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On two classes of approximation processes of integral type

Octavian Agratini[1] · Ali Aral[2] · Emre Deniz[2]

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Abstract The paper aims to study two classes of linear positive operators representing modifications of Picard and Gauss operators. The new operators reproduce both constants and a given exponential function. Approximation properties in polynomial weighted spaces are investigated and the speed of convergence is measured using a certain weighted modulus of smoothness. Also, the asymptotic behavior of the integral operators are established. Finally, aspects on generalized convexity are analyzed.

Keywords Linear positive operator · Picard operator · Gauss operator · Weighted space · Voronovskaja formula

Mathematics Subject Classification 41A36 · 41A25

1 Introduction

The study of the linear methods of approximation, which are given by sequences of linear positive operators, became a strongly ingrained part of Approximation Theory.

✉ Octavian Agratini
agratini@math.ubbcluj.ro

Ali Aral
aliaral73@yahoo.com

Emre Deniz
emredeniz--@hotmail.com

[1] Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Street M. Kogălniceanu 1, 400084 Cluj-Napoca, Romania

[2] Department of Mathematics, Faculty of Science and Arts, Kirikkale University, Yahsihan, 71450 Kirikkale, Turkey

Due to their special properties, over time, these approximation processes have been proved very useful in approximating various signals. Our paper will bring into light two sequences of integral operators known in the literature as Picard ($P_n,\ n \geq 1$), respectively Gauss ($W_n,\ n \geq 1$) operators. Their classical forms are described by the following formulas

$$(P_n f)(x) = \frac{n}{2}\int_{\mathbb{R}} f(x+t)e^{-n|t|}dt, \quad x \in \mathbb{R}, \tag{1}$$

$$(W_n f)(x) = \sqrt{\frac{n}{\pi}}\int_{\mathbb{R}} f(x+t)e^{-nt^2}dt, \quad x \in \mathbb{R}, \tag{2}$$

where the function f is selected such that the integrals are finite.

These operators have been investigated in several works. We mention the monograph [2] and the references therein. By using probabilistic schemes, Gauss-Weierstrass operators are reconstructed in [1, *Section 5.2.9*]. For each $n \in \mathbb{N}$, both operators are linear and positive. Moreover,

$$(P_n e_0)(x) = (W_n e_0)(x) = 1, \quad x \in \mathbb{R}, \tag{3}$$

where e_0 represents the constant function on $\mathbb{R}$ of constant value 1.

Throughout the paper e_j stands for monomial of j-degree, $e_j(t) = t^j$, $t \in \mathbb{R}$.

We amend the classical operators defined by (1) and (2), such that they will be able to reproduce not only e_0 but also a certain exponential function. The proposed generalizations of the above operators are defined as follows:

$$(P_n^* f)(x) = \frac{\sqrt{n}}{2}\int_{\mathbb{R}} f(\alpha_n(x)+t)e^{-\sqrt{n}|t|}dt, \quad n \geq n_a, \quad x \in \mathbb{R}, \tag{4}$$

and

$$(W_n^* f)(x) = \sqrt{\frac{n}{\pi}}\int_{\mathbb{R}} f(\beta_n(x)+t)e^{-nt^2}dt, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}, \tag{5}$$

where

$$\alpha_n(x) = x - \frac{1}{2a}\log\left(\frac{n}{n-4a^2}\right), \quad n \geq n_a, \tag{6}$$

$$\beta_n(x) = x - \frac{a}{2n}, \quad n \geq 1, \tag{7}$$

and $a > 0$. In the above $n_a = [4a^2]+1$, $[\cdot]$ indicating the integer part function or the so-called floor function. The domains of the sequences $P^* = (P_n^*)_{n\geq n_a}$, $W^* = (W_n^*)_{n\geq 1}$ are denoted by $\mathcal{F}(P^*)$ and $\mathcal{F}(W^*)$, respectively.

Also, we introduce the function φ_a given by formula

$$\varphi_a(x) = e^{2ax}, \quad x \in \mathbb{R}. \tag{8}$$

For a tending to zero, the original versions of the operators are reobtained.

Relating to operators defined by (4) and (5) we study their approximation properties in polynomial weighted spaces including Voronovskaja-type formulas. The final section is devoted to bringing to light properties of these operators that spring from the notion of generalized convexity.

2 Preliminary results

At first we calculate all the moments of both classes of operators.

