Positivity (2017) 21:1201–1212
DOI 10.1007/s11117-016-0461-x

Positivity

Positive solutions for nonlinear fractional differential equations

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Received: 8 May 2016 / Accepted: 28 November 2016 / Published online: 5 December 2016

Abstract We study the existence and uniqueness of positive solutions of the nonlinear fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f(t, x(t)) + {}^{C}D^{\alpha-1}g(t, x(t)), \ 0 < t \leq T, \\ x(0) = \theta_1 > 0, \ x'(0) = \theta_2 > 0, \end{cases}$$

where $1 < \alpha \leq 2$. In the process we convert the given fractional differential equation into an equivalent integral equation. Then we construct appropriate mapping and employ Schauder fixed point theorem and the method of upper and lower solutions to show the existence of a positive solution of this equation. We also use the Banach fixed point theorem to show the existence of a unique positive solution. The results obtained here extend the work of Matar (AMUC 84(1):51–57, 2015 [7]). Finally, an example is given to illustrate our results.

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Keywords Fractional differential equations · Positive solutions · Upper and lower solutions · Existence · Uniqueness · Fixed point theorems

Mathematics Subject Classification Primary 26A33; Secondary 34A12 · 34G20

1 Introduction

The history of the fractional derivatives of non-integer order turned out to be very attractive to mathematicians as well as physicists, biologists, engineers and economists. It as spreads from the end of seventh century until now. The number of publications and scientific meetings in the recent period devoted to, demonstrates the importance of the problems of this concept has raised both; more theoretical than applied. We can say that it has become a whole discipline. Specialists agree to raise the beginning of this story at the end of the year 1695 when Leibniz, issuing a letter to the Hospital, initiating a reflection on a possible theory of not entire fractional derivative of a function. In its response, the Hospital has questioned about the significance we could give to the derivative of order $1/2$ (see [5,8,9]).

Fractional differential equations arise from a variety of applications including in various fields of science and engineering. In particular, problems concerning qualitative analysis of the positivity of such solutions for fractional differential equations (FDE) have received the attention of many authors, see [1–4,6,7,11–14] and the references therein.

Recently, Zhang in [14] investigated the existence and uniqueness of positive solutions for the nonlinear fractional differential equation

$$\begin{cases} D^{\alpha}x\,(t) = f(t, x(t)),\ 0 < t \leq 1, \\ x\,(0) = 0, \end{cases}$$

where D^{α} is the standard Riemann Liouville fractional derivative of order $0 < \alpha < 1$, and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a given continuous function. By using the method of the upper and lower solution and cone fixed-point theorem, the author obtained the existence and uniqueness of a positive solution.

The nonlinear fractional differential equation boundary value problem

$$\begin{cases} D^{\alpha}x\,(t) + f(t, x(t)) = 0,\ 0 < t < 1, \\ x\,(0) = x\,(1) = 0, \end{cases}$$

has been investigated in [1], where $1 < \alpha \leq 2$, and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a given continuous function. By means of some fixed-point theorems on cone, some existence and multiplicity results of positive solutions have been established.

In [7], Matar discussed the existence and uniqueness of the positive solution of the following nonlinear fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x\,(t) = f(t, x(t)),\ 0 < t \leq 1, \\ x\,(0) = 0,\ x'\,(0) = \theta > 0, \end{cases}$$

where ${}^{C}D^{\alpha}$ is the standard Caputo's fractional derivative of order $1 < \alpha \leq 2$, and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a given continuous function. By employing the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the author obtained positivity results.

In this paper, we are interested in the analysis of qualitative theory of the problems of the positive solutions to fractional differential equations. Inspired and motivated by the works mentioned above and the papers [1–4,6,7,11–14] and the references therein, we concentrate on the positivity of the solutions for nonlinear fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f(t, x(t)) + {}^{C}D^{\alpha-1}g(t, x(t)), \ 0 < t \leq T, \\ x(0) = \theta_1 > 0, \ x'(0) = \theta_2 > 0, \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2$, $g, f : [0, T] \times [0, \infty) \to [0, \infty)$ are given continuous functions, g is non-decreasing on x and $\theta_2 \geq g(0, \theta_1)$. To show the existence and uniqueness of the positive solution, we transform (1.1) into an integral equation and then by the method of upper and lower solutions and use Schauder and Banach fixed point theorems.

