Positivity
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Positivity

On order bounded weighted composition operators between Dirichlet spaces

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Abstract Recently, Gao et al. (Chin Ann Math 37B:585–594, 2016) proved a sufficient condition for order boundedness of a weighted composition operator acting between Dirichlet spaces. In this paper, we prove that their condition is a necessary and sufficient condition for order boundedness of a weighted composition operator acting between these spaces.

Keywords Banach lattice · Hardy space · Dirichlet space · Weighted Bergman space · Weighted composition operator · Order bounded operator

Mathematics Subject Classification 47B38 · 30H10 · 30H20 · 46E15

1 Introduction

Recall that a *partially ordered vector space* is a real vector space X equipped with an order relation $\leq$ that is compatible with the algebraic structure as follows:

1. If $x \leq y$, then $x + z \leq y + z$ for all $z \in X$.
2. If $x \leq y$, then $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

A partially ordered vector space X is called a *Riesz space* (or a *vector lattice*) if for each pair of vectors $x, y \in X$, both the supremum $(x \vee y)$ and the infimum $(x \wedge y)$ of the set $\{x, y\}$ exist in X. Note that, for $x \in X$, we use the standard notations $x^+ := x \vee 0$, $x^- = x \wedge 0$, $x := x^+ - x^-$ and $|x| := x^+ + x^-$. A norm $\|\cdot\|$ on a Riesz space is

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said to be a lattice norm if $\|x\| \le \|y\|$, whenever $|x| \le |y|$. A Riesz space equipped with a lattice norm is called *normed Riesz space*. If a normed Riesz space is also norm complete, then it is called a *Banach lattice*.

For example, if X is a compact Hausdorff space then $C(X)$ the set of all real-valued continuous functions on X with pointwise addition and scalar multiplication and with the norm $\|f\| = \sup_{x\in X}\{|f(x)| : x \in X\}$ is a Banach lattice with an ordering defined as: $f \le g$ if and only if $f(x) \le g(x)$ for all x.

A subset S of a vector lattice X is *order bounded* if there exist an element $y \in X$ such that $|x| \le y$ holds for all $x \in S$. Let X and Y be two Banach lattices. Then an operator $T : X \to Y$ is *order bounded* if T maps order bounded sets in X to order bounded sets in Y.

Since $\mathbb{C}$ is not an ordered field, so there does not exist an ordering between two holomorphic functions in general. In particular, there does not exist an ordering between two functions in a Hardy space or a weighted Bergman space or a Dirichlet type space. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and

$$L^p(\mu) = L^p(\Omega, \mathcal{A}, \mu) = \left\{ f | f : \Omega \to \mathbb{C} \text{ is measurable and } \int_\Omega |f|^p d\mu < \infty \right\}.$$

Then for any non-negative $h \in L^p(\mu)$ and a measurable function g such that $0 \le g \le h$, we have that

$$\int_\Omega |g|^p d\mu \le \int_\Omega |h|^p d\mu < \infty.$$

Thus $g \in L^p(\mu)$. That is, $[0, h]$ is an ordered interval in $L^p(\mu)$. Thus motivated by the standard definition of order bounded operators between two quasi-Banach lattices, a property of operators which is closely related to the notion of boundedness known as order boundedness, was introduced by Hunziker and Jarchow in [7]. See [2,10,11] also.

Definition 1.1 An operator T from a quasi-Banach space X to a subspace of a quasi-Banach lattice L is *order bounded* if T maps the unit ball B_X of X into an ordered interval of L, that is, there exists a non-negative element g in L such that $|Tf| \le g$ for all $f \in B_X$.