Lemma 1 *Let P_n^*, $n \geq n_a$, be the operators given at* (4) *and* (6)*. For each integer p, $p \geq 0$, we have*

$$(P_n^* e_p)(x) = \sum_{s=0}^{[p/2]} \frac{(2s)!}{n^s} \binom{p}{2s} \alpha_n^{p-2s}(x), \quad x \in \mathbb{R}. \tag{9}$$

Proof Setting $I_k = \int_{\mathbb{R}} t^k e^{-\sqrt{n}|t|} dt$, for k odd we deduce $I_k = 0$. For k even, $k = 2s$, we obtain

$$I_{2s} = 2 \frac{(2s)!}{\left(\sqrt{n}\right)^{2s+1}}, \quad 0 \leq 2s \leq p. \tag{10}$$

Further,

$$\begin{aligned}(P_n^* e_p)(x) &= \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \sum_{k=0}^{p} \binom{p}{k} \alpha_n^{p-k}(x) t^k e^{-\sqrt{n}|t|} dt \\ &= \sqrt{n} \sum_{s=0}^{[p/2]} \binom{p}{2s} \alpha_n^{p-2s}(x) \frac{(2s)!}{\left(\sqrt{n}\right)^{2s+1}},\end{aligned}$$

and thus we arrive at relation (9). □

As particular cases we obtain

$$P_n^* e_0 = e_0, \quad P_n^* e_1 = \alpha_n, \quad P_n^* e_2 = \alpha_n^2 + \frac{2}{n}. \tag{11}$$

Lemma 2 *Let W_n^*, $n \geq 1$, be the operators given at* (5) *and* (7)*. The moments of these operators have the following values*

$$\begin{aligned}&(W_n^* e_0)(x) = 1, \quad (W_n^* e_1)(x) = \beta_n(x), \\ &(W_n^* e_p)(x) = \beta_n^p(x) + \sum_{s=1}^{[p/2]} \frac{(2s-1)!!}{(2n)^s} \binom{p}{2s} \beta_n^{p-2s}(x), \quad p \geq 2,\end{aligned} \tag{12}$$

where $x \in \mathbb{R}$.

Proof For $p = 0$ and $p = 1$ identities are established immediately. Let $p \geq 2$ be fixed. Setting $J_k = \int_{\mathbb{R}} t^k e^{-nt^2} dt$, for k odd we get $J_k = 0$. For k even, $k = 2s$, we have

$$J_{2s} = \frac{(2s-1)!!}{(2n)^s} J_0 \quad \text{and} \quad J_0 = \sqrt{\frac{\pi}{n}}, \tag{13}$$

where $s \in \mathbb{N}$, $1 \leq 2s \leq p$.

Further we can write

$$(W_n^* e_p)(x) = \sqrt{\frac{n}{\pi}} \left(\beta_n^p(x) J_0 + \sum_{s=1}^{[p/2]} \binom{p}{2s} \beta_n^{p-2s}(x) J_{2s} \right)$$

which leads us to the desired relation. □

As particular case we obtain

$$W_n^* e_2 = \beta_n^2 + \frac{1}{2n}. \tag{14}$$

Denoting by $\mu_r(L_n; \cdot)$ the central moment of r order of the operator L_n, this means $\mu_r(L_n, x) = L_n((\cdot - x)^r; x)$, $r = 0, 1, 2, \ldots$, we can enunciate

Lemma 3 *Let P_n^* and W_n^* be the operators defined by (4) and (5), respectively.*

(i) $\mu_0(P_n^*; x) = 1$, $\mu_1(P_n^*; x) = \alpha_n(x) - x$, $\mu_2(P_n^*; x) = (\alpha_n(x) - x)^2 + \dfrac{2}{n}$, $n \geq n_a$,

(ii) $\mu_0(W_n^*; x) = 1$, $\mu_1(W_n^*; x) = \beta_n(x) - x$, $\mu_2(W_n^*; x) = (\beta_n(x) - x)^2 + \dfrac{1}{2n}$, $n \geq 1$,

where α_n and β_n are defined by (6) and (7), respectively.