This paper is organized as follows. In Sect. 2, we introduce some notations and lemmas, and state some preliminaries results needed in later section. Also, we present the inversion of (1.1) and the Banach and Schauder fixed point theorems. For details on Banach and Schauder theorems we refer the reader to [10]. In Sect. 3, we give and prove our main results on positivity, and we provide an example to illustrate our results. The results presented in this paper extend the main results in [7].

2 Preliminaries

Let $X = C([0, T])$ be the Banach space of all real-valued continuous functions defined on the compact interval $[0, T]$, endowed with the maximum norm. Define the subspace $\mathcal{A} = \{x \in X : x(t) \geq 0, \ t \in [0, T]\}$ of X. By a positive solution $x \in X$, we mean a function $x(t) > 0$, $0 \leq t \leq T$.

Let $a, b \in \mathbb{R}^{+}$ such that $b > a$. For any $x \in [a, b]$, we define the upper-control function $U(t, x) = \sup\{f(t, \lambda) : a \leq \lambda \leq x\}$, and lower-control function $L(t, x) = \inf\{f(t, \lambda) : x \leq \lambda \leq b\}$. Obviously, $U(t, x)$ and $L(t, x)$ are monotonous non-decreasing on the argument x and $L(t, x) \leq f(t, x) \leq U(t, x)$.

We introduce some necessary definitions, lemmas and theorems which will be used in this paper. For more details, see [5,8,9].

Definition 2.1 ([5,9]) The fractional integral of order $\alpha > 0$ of a function $x : \mathbb{R}^{+} \to \mathbb{R}$ is given by

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} x(s) ds,$$

provided the right side is pointwise defined on $\mathbb{R}^{+}$.

Definition 2.2 ([5,9]) The Caputo fractional derivative of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$^{C}D^{\alpha}x(t) = I^{n-\alpha}x^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds,$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $\mathbb{R}^+$.

Lemma 2.3 ([5,9]) *Let* $\Re(\alpha) > 0$. *Suppose* $x \in C^{n-1}[0, +\infty)$ *and* $x^{(n)}$ *exists almost everywhere on any bounded interval of* $\mathbb{R}^+$. *Then*

$$\left(I^{\alpha}\, {}^{C}D^{\alpha}x\right)(t) = x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}t^k.$$

In particular, when $1 < \Re(\alpha) < 2$, $\left(I^{\alpha}\, {}^{C}D^{\alpha}x\right)(t) = x(t) - x(0) - x'(0)t$.

The following lemma is fundamental to our results.

Lemma 2.4 *Let* $x \in C^1([0, T])$, $x^{(2)}$ *and* $\frac{\partial g}{\partial t}$ *exist, then* x *is a solution of (1.1) if and only if*

$$x(t) = \theta_1 + (\theta_2 - g(0, \theta_1))\,t + \int_0^t g(s, x(s))\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s))ds. \tag{2.1}$$

Proof Let x be a solution of (1.1). First we write this equation as

$$I^{\alpha}\, {}^{C}D^{\alpha}x(t) = I^{\alpha}\left(f(t, x(t)) + {}^{C}D^{\alpha-1}g(t, x(t))\right),\ 0 < t \le T.$$

From Lemma 2.3, we have

$$\begin{aligned} x(t) - x(0) - x'(0)t &= I^{\alpha}\, {}^{C}D^{\alpha-1}g(t, x(t)) + I^{\alpha}f(t, x(t)) \\ &= I I^{\alpha-1}\, {}^{C}D^{\alpha-1}g(t, x(t)) + I^{\alpha}f(t, x(t)) \\ &= I\left(g(t, x(t)) - g(0, x(0))\right) + I^{\alpha}f(t, x(t)) \\ &= \int_0^t g(s, x(s))\,ds - g(0, \theta_1)\,t \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s))ds, \end{aligned}$$

then, we obtain (2.1). Since each step is reversible, the converse follows easily. This completes the proof. □

Lastly in this section, we state the fixed point theorems which enable us to prove the existence and uniqueness of a positive solution of (1.1).