This concept plays an important role in understanding the properties of concrete linear operators acting between Banach spaces in general and holomorphic function spaces such as Hardy spaces, weighted Bergman spaces and Dirichlet spaces in particular. For example,

1. Kwapień [10] and Schwartz [11] proved the following result: If X is a Banach space, μ is any measure, $1 \le p < \infty$ and $T : X \to L^p(\mu)$ is order bounded, then T is p-integral. If T^* is p-summing then T is order-bounded. Recall that a bounded linear operator $T : X \to Y$ is *p-integral* if there exists a probability measure μ along with a factorization

$$\eta_Y \circ T : X \xrightarrow{v} L^\infty(\mu) \overset{i_p}{\hookrightarrow} L^p(\mu) \xrightarrow{w} Y^{**},$$

where i_p is the canonical embedding, v, w are bounded operators, $\eta_Y : X \to X^{**} : x \mapsto \langle \cdot, x \rangle$ is a linear and isometric embedding and the symbol $X \hookrightarrow Y$ means that X is contained in Y as a set and canonical embedding is continuous. Also recall that an operator T from a Banach space X to a Banach space Y is *p-summing* if it maps each weak l^p sequence in X to a strong l^p sequence in Y.

2. Hunziker and Jarchow [7] proved that for $\beta \geq 1$, the order boundedness of composition operator acting from a Hardy space H^p to $L^{\beta p}(\partial\mathbb{D}, \mathcal{M}, dm)$ implies compactness of composition operator acting from H^p to $H^{\beta p}$, where dm is the normalized Lebesgue measure on the σ-algebra of Lebesgue measurable sets on the unit circle $\partial\mathbb{D}$.
3. Ueki [12] proved that every order-bounded weighted composition operator acting between weighted Bergman spaces is bounded.
4. Ueki [12] proved that for $\alpha, \beta \in (-1, \infty)$, the weighted composition operator acting from weighted Bergman space $\mathcal{A}^2_\alpha$ to $\mathcal{A}^2_\beta$ is order-bounded if and only if it acts as Hilbert–Schmidt operator between these spaces.

Thus the concept of order-boundedness from a quasi-Banach space to a subspace of a quasi-Banach lattice plays an important role in the study of composition and weighted composition operators. In this paper, we characterize order-bounded weighted composition operators acting between Dirichlet spaces.

2 Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$. Let $\psi \in H(\mathbb{D})$ and φ be a holomorphic self-map on $\mathbb{D}$. Then a linear operator $W_{\psi,\varphi}$ with symbols ψ and φ, known as the *weighted composition operator* is defined by

$$W_{\psi,\varphi} f = \psi \cdot (f \circ \varphi), \qquad f \in H(\mathbb{D}).$$

This operator has been studied extensively for the role this operator plays in the identification of the isometries of Banach spaces. Indeed, the surjective isometries of several functional Banach spaces have been shown to be weighted composition operators.

The *Hardy space* H^p $(0 < p < \infty)$ is the space of functions $f \in H(\mathbb{D})$ which satisfy

$$\|f\|^p_{H^p} = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Let dA denote the normalized area measure on $\mathbb{D}$. For each $\alpha \in (-1, \infty)$, if we set

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z), \; z \in \mathbb{D},$$

then dA_α is a measure on $\mathbb{D}$ such that $A_\alpha(\mathbb{D}) = 1$. For $p \in (0, \infty)$, $\alpha \in (-1, \infty)$, let $L^p(dA_\alpha)$ be the weighted Lebesgue space consisting of the measurable functions f

on $\mathbb{D}$ such that $\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty$. Also denote by $\mathcal{A}^p_\alpha$ the *weighted Bergman space* $L^p(dA_\alpha) \cap H(\mathbb{D})$ with norm defined as

$$\|f\|_{\mathcal{A}^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \right)^{1/p} < \infty.$$

For $p \geq 1$, $\mathcal{A}^p_\alpha$ is a Banach space. If $0 < p < 1$, then despite the norm notation, $\|f\|_{\mathcal{A}^p_\alpha}$ fails to satisfy the properties of norm. However, in this case $(f, g) = \|f - g\|_{\mathcal{A}^p_\alpha}$ defines a translation invariant metric on $\mathcal{A}^p_\alpha$ that turns $\mathcal{A}^p_\alpha$ into a complete metric space. In particular, the space $\mathcal{A}^p_0$ is the classical Bergman space $\mathcal{A}^p$. For $p \in (0, \infty)$, $\alpha \in (-1, \infty)$ and a fixed $z \in \mathbb{D}$, let

$$K_z(w) = \frac{(1 - |z|^2)^{\alpha+2/p}}{(1 - \overline{z}w)^{2(\alpha+2)/p}}, \quad w \in \mathbb{D}. \tag{1}$$