Proof All the above identities are implied by relations (11), (12) and (14). □

Lemma 4 *Let P_n^* and W_n^* be the operators defined by (4) and (5), respectively. The following relations take place:*

(i) $\mu_6(P_n^*; x) = (\alpha_n(x) - x)^6 + \dfrac{30}{n}(\alpha_n(x) - x)^4 + \dfrac{360}{n^2}(\alpha_n(x) - x)^2 + \dfrac{720}{n^3}$,

(ii) $\mu_6(W_n^*; x) = (\beta_n(x) - x)^6 + \dfrac{15}{2n}(\beta_n(x) - x)^4 + \dfrac{45}{4n^2}(\beta_n(x) - x)^2 + \dfrac{15}{8n^3}$,

(iii) $\lim\limits_{n\to\infty} \dfrac{\mu_6(P_n^*; x)}{\mu_2(P_n^*; x)} = 0$, $\lim\limits_{n\to\infty} \dfrac{\mu_6(W_n^*; x)}{\mu_2(W_n^*; x)} = 0$.

Proof (i)

$$\mu_6(P_n^*;x) = \frac{\sqrt{n}}{2}\int_{\mathbb{R}}((\alpha_n(x)-x)+t)^6 e^{-\sqrt{n}|t|}dt$$
$$= (\alpha_n(x)-x)^6(P_n^* e_0)(x) + \frac{\sqrt{n}}{2}(15(\alpha_n(x)-x)^4 I_2$$
$$+15(\alpha_n(x)-x)^2 I_4 + I_6),$$

where I_{2s}, $s \in \mathbb{N}$, are indicated at (10).

(ii) $\mu_6(W_n^*;x)$ is computed in the same manner taking into account the relation (13).

(ii) For the sake of simplicity, we denote $\alpha_n(x) - x = a_n$, where

$$a_n = -\frac{1}{2a}\log\left(\frac{n}{n-4a^2}\right), \quad n \ge n_a.$$

We get

$$\frac{\mu_6(P_n^*;x)}{\mu_2(P_n^*;x)} = \frac{a_n^6 + 30n^{-1}a_n^4 + 360n^{-2}a_n^2 + 720n^{-3}}{a_n^2 + 2n^{-1}}, \quad n \ge n_a.$$

Since $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} na_n^2 = 0$, the shown identity occurs. Similarly we proceed to second limit.

□

3 Weighted approximation

For proceed further, we need a result due to Gadzhiev [3]. The author considered a continuous and strictly increasing function φ defined on $\mathbb{R}$ and $\rho(x) = 1+\varphi^2(x)$ such that $\lim_{x\to\pm\infty}\rho(x) = \infty$.

Set

$$B_\rho(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} :\ |f(x)| \le M_f\,\rho(x)\},$$

where M_f is a constant depending on f,

$$C_\rho(\mathbb{R}) = B_\rho(\mathbb{R}) \cap C(\mathbb{R}),$$
$$C_\rho^*(\mathbb{R}) = \left\{ f \in C_\rho(\mathbb{R}) :\ \lim_{|x|\to\infty}\frac{f(x)}{\rho(x)} \text{ exists and it is finite}\right\}.$$

If the space $B_\rho(\mathbb{R})$ is endowed with the norm $\|\cdot\|_\rho$ defined by

$$\|f\|_\rho = \sup_{x\in\mathbb{R}}\frac{|f(x)|}{\rho(x)}, \tag{15}$$

then the same norm is considered in the other two spaces defined above.

Theorem 1 [3, *Theorem 2*] *Let* $(A_n)_{n\geq 1}$ *be a sequence of linear positive operators mapping* $C_\rho(\mathbb{R})$ *into* $B_\rho(\mathbb{R})$. *If*

$$\lim_{n\to\infty} \|A_n\varphi^\nu - \varphi^\nu\|_\rho = 0, \quad \nu = 0, 1, 2, \tag{16}$$

then, for any $f \in C_\rho^*(\mathbb{R})$ *we have*

$$\lim_{n\to\infty} \|A_n f - f\|_\rho = 0. \tag{17}$$

Our aim is to study the approximation property of P_n^* and W_n^* operators on some weighted spaces. We consider a weight commonly used in defining spaces of function with polynomial growth. We choose

$$\varphi(x) = x \quad \text{and} \quad \rho(x) = 1 + x^2, \ x \in \mathbb{R}. \tag{18}$$

This choice meets the conditions specified formerly.

Theorem 2 *Let* P_n^*, $n \geq n_a$, *be the operators defined by* (4) *and* (6). *For each* $f \in C_\rho^*(\mathbb{R})$ *the following relation*

$$\lim_{n\to\mathbb{R}} \|P_n^* f - f\|_\rho = 0 \tag{19}$$

holds, where ρ *is stated at* (18).

