounded operators from a Banach lattice into itself form a Banach lattice algebra. How does this tie in with order theoretically defined spaces of operators on a Banach lattice?

Problem 36 In [9], there is a duality theory for multi-norms that gives a multi-norm (not a dual multi-norm). Is there a similar duality theory for p-multi-norms that gives a p-multi-norm (not a p'-multi-norm)?

Problem 37 A 2-convex 2-multi-norm is just an *operator sequence space* in the sense of [12]. Do any results of [12] generalize to suitable p-multi-norms?

Problem 38 A p-multi-norm is a special case of a power-norm, as defined in [8] and studied in [4]. An example of a power-norm $(\|\cdot\|_n)$ based on a Banach space E is given by

$$\|(x_1, \ldots, x_n)\|_n = \int_0^1 \left\| \sum_{k=1}^{n} r_k(t) x_k \right\| \, dt \quad (x_1, \ldots, x_n \in E, n \in \mathbb{N}),$$

where (r_k) is the sequence of Rademacher functions. Does this example fit into a version of the p-multi-norm theory? Does it relate to Rademacher-bounded sequences of operators, rather extensively studied in the last decade?

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