Definition 2.5 Let $(X, \|.\|)$ be a Banach space and $\Phi : X \to X$. The operator Φ is a contraction operator if there is an $\lambda \in (0, 1)$ such that $x, y \in X$ imply

$$\|\Phi x - \Phi y\| \leq \lambda \|x - y\|.$$

Theorem 2.6 *(*Banach [10]) *Let $\mathcal{C}$ be a nonempty closed convex subset of a Banach space X and $\Phi : \mathcal{C} \to \mathcal{C}$ be a contraction operator. Then there is a unique $x \in \mathcal{C}$ with $\Phi x = x$.*

Theorem 2.7 (Schauder [10]) *Let $\mathcal{C}$ be a nonempty closed convex subset of a Banach space X and $\Phi : \mathcal{C} \to \mathcal{C}$ be a continuous compact operator. Then Φ has a fixed point in $\mathcal{C}$.*

3 Main results

In this section, we consider the results of existence problem for many cases of the FDE (1.1). Moreover, we introduce the sufficient conditions of the uniqueness problem of (1.1).

To transform Eq. (2.1) to be applicable to Schauder fixed point, we define an operator $\Phi : \mathcal{A} \longrightarrow X$ by

$$\begin{aligned}(\Phi x)(t) &= \theta_1 + (\theta_2 - g(0, \theta_1))\, t + \int_0^t g(s, x(s))\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds, \; t \in [0, T],\end{aligned} \tag{3.1}$$

where the figured fixed point must satisfy the identity operator equation $\Phi x = x$.

The following assumptions are needed for the next results.

$(H1)$ Let $x^*, x_* \in \mathcal{A}$, such that $a \leq x_*(t) \leq x^*(t) \leq b$ and

$$\begin{cases} {}^C D^{\alpha} x^*(t) - {}^C D^{\alpha-1} g(t, x^*(t)) \geq U(t, x^*(t)), \\ {}^C D^{\alpha} x_*(t) - {}^C D^{\alpha-1} g(t, x_*(t)) \leq L(t, x_*(t)), \end{cases}$$

for any $t \in [0, T]$.

$(H2)$ For $t \in [0, T]$ and $x, y \in X$, there exist positive real numbers $\beta_1, \beta_2 < 1$ such that

$$|g(t, y) - g(t, x)| \leq \beta_1 \|y - x\|,$$
$$|f(t, y) - f(t, x)| \leq \beta_2 \|y - x\|.$$

The functions x^* and x_* are respectively called the pair of upper and lower solutions for Eq. (1.1).

Theorem 3.1 *Assume that $(H1)$ is satisfied, then the FDE (1.1) has at least one solution $x \in X$ satisfying $x_*(t) \leq x(t) \leq x^*(t)$, $t \in [0, T]$.*

Proof Let $\mathcal{C} = \{x \in \mathcal{A} : x_*(t) \leq x(t) \leq x^*(t),\ t \in [0, T]\}$, endowed with the norm $\|x\| = \max_{t\in[0,T]} |x(t)|$, then we have $\|x\| \leq b$. Hence, $\mathcal{C}$ is a convex, bounded, and closed subset of the Banach space X. Moreover, the continuity of g and f implies the continuity of the operator Φ on $\mathcal{C}$ defined by (3.1). Now, if $x \in \mathcal{C}$, there exist positive constants c_f and c_g such that