Proof Based on (15), for linear positive operators P_n^* defined on $C_\rho(\mathbb{R})$, we have

$$|(P_n^* f)(x)| \leq \|f\|_\rho (P_n^*\rho)(x), \ x \in \mathbb{R}.$$

Lemma 1 guarantees that our operators map $C_\rho(\mathbb{R})$ into $C_\rho(\mathbb{R}) \subset B_\rho(\mathbb{R})$.

We check the three conditions of relation (16).

Since $P_n^* e_0 = e_0$, for $\nu = 0$ the condition is fulfilled.

For $\nu = 1$, on the basis of (11), we have

$$\begin{aligned}\|P_n^* e_1 - e_1\|_\rho &= \sup_{x\in\mathbb{R}} \frac{|(P_n^* e_1)(x) - x|}{1+x^2} \\ &= \sup_{x\in\mathbb{R}} \frac{\left|\frac{1}{2a}\log\frac{n}{n-4a^2}\right|}{1+x^2} \leq \frac{1}{2a}\log\frac{n}{n-4a^2}.\end{aligned}$$

Consequently, $\lim_{n\to\infty} \|P_n^* e_1 - e_1\|_\rho = 0$.

Finally, for $\nu = 2$, on the basis of (11), we get

$$\begin{aligned}\|P_n^* e_2 - e_2\|_\rho &= \sup_{x\in\mathbb{R}} \frac{\left|\left(x - \frac{1}{2a}\log\frac{n}{n-4a^2}\right)^2 + \frac{2}{n} - x^2\right|}{1+x^2} \\ &= \sup_{x\in\mathbb{R}} \frac{\left|-\frac{x}{a}\log\frac{n}{n-4a^2} + \frac{1}{4a^2}\log^2\frac{n}{n-4a^2} + \frac{2}{n}\right|}{1+x^2} \\ &\le \frac{1}{a}\log\frac{n}{n-4a^2} + \frac{1}{4a^2}\log^2\frac{n}{n-4a^2} + \frac{2}{n}.\end{aligned}$$

Again, $\lim_{n\to\infty}\|P_n^* e_2 - e_2\|_\rho = 0$.
In view of Theorem 1, relation (19) follows. □

Following the same route and using relations (12) and (14) we can formulate

Theorem 3 *Let W_n^*, $n \ge 1$, be the operators defined by (5) and (7). For each $f \in C_\rho^*(\mathbb{R})$ the following relation*

$$\lim_{n\to\infty}\|W_n^* f - f\|_\rho = 0$$

holds, where ρ is stated at (18).

4 Quantitative Voronovskaja formulas

In this section we establish the asymptotic behavior for our operators.

In order to measure the rate of convergence on $C_\rho^*(\mathbb{R})$ we use a weighted modulus of smoothness. Following [4] we consider

$$\Omega(f,\delta) = \sup_{\substack{x\in\mathbb{R}\\ |h|\le\delta}} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)}, \quad f \in C_\rho^*(\mathbb{R}). \tag{20}$$

Among its properties we recall the following: $\lim_{\delta\to 0^+}\Omega(f,\delta) = 0$, $\Omega(f,\cdot)$ is an increasing function and for each $\lambda > 0$

$$\Omega(f,\lambda\delta) \le 2(1+\lambda)(1+\delta^2)\Omega(f,\delta). \tag{21}$$

Lemma 5 *For each $f \in C_\rho^*(\mathbb{R})$ let $\Omega(f,\cdot)$ be defined by (20). For any $(t,x) \in \mathbb{R}\times\mathbb{R}$ and any $\delta > 0$ the following relation*

$$|f(t) - f(x)| \le 4\left(1 + \frac{(t-x)^4}{\delta^4}\right)(1+\delta^2)^2(1+x^2)\Omega(f,\delta). \tag{22}$$

holds.

