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On existence of positive solutions for a class of discrete fractional boundary value problems

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Abstract Motivated by some recent developments in the existence theory of fractional difference equations, in this paper we consider boundary value problem

$$\begin{aligned} -\Delta^{\nu}_{\nu-2}u(t) &= f(t+\nu-1, u(t+\nu-1)), \quad 1<\nu\leq 2, \\ u(\nu-2) &= 0, \quad \Delta^{\nu-1}_{\nu-1}u(\nu+N)=0, \end{aligned}$$

where $t \in [0, N+1]_{\mathbb{N}_0}$ and N ($N \geq 2$) is an integer. The nonlinear function $f : [\nu-1, \nu+N]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}^+$ is assumed to be continuous. We establish some useful inequalities satisfied by the Green's function associated with above boundary value problem. Sufficient conditions are developed to ensure the existence and nonexistence of positive solutions for the boundary value problem.

Keywords Discrete fractional calculus · Fractional summation · Fractional difference · Positive solutions

Mathematics Subject Classification 39A05 · 39A99 · 26A33

1 Introduction

In recent years the continuous fractional calculus has seen tremendous growth due to the fact that many problems in science and engineering can be modeled by fractional differential equations. In many situations, the mathematical models based on fractional

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operators provide more suitable results than analogous models involving classical integer order depravities. The qualitative theory of fractional differential equations has been studied comprehensively in [1–4] and references therein. However quite recently the discrete fractional calculus has gained a great deal of interest by several researchers. Miller and Ross [5] initiated the study of discrete fractional calculus. Recently, important developments in this direction has been reported by Atici and Eloe [6–8], Holm [9,10], Abdeljawad [11], Goodrich [12], Jia et al. [13], Goodrich and Peterson [14].

There are few papers dealing with the existence of positive solutions for discrete fractional boundary value problems. In [15], Atici and Eloe obtained sufficient conditions for the existence of positive solutions of the following two-point boundary value problem for a finite fractional difference equation

$$\begin{aligned} &-\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)), \quad t = 1, 2, 3, \cdots, b+1, \\ &y(\nu-2) = 0, \quad y(\nu+b+1) = 0, \end{aligned}$$

where $1 < \nu \leq 2$ is a real number and, $b \geq 2$ an integer and $f : [\nu-1, \nu+b-1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous.

Goodrich [16] considered a discrete fractional boundary value problem of the form

$$\begin{aligned} &-\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)), \quad 1 < \nu \leq 2, \\ &y(\nu-2) = g(y), \quad y(\nu+b) = 0, \end{aligned}$$

where $f : [\nu-1, \nu+b-1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous, $g : C([\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}, \mathbb{R})$ is a given functional. He proved the existence and uniqueness of solutions using tools from nonlinear functional analysis.

Goodrich [17] investigated a nonlinear discrete fractional boundary value problem given by

$$\begin{aligned} &-\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)), \quad 1 < \nu \leq 2, \\ &y(\nu-2) = 0, \quad \Delta y(\nu+b) = 0, \end{aligned}$$

where $t \in [0, b+1]_{\mathbb{N}_0}$, $f : [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$, and $b \in \mathbb{N}_0$. The Green's function for this problem is studied and some new results are obtained for $1 < \nu < 2$.

Inspired by the above cited works, we consider the following class of two-point boundary value problem for fractional difference equations

$$\begin{aligned} &-\Delta^{\nu}_{\nu-2} u(t) = f(t+\nu-1, u(t+\nu-1)), \quad t \in \mathbb{N}_0, \\ &u(\nu-2) = 0, \quad \Delta^{\nu-1}_{\nu-1} u(\nu+N) = 0, \end{aligned} \tag{1}$$

where $1 < \nu \leq 2$ is a real number and, $N \geq 2$ an integer and $f : [\nu-1, \nu+N]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous. This boundary value problem is similar to the problem discussed in [17]. The difference here is that the boundary condition at $\nu+N$ involves a difference of order $\nu-1$ rather than order one. We obtain a different Green's function as compared to one obtained in [17]. We shall establish an equivalent summation representation of above problem and obtain various existence and non-existence results for positive solutions. Furthermore, for eigenvalue problem, intervals for parameter (eigenvalues) are obtained for which there exist positive solutions or no positive solution.

Rest of the paper is organized as follows: In Sect. 2, we shall list some basic definitions and properties of discrete fractional operators. In Sect. 3 we shall obtain an equivalent summation representation of the boundary value problem (1). Some useful inequality will be established for the Green's function. Finally in Sect. 4 we shall use tools from functional analysis to establish several existence results for positive solutions.

2 Preliminaries

In this section we review some basic definitions and properties of discrete fractional operators. For details, we refer the reader to [14].

Definition 1 For $\nu > 0$, the ν-th fractional sum of a function $u : \mathbb{N}_a \to \mathbb{R}$ is defined as

$$\Delta_a^{-\nu} u(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} u(s),$$

for $t \in \mathbb{N}_{a+\nu} := \{a+\nu, a+\nu+1, \ldots\}$. Also the fractional difference of order $\nu > 0$ is defined by $\Delta_a^{\nu} u(t) := \Delta^n \Delta_a^{\nu-n} u(t)$ where $n-1 < \nu \leq n$ with $n \in \mathbb{N}$ and $t \in N_{a+n-\nu}$. The fractional difference operator Δ_a^{ν} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+n-\nu}$.