$$\max\{f(t, x(t)) : t \in [0, T],\ x(t) \leq b\} < c_f,$$

and

$$\max\{g(t, x(t)) : t \in [0, T],\ x(t) \leq b\} < c_g.$$

Then

$$\begin{aligned}
|(\Phi x)(t)| &\leq |\theta_1 + (\theta_2 - g(0, \theta_1))\, t| + \int_0^t |g(s, x(s))|\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s))|\, ds \\
&\leq \theta_1 + \left(\theta_2 + c_0 + c_g\right) T + \frac{c_f T^\alpha}{\Gamma(\alpha+1)},
\end{aligned}$$

where $c_0 = |g(0, \theta_1)|$. Thus,

$$\|\Phi x\| \leq \theta_1 + \left(\theta_2 + c_0 + c_g\right) T + \frac{c_f T^\alpha}{\Gamma(\alpha+1)}.$$

Hence, $\Phi(\mathcal{C})$ is uniformly bounded. Next, we prove the equicontinuity of $\Phi(\mathcal{C})$. Let $x \in \mathcal{C}$, $\varepsilon > 0$, $\delta > 0$, and $0 \leq t_1 < t_2 \leq T$ such that $|t_2 - t_1| < \delta$. If

$$\delta = \min\left\{1, \frac{\varepsilon\Gamma(\alpha+1)}{2\left(\left(\theta_2 + c_0 + c_g\right)\Gamma(\alpha+1) + 2c_f\right)}, \left(\frac{\varepsilon\Gamma(\alpha+1)}{4c_f}\right)^{\frac{1}{\alpha}}\right\},$$

then

$$\begin{aligned}
&|(\Phi x)(t_1) - (\Phi x)(t_2)| \\
&\leq (\theta_2 + c_0)(t_2 - t_1) + \left|\int_0^{t_1} g(s, x(s))ds - \int_0^{t_2} g(s, x(s))ds\right| \\
&\quad + \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, x(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, x(s))ds\right| \\
&\leq (\theta_2 + c_0)(t_2 - t_1) + \left|\int_{t_1}^{t_2} g(s, x(s))ds\right| \\
&\quad + \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1} \left((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\right) f(s, x(s))ds\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,x(s))ds\right|\\
&\le\left(\theta_2+c_0+c_g\right)(t_2-t_1)+\frac{c_f}{\Gamma(\alpha+1)}\left(t_2^\alpha-t_1^\alpha+2\,(t_2-t_1)^\alpha\right)\\
&\le\left(\theta_2+c_0+c_g+\frac{2c_f}{\Gamma(\alpha+1)}\right)\delta+\frac{2c_f\delta^\alpha}{\Gamma(\alpha+1)}\\
&<\varepsilon.
\end{aligned}
$$

Therefore, $\Phi(\mathcal{C})$ is equicontinuous. The Arzelè-Ascoli Theorem implies that $\Phi:\mathcal{C}\longrightarrow X$ is compact. The only thing to apply Schauder fixed point is to prove that $\Phi(\mathcal{C})\subseteq\mathcal{C}$. Let $x\in\mathcal{C}$, then by hypotheses, we have

$$
\begin{aligned}
(\Phi x)(t)&=\theta_1+(\theta_2-g\,(0,\theta_1))\,t+\int_0^t g\,(s,x\,(s))\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))ds\\
&\le\theta_1+(\theta_2-g\,(0,\theta_1))\,t+\int_0^t g\left(s,x^*\,(s)\right)ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}U(s,x(s))ds\\
&\le\theta_1+(\theta_2-g\,(0,\theta_1))\,t+\int_0^t g\left(s,x^*\,(s)\right)ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}U(s,x^*(s))ds\\
&\le x^*(t),
\end{aligned}
$$

and

$$
\begin{aligned}
(\Phi x)(t)&=\theta_1+(\theta_2-g\,(0,\theta_1))\,t+\int_0^t g\,(s,x\,(s))\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))ds\\
&\ge\theta_1+(\theta_2-g\,(0,\theta_1))\,t+\int_0^t g\,(s,x_*\,(s))\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}L(t,x(s))ds\\
&\ge\theta_1+(\theta_2-g\,(0,\theta_1))\,t+\int_0^t g\,(s,x_*\,(s))\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}L(t,x_*(s))ds\\
&\ge x_*(t).
\end{aligned}
$$