Proof Let $\delta > 0$ be arbitrary fixed and $(x, t) \in \mathbb{R} \times \mathbb{R}$. Set $t - x = h$.

$$\frac{|f(t) - f(x)|}{(1 + (t - x)^2)(1 + x^2)} \leq \sup_{\substack{x \in \mathbb{R} \\ |h| = |t - x|}} \frac{|f(x + h) - f(x)|}{(1 + h^2)(1 + x^2)}$$
$$\leq \sup_{\substack{x \in \mathbb{R} \\ |\widetilde{h}| \leq |t - x|}} \frac{|f(x + \widetilde{h}) - f(x)|}{(1 + \widetilde{h}^2)(1 + x^2)} = \Omega(f, |t - x|)$$
$$\leq 2\left(1 + \frac{|t - x|}{\delta}\right)(1 + \delta^2)\Omega(f, \delta).$$

In the last increase we used (21) with $\lambda := |t - x|/\delta$. We got

$$|f(t) - f(x)| \leq 2\left(1 + \frac{|t - x|}{\delta}\right)(1 + (t - x)^2)(1 + x^2)(1 + \delta^2)\Omega(f, \delta).$$

If we prove

$$\left(1 + \frac{|t - x|}{\delta}\right)(1 + (t - x)^2) \leq 2\left(1 + \frac{(t - x)^4}{\delta^4}\right)(1 + \delta^2), \tag{23}$$

i.e., $(1 + y)(1 + (t - x)^2) \leq 2(1 + y^4)(1 + \delta^2)$, where $y = |t - x|/\delta$, then (22) is true. We justify (23) on two cases.

For $y \leq 1$, $(1 + y)(1 + (t - x)^2) \leq 2(1 + \delta^2)$ and (23) is evident.

For $1 < y$, $(1 + y)(1 + (t - x)^2) \leq 2y(y^2 + \delta^2 y^2) = 2y^3(1 + \delta^2)$ and again (23) is true. The proof is completed. □

Theorem 4 *Let P_n^*, $n \geq n_a$, be given by (4) and (6). Let $f \in C_\rho^*(\mathbb{R})$ such that f is twice differentiable and f', f'' belong to $C_\rho^*(\mathbb{R})$. For any $x \in \mathbb{R}$ we have*

(i)

$$|n((P_n^* f)(x) - f(x)) + 2af'(x) - f''(x)| \leq |A_n(x)||f'(x)|$$
$$+|B_n(x)||f''(x)| + 16n(1 + x^2)\mu_2(P_n^*; x)\Omega\left(f''; \sqrt[4]{\frac{\mu_6(P_n^*; x)}{\mu_2(P_n^*; x)}}\right),$$

where

$$A_n(x) = n\mu_1(P_n^*; x) + 2a \quad \textit{and} \quad B_n(x) = \frac{n}{2}\mu_2(P_n^*; x) - 1.$$

(ii) $\lim\limits_{n \to \infty} n((P_n^* f)(x) - f(x)) = -2af'(x) + f''(x).$

Proof (i) Let x be arbitrarily fixed and $t \in \mathbb{R}$. By Taylor's formula with Lagrange form of the remainder, we have

$$f(t) = f(x) + (t - x)f'(x) + \frac{(t - x)^2}{2}f''(x) + \frac{(t - x)^2}{2}h(\xi_{t,x}), \tag{24}$$

where $\xi_{t,x}$ is a certain real number between t and x. In the above

$$h(\xi_{t,x}) = f''(\xi_{t,x}) - f''(x) \tag{25}$$

is a continuous function. If $t \to x$, then $\xi_{t,x} \to x$ and h vanishes at x. Applying the operator P_n^* to both sides of identity (24), knowing that $P_n^* e_0 = e_0$, we obtain

$$(P_n^* f)(x) - f(x) = \mu_1(P_n^*; x) f'(x) + \mu_2(P_n^*; x) \frac{f''(x)}{2} + \frac{1}{2} P_n^*((\cdot - x)^2 h; x).$$

This identity can be rewritten in the following way

$$\begin{aligned} &|n((P_n^* f)(x) - f(x)) + 2af'(x) - f''(x)| \\ &\quad \le |A_n(x)||f'(x)| + |B_n(x)||f''(x)| + \frac{n}{2} P_n^*((\cdot - x)^2 |h|, x). \end{aligned} \tag{26}$$

By using both (24) and (22) applied for f'', we get

$$|h(\xi_{t,x})| = |f''(\xi_{t,x}) - f''(x)| \le 4 \left(1 + \frac{(t-x)^4}{\delta^4}\right) (1+\delta^2)(1+x^2)\Omega(f''; \delta)$$

and, in the factor $(1+\delta^2)^2$ considering $\delta \le 1$, we can write

$$n P_n^*((\cdot - x)^2 |h|; x) \le 16n(1+x^2)\mu_2(P_n^*; x) \left(1 + \frac{\mu_6(P_n^*; x)}{\delta^4 \mu_2(P_n^*; x)}\right) \Omega(f''; \delta).$$

Further, a rank $N_1 \ge n_a$ exists such that for any $n \ge N_1$ we can choose $\delta^4 = \mu_6(P_n^*; x)/\mu_2(P_n^*; x) \le 1$. This choice is allowed because of Lemma 4(iii). Returning at (26) the required inequality is proved.