Then, using the well known identity

$$1 - |\sigma_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{z}w|^2},$$

and making the change of variables $\zeta = \sigma_z(w)$, we see that $K_z \in \mathcal{A}^p_\alpha$ and

$$\begin{aligned} \|K_z\|^p_{\mathcal{A}^p_\alpha} &= (\alpha + 1) \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha (1 - |z|^2)^{\alpha+2}}{|1 - \overline{z}w|^{2(\alpha+2)}} \, dA(w) \\ &= (\alpha + 1) \int_{\mathbb{D}} (1 - |\zeta|^2)^\alpha \, dA(\zeta) = 1. \end{aligned} \tag{2}$$

For $0 < p < \infty$, the spaces of *Dirichlet type space* $\mathcal{D}^p_{p-1}$ consist of functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{D}^p_{p-1}} = \left(|f(0)|^p + \int_{\mathbb{D}} |f(z)|^p dA_{p-1}(z) \right)^{1/p} < \infty. \tag{3}$$

It is well known that $\mathcal{D}^p_{p-1} \subset H^p$ if $0 < p \leq 2$, and $H^p \subset \mathcal{D}^p_{p-1}$ if $2 \leq p < \infty$ and the inclusions are strict when $p \neq 2$. Moreover, $H^p \subset \mathcal{D}^p_{p-1} \subset \mathcal{A}^{2p}$ for $2 \leq p < \infty$ and $\mathcal{D}^p_{p-1} \subset H^p \subset \mathcal{A}^{2p}$ for $0 < p \leq 2$. For more about these spaces, we refer the readers to [1,5,13,14].
The order boundedness of composition operators on Hardy spaces was first considered by Hunziker and Jarchow in [7]. Hibschweiler [6] studied the order-bounded weighted composition operators mapping into $L^p(\partial\mathbb{D}, \mathcal{M}, dm)$. For some recent work on the subject, we refer the interested reader to [2–4,6–9,12]. Ueki [12] studied order-boundedness of weighted composition operators mapping into Bergman spaces. Recently, Guo et al. [4] considered order-bounded weighted composition operators mapping into Dirichlet spaces. They also provided a sufficient condition for order

boundedness of $W_{\psi,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$. In fact they proved the following result. See Corollary 3.1 in [4].

Theorem 2.1 *Let* $0 < p, q < \infty$, $\psi \in H(\mathbb{D})$ *and* φ *be a holomorphic self-map of* $\mathbb{D}$. *If* ψ *and* φ *satisfy the condition*

$$\int_{\mathbb{D}} \frac{|\psi'(z)|^q}{(1-|\varphi(z)|^2)^{q/p}} dA_{q-1}(z) + \int_{\mathbb{D}} \frac{|\psi(z)|^q |\varphi'(z)|^q}{(1-|\varphi(z)|^2)^{(p+1)q/p}} dA_{q-1}(z) < \infty, \tag{4}$$

then $W_{\psi,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$ *is order bounded.*

In this article, we prove that the condition (4) provided by Gao, Kumar and Zhou is also a necessary condition for the order boundedness of $W_{\psi,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$.

Throughout this paper constants are denoted by C, they are positive and not necessarily the same at each occurrence.

3 Order boundedness of $W_{\psi,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$

In this section, we characterize order boundedness of $W_{\psi,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$. In fact, our aim is to prove the following result.

Theorem 3.1 *Let* $0 < p, q < \infty$, $\psi \in H(\mathbb{D})$ *and* φ *be a holomorphic self-map of* $\mathbb{D}$. *Then* $W_{\psi,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$ *is order bounded if and only if* ψ *and* φ *satisfy* (4).