Hence, $x_*(t) \leq (\Phi x)(t) \leq x^*(t)$, $t \in [0, T]$, that is, $\Phi(\mathcal{C}) \subseteq \mathcal{C}$. According to Schauder fixed point theorem, the operator Φ has at least one fixed point $x \in \mathcal{C}$. Therefore, the FDE (1.1) has at least one positive solution $x \in X$ and $x_*(t) \leq x(t) \leq x^*(t),\ t \in [0, T]$. □

Next, we consider many particular cases of the previous theorem.

Corollary 3.2 *Assume that there exist continuous functions k_1, k_2, k_3 and k_4 such that*

$$\begin{aligned} &0 < k_1(t) \leq g(t, x(t)) \leq k_2(t) < \infty,\ (t, x(t)) \in [0, T] \times [0, +\infty), \\ &\theta_2 \geq k_1(0),\ \theta_2 \geq k_2(0), \end{aligned} \tag{3.2}$$

and

$$0 < k_3(t) \leq f(t, x(t)) \leq k_4(t) < \infty,\ (t, x(t)) \in [0, T] \times [0, +\infty). \tag{3.3}$$

Then, the FDE (1.1) has at least one positive solution $x \in X$. Moreover,

$$\begin{aligned} &\theta_1 + (\theta_2 - k_1(0))\, t + \int_0^t k_1(s)ds + I^\alpha k_3(t) \\ &\quad \leq x(t) \\ &\quad \leq \theta_1 + (\theta_2 - k_2(0))\, t + \int_0^t k_2(s)ds + I^\alpha k_4(t). \end{aligned} \tag{3.4}$$

Proof By the given assumption (3.3) and the definition of control function, we have $k_3(t) \leq L(t, x) \leq U(t, x) \leq k_4(t)$, $(t, x(t)) \in [0, T] \times [a, b]$. Now, we consider the equations

$$\begin{cases} {}^C D^\alpha x(t) = k_3(t) + {}^C D^{\alpha-1} k_1(t),\ x(0) = \theta_1,\ x'(0) = \theta_2, \\ {}^C D^\alpha x(t) = k_4(t) + {}^C D^{\alpha-1} k_2(t),\ x(0) = \theta_1,\ x'(0) = \theta_2. \end{cases} \tag{3.5}$$

Obviously, Eq. (3.5) are equivalent to

$$x(t) = \theta_1 + (\theta_2 - k_1(0))\, t + \int_0^t k_1(s)ds + I^\alpha k_3(t),$$
$$x(t) = \theta_1 + (\theta_2 - k_2(0))\, t + \int_0^t k_2(s)ds + I^\alpha k_4(t).$$

Hence, the first implies

$$x(t) - \theta_1 - (\theta_2 - k_1(0))\, t - \int_0^t k_1(s)ds = I^\alpha k_3(t) \leq I^\alpha (L(t, x(t))),$$

and the second implies

$$x(t) - \theta_1 - (\theta_2 - k_2(0))\, t - \int_0^t k_2(s)ds = I^\alpha k_4(t) \geq I^\alpha (U(t, x(t))),$$

which are the upper and lower solutions of Eq. (3.5), respectively. An application of Theorem 3.1 yields that the FDE (1.1) has at least one solution $x \in X$ and satisfies Eq. (3.4). □

Corollary 3.3 *Assume that (3.2) holds and* $0 < \sigma < k(t) = \lim_{x\to\infty} f(t,x) < \infty$ *for* $t \in [0,T]$. *Then the FDE (1.1) has at least a positive solution* $x \in X$.