(ii) Easily obtain

$$\lim_{n\to\infty} A_n(x) = 0, \quad \lim_{n\to\infty} B_n(x) = 0, \quad \lim_{n\to\infty} n\mu_2(P_n^*; x) = 2$$

and taking into account Lemma 4(iii), the statement follows.

□

Theorem 5 *Let W_n^*, $n \ge 1$, be given by (5) and (7). Let $f \in C_\rho^*(\mathbb{R})$ such that f is twice differentiable and f', f'' belong to $C_\rho^*(\mathbb{R})$. For any $x \in \mathbb{R}$ we have*

(i)

$$\begin{aligned} &\left| n((W_n^* f)(x) - f(x)) + \frac{a}{2} f'(x) - \frac{1}{4} f''(x) \right| \le |C_n(x)||f'(x)| \\ &+ |D_n(x)||f''(x)| + 16n(1+x^2)\mu_2(W_n^*; x)\Omega\left(f''; \sqrt[4]{\frac{\mu_6(W_n^*; x)}{\mu_2(W_n^*; x)}}\right) \end{aligned}$$

where

$$C_n(x) = n\mu_1(W_n^*; x) + \frac{a}{2} \quad and \quad D_n(x) = \frac{n}{2}\mu_2(W_n^*; x) - \frac{1}{4}.$$

(ii) $\lim_{n\to\infty} n((P_n^* f)(x) - f(x)) = -\frac{a}{2} f'(x) + \frac{1}{4} f''(x).$

For achieving the proof we appeal, inter alia, at relation (24), the central moments $\mu_k(W_n^*; \cdot)$, $k \in \{1, 2\}$, Lemmas 4 and 6. Actually, the technique proceed with arguments identical with those used in the proof of Theorem 4, consequently we omit it.

5 A property implied by generalized convexity

Lemma 6 *The operators P_n^*, $n \geq n_a$, and W_n^*, $n \geq 1$, reproduce the function φ_a defined by* (8).

Proof We have

$$\begin{aligned}(P_n^* \varphi_a)(x) &= e^{2a\alpha_n(x)} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} e^{2at-\sqrt{n}|t|} dt \\ &= e^{2a\alpha_n(x)} \frac{n}{n-4a^2} = \varphi_a(x).\end{aligned}$$

Similarly, relation $(W_n^* \varphi_a)(x) = \varphi_a(x)$ is deduced by direct calculation. □

This way, we infer that besides the function e_0, function φ_a is also a fixed point for all operators P_n^*, $n \geq n_a$, and W_n^*, $n \geq 1$. Further, we use the couple (e_0, φ_a).

On the basis of [5, *Definition 2*] and taken in view Ziegler's remark [5, *page 426*] we present the following

Definition A function f defined on $\mathbb{R}$ is said to be convex with respect to (e_0, φ_a), provided

$$\begin{vmatrix} 1 & 1 & 1 \\ \varphi_a(x_1) & \varphi_a(x_2) & \varphi_a(x_3) \\ f(x_1) & f(x_2) & f(x_3) \end{vmatrix} \geq 0, \quad -\infty < x_1 < x_2 < x_3 < \infty. \tag{27}$$

The set of functions satisfying (27) is denoted by $\mathcal{C}(e_0, \varphi_a)$.

Theorem 6 *Let the operators P_n^*, $n \geq n_a$, W_n^*, $n \geq 1$, be given. For every function $f \in C(\mathbb{R}) \cap \mathcal{C}(e_0, \varphi_a)$, we have*

$$(P_n^* f)(x) \geq f(x) \quad and \quad (W_n^* f)(x) \geq f(x), \; x \in \mathbb{R}.$$

Proof Since our operators reproduce the functions e_0 and φ_a, we can apply Theorem 2 of the paper [5]. We are considering the fact that this result of Ziegler also works for functions defined on unbounded intervals. We emphasize that the condition to be e_0 and φ_a fixed points for our operators are indispensable [5, *Theorem 3*]. □

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