Recall that $W_{\psi,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$ is order bounded if and only if we can find $g \in L^q(dA_{q-1}),\ g \geq 0$ such that for all $f \in B_{\mathcal{D}^p_{p-1}}$, we have that

$$|(W_{\psi,\varphi} f)'(z)| \leq g(z), \qquad \text{a.e. } [A_{q-1}].$$

In fact, under the assumptions of Theorem 3.1 on the parameters p, q, ψ and φ, the order bounded weighted composition operator $W_{\psi,\varphi}$ acting from $\mathcal{D}^p_{p-1}$ to $\mathcal{D}^q_{q-1}$ is bounded. This fact can be easily be proved by using the following lemma, see Lemma 3.1 in [4].

Lemma 3.1 *Let* $0 < p < \infty$ *and* $z \in \mathbb{D}$. *Then*

$$\sup \left\{ |f(z)| : f \in \mathcal{D}^p_{p-1}, \|f\|_{\mathcal{D}^p_{p-1}} \leq 1 \right\} = \frac{1}{(1-|z|^2)^{1/p}}. \tag{5}$$

Indeed, if $W_{\psi,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$ is order bounded, then there exists a non-negative function $g \in L^q(dA_{q-1})$ such that for all $f \in B_{\mathcal{D}^p_{p-1}}$, we have that

$$|(W_{\psi,\varphi} f)'(z)| \leq g(z), \quad \text{a.e. } [A_{q-1}].$$

Thus for each $f \in \mathcal{D}^p_{p-1}$ with $\|f\|_{\mathcal{D}^p_{p-1}} \leq 1$, by (5) in Lemma 3.1, we have that

$$\begin{aligned}\|W_{\psi,\varphi} f\|^q_{\mathcal{D}^q_{q-1}} &= |(W_{\psi,\varphi} f)(0)|^q + \int_{\mathbb{D}} |(W_{\psi,\varphi} f)'(z)|^q \, dA_{q-1}(z) \\ &\leq \frac{|\psi(0)|^q}{(1-|\varphi(0)|)^{q/p}} + \int_{\mathbb{D}} g(z)^q \, dA_{q-1}(z) \\ &\leq \|g_0\|^q_{L^q_{q-1}} + \|g\|^q_{L^q_{q-1}},\end{aligned}$$

where $g_0 = |\psi(0)|/(1-|\varphi(0)|)^{1/p}$. Thus, $W_{\psi,\varphi} f \in \mathcal{D}^q_{q-1}$. Moreover, by taking the supremum over all functions f in $B_{\mathcal{D}^p_{p-1}}$, we see that $W_{\psi,\varphi}$ is bounded and $\|W_{\psi,\varphi}\| \leq (\|g_0\|^q_{L^q_{q-1}} + \|g\|^q_{L^q_{q-1}})^{1/q}$.

Proof of Theorem 2.1. In view of Theorem 2.1 we only need to show that if $W_{\psi,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$ is order bounded, then (4) holds. Suppose that $W_{\psi,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$ is order bounded. Then there exists a non-negative function $g \in L^q_{q-1}$ such that $|(W_{\psi,\varphi})' f(z)| \leq g(z)$ for almost every $z \in \mathbb{D}$ and all $f \in \mathcal{D}^p_{p-1}$ with $\|f\|_{\mathcal{D}^p_{p-1}} \leq 1$. Let

$$f_{\varphi(z)}(w) = \frac{(1-|\varphi(z)|^2)^{(p+1)/p}}{(1-\overline{\varphi(z)}w)^{-1+2(p+1)/p}}, \quad w \in \mathbb{D}.$$

A straightforward computation shows that $|f_{\varphi(z)}(0)| = (1-|\varphi(z)|^2)^{(p+1)/p} \leq 1$ and

$$f'_{\varphi(z)}(w) = \frac{2+p}{p}\overline{\varphi(z)}\left\{\frac{1-|\varphi(z)|^2}{(1-\overline{\varphi(z)}w)^2}\right\}^{(p+1)/p}.$$