Proof By assumption, if $x > \rho > 0$, then $0 \le |f(t,x) - k(t)| < \sigma$ for any $t \in [0,T]$. Hence, $0 < k(t) - \sigma \le f(t,x) \le k(t) + \sigma$ for $t \in [0,T]$ and $\rho < x < +\infty$. Now if $\max\{f(t,x) : t \in [0,T],\ x \le \rho\} \le \nu$, then $k(t) - \sigma \le f(t,x) \le k(t) + \sigma + \nu$ for $t \in [0,T]$, and $0 < x < +\infty$. By Corollary 3.3, the FDE (1.1) has at least one positive solution $x \in X$ satisfying

$$\begin{aligned}&\theta_1 + (\theta_2 - k_1(0))\,t + \int_0^t k_1(s)ds + I^\alpha k(t) - \frac{\sigma t^\alpha}{\Gamma(\alpha+1)}\\ &\quad\le x(t)\\ &\quad\le \theta_1 + (\theta_2 - k_2(0))\,t + \int_0^t k_2(s)ds + I^\alpha k(t) + \frac{(\sigma+\nu)\,t^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

□

Corollary 3.4 *Assume that* $0 < \sigma < f(t,x(t)) \le \gamma x(t) + \eta < \infty$ *for* $t \in [0,T]$, *and* σ, η *and* γ *are positive constants. Then, the FDE (1.1) has at least one positive solution* $x \in C([0,\delta])$, *where* $0 < \delta < 1$.

Proof Consider the equation

$$\begin{cases} {}^C D^\alpha x(t) - {}^C D^{\alpha-1} g\,(t, x(t)) = \gamma x(t) + \eta, & 0 < t \le T,\\ x(0) = \theta_1 > 0,\ x'(0) = \theta_2 > 0. \end{cases} \tag{3.6}$$

Equation (3.6) is equivalent to integral equation

$$\begin{aligned}x(t) &= \theta_1 + (\theta_2 - g\,(0,\theta_1))\,t + \int_0^t g\,(s, x\,(s))\,ds\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,(\gamma x(s) + \eta)\,ds\\ &= \theta_1 + (\theta_2 - g\,(0,\theta_1))\,t + \int_0^t g\,(s, x\,(s))\,ds\\ &\quad + \frac{\eta t^\alpha}{\Gamma(\alpha+1)} + \frac{\gamma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,x(s)ds.\end{aligned}$$

Let ω and ϕ be positive real numbers. Choose an appropriate $\delta \in (0,1)$ such that $0 < \frac{\gamma\delta^\alpha}{\Gamma(\alpha+1)} < \phi < 1$ and $\omega > (1-\phi)^{-1}\left(\theta_1 + \left(\theta_2 + c_0 + c_g\right)\delta + \frac{\eta\delta^\alpha}{\Gamma(\alpha+1)}\right)$. Then if $0 \le t \le \delta$, the set $B_\omega = \{x \in X : |x(t)| \le \omega,\ 0 \le t \le \delta\}$ is convex, closed, and bounded subset of $C([0,\delta])$. The operator $F : B_\omega \longrightarrow B_\omega$ given by

$$(Fx)(t) = \theta_1 + (\theta_2 - g(0, \theta_1))\, t + \int_0^t g(s, x(s))\, ds$$
$$+ \frac{\eta t^\alpha}{\Gamma(\alpha+1)} + \frac{\gamma}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

is compact as in the proof of Theorem 3.1. Moreover,

$$|(Fx)(t)| \leq \theta_1 + \left(\theta_2 + c_0 + c_g\right) T + \frac{\eta T^\alpha}{\Gamma(\alpha+1)} + \frac{\gamma T^\alpha}{\Gamma(\alpha+1)} \|x\|.$$

If $x \in B_\omega$, then

$$|(Fx)(t)| \leq (1-\phi)\,\omega + \phi\omega = \omega,$$

that is $\|Fx\| \leq \omega$. Hence, the Schauder fixed theorem ensures that the operator F has at least one fixed point in B_ω, and then Eq. (3.6) has at least one positive solution $x^*(t)$, where $0 < t < \delta$. Therefore, if $t \in [0, T]$ one can asserts that

$$x^*(t) = \theta_1 + (\theta_2 - g(0, \theta_1))\, t + \int_0^t g\left(s, x^*(s)\right) ds$$
$$+ \frac{\eta t^\alpha \Gamma(\alpha+1)}{+} \frac{\gamma}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x^*(s) ds.$$