Thus using (1), (2) with $\alpha = p-1$, the fact that $|\varphi(z)| < 1$ and (3), we have that $\|f\|_{\mathcal{D}^p_{p-1}} \leq \{1+(2+p)/p\}^{1/p} := c_0$. So by taking $\tau_{\varphi(z)}(w) = f_{\varphi(z)}(w)/c_0$, we have that

$$\|\tau_{\varphi(z)}\|_{\mathcal{D}^p_{p-1}} = \frac{\|f\|_{\mathcal{D}^p_{p-1}}}{c_0} \leq 1.$$

Therefore, $\tau_{\varphi(z)} \in B_{\mathcal{D}^p_{p-1}}$ and for almost all $z \in \mathbb{D}$. Moreover

$$\begin{aligned}|(W_{\psi,\varphi})' \tau_{\varphi(z)}(w)| = \Bigg| &\psi'(w) \frac{(1-|\varphi(z)|^2)^{(p+1)/p}}{(1-\overline{\varphi(z)}\varphi(w))^{-1+2(p+1)/p}} \\ &+ \frac{2+p}{p}\overline{\varphi(z)}\psi(w)\varphi'(z)\frac{(1-|\varphi(z)|^2)^{(p+1)/p}}{(1-\overline{\varphi(z)}\varphi(w))^{2(p+1)/p}}\Bigg|.\end{aligned}$$

In particular, by taking $w = z$, we have that

$$\begin{aligned}
&\left|\frac{\psi'(z)}{(1-|\varphi(z)|^2)^{1/p}}\right| - \frac{2+p}{p}\left|\frac{\psi(z)\overline{\varphi(z)}\varphi'(z)}{(1-|\varphi(z)|^2)^{1+1/p}}\right| \\
&\quad \leq \left|\frac{\psi'(z)}{(1-|\varphi(z)|^2)^{1/p}} + \frac{2+p}{p}\frac{\psi(z)\overline{\varphi(z)}\varphi'(z)}{(1-|\varphi(z)|^2)^{1+1/p}}\right| \\
&\quad = |\psi'(z) f_{\varphi(z)}(\varphi(z)) + \psi(z)\varphi'(z) f'_{\varphi(z)}(\varphi(z))| \\
&\quad = |(W_{\psi,\varphi} f_{\varphi(z)})'(z)| \leq g(z).
\end{aligned}$$

Using the fact that $|\varphi(z)| < 1$, for almost all $z \in \mathbb{D}$ we have that

$$\left|\frac{\psi'(z)}{(1-|\varphi(z)|^2)^{1/p}}\right| \leq g(z) + \frac{2+p}{p}\left|\frac{\psi(z)\varphi'(z)}{(1-|\varphi(z)|^2)^{1+1/p}}\right|. \tag{6}$$

Next, fix such a $z \in \mathbb{D}$ and consider the function

$$u_{\varphi(z)}(w) = \frac{(1-|\varphi(z)|^2)^{(p+1)/p}}{(1-\overline{\varphi(z)}w)^{-1+2(p+1)/p}} - \frac{(1-|\varphi(z)|^2)^{(2p+1)/p}}{(1-\overline{\varphi(z)}w)^{2(p+1)/p}}, \quad w \in \mathbb{D},$$

Then

$$u'_{\varphi(z)}(w) = \overline{\varphi(z)}\left\{\frac{p+2}{p}\frac{(1-|\varphi(z)|^2)^{(p+1)/p}}{(1-\overline{\varphi(z)}w)^{2(p+1)/p}} - \frac{2(p+1)}{p}\frac{(1-|\varphi(z)|^2)^{(2p+1)/p}}{(1-\overline{\varphi(z)}w)^{(3p+2)/p}}\right\}.$$

A routine calculation using (1), (2) with $\alpha = p-1$ in (3) shows that for $z \in \mathbb{D}$, $u_{\varphi(z)} \in \mathcal{D}^p_{p-1}$ and $\|u_{\varphi(z)}\|_{\mathcal{D}^p_{p-1}} \leq (2^p + 3(p+2)/p)^{1/p} := d_0$. Once again it is easy to show that the function $h_{\varphi(z)}$, defined by $h_{\varphi(z)}(w) = h_{\varphi(z)}(w)/d_0$, is in the unit ball of $\mathcal{D}^p_{p-1}$ and satisfies $h_{\varphi(z)}(\varphi(z)) = 0$. Moreover,

$$h'_{\varphi(z)}(w) = \frac{\overline{\varphi(z)}}{d_0}\left\{\frac{p+2}{p}\frac{(1-|\varphi(z)|^2)^{(p+1)/p}}{(1-\overline{\varphi(z)}w)^{2(p+1)/p}} - \frac{2(p+1)}{p}\frac{(1-|\varphi(z)|^2)^{(2p+1)/p}}{(1-\overline{\varphi(z)}w)^{(3p+2)/p}}\right\}$$

and

$$h'_{\varphi(z)}(\varphi(z)) = -\frac{1}{d_0}\frac{\overline{\varphi(z)}}{(1-|\varphi(z)|^2)^{(p+1)/p}}.$$

Thus

$$\begin{aligned}
\frac{1}{d_0}\frac{|\psi(z)\overline{\varphi(z)}\varphi'(z)|}{(1-|\varphi(z)|^2)^{(p+1)/p}} &= |\psi'(z)h_{\varphi(z)}(\varphi(z)) + \psi(z)\varphi'(z)h'_{\varphi(z)}(\varphi(z))| \\
&= |W_{\psi,\varphi}h_{\varphi(z)}(z)| \leq g(z).
\end{aligned}$$

Therefore, for almost all $z \in \mathbb{D}$ such that $|\varphi(z)| > 1/2$, we have

$$\frac{|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{(p+1)/p}} \leq d_0 g(z). \tag{7}$$

On the other hand, by the continuity of the function $1/(1-|\varphi|^2)^{(p+1)/p}$, there is a constant $L > 0$ such that

$$\frac{1}{(1-|\varphi(z)|^2)^{(p+1)/p}} \leq L \tag{8}$$

for all $z \in \mathbb{D}$ such that $|\varphi(z)| \leq 1/2$. Taking the constant function 1 and the function $p(z) = z$, as test functions in $\mathcal{D}^p_{p-1}$, we see that for almost all $z \in \mathbb{D}$, $|\psi'(z)| = |W_{\psi,\varphi}1| \leq g(z)$ and

$$|\psi'(z)\varphi(z) + \psi(z)\varphi'(z)| = |W_{\psi,\varphi}p| \leq g(z).$$

By the triangle inequality, we deduce

$$|\psi(z)\varphi'(z)| \leq |\psi'(z)\varphi(z)| + |\psi'(z)\varphi(z) + \psi(z)\varphi'(z)| \leq 2g(z). \tag{9}$$

Thus, from (8) and (9), we see that for almost all z such that $|\varphi(z)| \leq 1/2$,

$$\frac{|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{(p+1)/p}} \leq 2Lg(z). \tag{10}$$

Consequently, from (7) and (10), it follows that for almost every $z \in \mathbb{D}$,

$$\frac{|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{(p+1)/p}} \leq \max\{2L, d_0\}g(z). \tag{11}$$

Since $g \in L^q_{q-1}$, (11) implies that

$$\int_{\mathbb{D}} \frac{|\psi(z)|^q |\varphi'(z)|^q}{(1-|\varphi(z)|^2)^{(p+1)q/p}} dA_{q-1}(z) < \infty, \tag{12}$$

Moreover, using an elementary inequality, (6), (12) and the fact that $g \in L^q_{q-1}$, we see that

$$\int_{\mathbb{D}} \frac{|\psi'(z)|^q}{(1-|\varphi(z)|^2)^{q/p}} dA_{q-1}(z) < \infty, \tag{13}$$

Adding (12) and (13), we see that (4) holds, and the proof is accomplished. □

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