The definition of control function implies

$$U(t, x^*(t)) \leq \gamma x^*(t) + \eta =^C D^\alpha x^*(t) -^C D^{\alpha-1} g\left(t, x^*(t)\right),$$

then x^* is an upper positive solution of the FDE (1.1). Moreover, one can consider

$$x_*(t) = \theta_1 + (\theta_2 - g(0, \theta_1))\, t + \int_0^t g(s, x_*(s))\, ds + \frac{\sigma t^\alpha}{\Gamma(\alpha+1)}$$

as a lower positive solution of Eq. (1.1). By Theorem 3.1, the FDE (1.1) has at least one positive solution $x \in C([0, \delta])$, where $0 < \delta < 1$ and $x_*(t) \leq x(t) \leq x^*(t)$. □

The last result is the uniqueness of the positive solution of (1.1) using Banach contraction principle.

Theorem 3.5 *Assume that* $(H1)$ *and* $(H2)$ *are satisfied and*

$$\left(T\beta_1 + \frac{\beta_2 T^\alpha}{\Gamma(\alpha+1)}\right) < 1. \tag{3.7}$$

Then the FDE (1.1) has a unique positive solution $x \in \mathcal{C}$.

Proof From Theorem 3.1, it follows that the FDE (1.1) has at least one positive solution in $\mathcal{C}$. Hence, we need only to prove that the operator defined in (3.1) is a contraction on X. In fact, for any $x, y \in X$, we have

$$\begin{aligned} &|(\Phi x)(t) - (\Phi y)(t)| \\ &\leq \int_0^t |g(s, x(s)) - g(s, y(s))| \, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s)) - f(s, y(s))| \, ds \\ &\leq \left(T\beta_1 + \frac{\beta_2 T^\alpha}{\Gamma(\alpha+1)} \right) \|x - y\| . \end{aligned}$$

Hence, the operator Φ is a contraction mapping by (3.7). Therefore, the FDE (1.1) has a unique positive solution $x \in \mathcal{C}$. □

Finally, we give an example to illustrate our results.

Example 3.6 We consider the nonlinear fractional differential equation

$$\begin{cases} {}^C D^{\frac{5}{4}} x(t) - {}^C D^{\frac{1}{4}} \frac{x(t)}{3+x(t)} = \frac{1}{1+t} \left(1 + \frac{tx(t)}{2+x(t)}\right), \ 0 < t \leq 1, \\ x(0) = 1, \ x'(0) = \theta_2 \geq 1, \end{cases} \tag{3.8}$$

where $\theta_1 = 1$, $T = 1$, $g(t, x) = \frac{x}{3+x}$ and $f(t, x) = \frac{1}{1+t}\left(1 + \frac{tx}{2+x}\right)$. Since g is non-decreasing on x,

$$\lim_{x \longrightarrow \infty} \frac{x}{3+x} = \lim_{x \longrightarrow \infty} \frac{1}{1+t} \left(1 + \frac{tx}{2+x}\right) = 1,$$

and

$$\begin{aligned} &\frac{1}{3} \leq g(t, x) \leq 1, \\ &\frac{1}{2} \leq \frac{1}{2}\left(1 + \frac{tx}{2+x}\right) \leq f(t, x) \leq 1 + \frac{tx}{2+x} \leq 1 + t \leq 2, \end{aligned}$$

for $(t, x) \in [0, 1] \times [0, +\infty)$, hence by any of the above Corollaries, the Eq. (3.8) has a positive solution. Also, we have

$$T\beta_1 + \frac{\beta_2 T^\alpha}{\Gamma(\alpha+1)} = \frac{5}{6} < 1,$$

then by Theorem 3.5, the Eq. (3.8) has a unique positive solution.

Acknowledgements The authors gratefully acknowledge the reviewers for their helpful comments.

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