Lemma 1 *Assume $\mu, \nu > 0$. Then following properties hold for fractional sum and difference*:

(i) $\Delta t^{(\nu)} = \nu t^{(\nu-1)}$,
(ii) $\nu^{(\nu)} = \Gamma(\nu+1)$,
(iii) $\Delta_{a+\mu}^{-\nu} (t-a)^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} (t-a)^{(\mu+\nu)}$ *and*
(iv) $\Delta_{a+\mu}^{\nu} (t-a)^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)} (t-a)^{(\mu-\nu)}$,

whenever expressions in (i)–(iv) *are well defined.*

Lemma 2 [14] *Assume u be a real valued function and $\mu, \nu > 0$. Then*

$$\Delta_{a+\mu}^{-\nu} [\Delta_a^{-\mu} u(t)] = \Delta_a^{-(\mu+\nu)} u(t) = \Delta_{a+\nu}^{-\mu} [\Delta_a^{-\nu} u(t)],$$

for all $t \in \mathbb{N}_{\nu+\mu}$.

Lemma 3 [9] *Let $u : \mathbb{N}_a \to \mathbb{R}$ and $n-1 < \nu \leq n$. Consider the ν-th-order discrete fractional equation*

$$\Delta_{a+\nu-n}^{\nu} u(t) = h(t), \quad t \in \mathbb{N}_a, \tag{2}$$

and the corresponding discrete fractional initial value problem

$$\begin{aligned}\Delta^{\nu}_{a+\nu-n}u(t) &= h(t), \quad t \in \mathbb{N}_a,\\ \Delta^{i}u(a+\nu-n) &= A_i, \quad i \in \{0,1,\ldots,n-1\}, \quad A_i \in \mathbb{R}.\end{aligned} \tag{3}$$

Then the general solution of (2) *is given by*

$$u(t) = \Delta_a^{-\nu}h(t) + c_1(t-a)^{(\nu-1)} + c_2(t-a)^{(\nu-2)} + \cdots + c_n(t-a)^{(\nu-n)},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $t \in \mathbb{N}_{a+\nu-n}$, *and the unique solution to* (3) *is given by*

$$u(t) = \Delta_a^{-\nu}h(t) + \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i}\sum_{k=0}^{i-j}\frac{(-1)^k(i-k)^{(n-\nu)}}{i!}\binom{i}{j}\binom{i-j}{k}A_j\right)(t-a)^{(i+\nu-n)},$$
$$t \in \mathbb{N}_{a+\nu-n}.$$

3 Green's function and its properties

To establish existence theorems, the boundary value problem for the fractional difference equation is reduced to fractional summation equation. This is standard practice in existence theory of fractional difference equations. In the following lemma we reduce the boundary value problem (1) to an equivalent summation equation.

Lemma 4 *Let* $1 < \nu \leq 2$ *and* $h \in C([\nu-1, \nu+N]_{\mathbb{N}_{\nu-1}})$. *Then a function* u *is solution of discrete fractional boundary value problem*

$$\begin{cases}\Delta^{\nu}_{\nu-2}u(t) + h(t+\nu-1) = 0,\\ u(\nu-2) = 0, \quad \Delta^{\nu-1}_{\nu-1}u(\nu+N) = 0,\end{cases}$$

if and only if $u(t)$, *for* $t \in [\nu-2, \nu+N]_{\mathbb{N}_{\nu-2}}$ *is solution of*

$$u(t) = \sum_{s=0}^{N+1} G(t,s)h(s+\nu-1),$$

where

$$G(t,s) = \begin{cases}\frac{t^{(\nu-1)}-(t-s-1)^{(\nu-1)}}{\Gamma(\nu)}, & s \leq t-\nu \leq N+1;\\ \frac{t^{(\nu-1)}}{\Gamma(\nu)}, & t-\nu < s \leq N+1.\end{cases} \tag{4}$$

Proof In view of Lemma 3, the general solution of the fractional difference equation $\Delta^{\nu}_{\nu-2}u(t) + h(t+\nu-1) = 0$ is given by

$$u(t) = -\Delta_0^{-\nu}h(t+\nu-1) + c_1t^{(\nu-1)} + c_2t^{(\nu-2)}, \tag{5}$$

where $t \in [\nu-1, \nu+N]_{\mathbb{N}_{\nu-1}}$ and $c_1, c_2 \in \mathbb{R}$. Now applying the boundary condition $u(\nu-2) = 0$, we immediately get $c_2 = 0$. Applying $\Delta_{\nu-1}^{\nu-1}$ on both sides of equation (5) and by using Lemmas 1 and 2, we arrive at

$$\Delta_{\nu-1}^{\nu-1}u(t) = -\Delta_0^{-1}h(t+\nu-1) + c_1\Gamma(\nu) = -\sum_{s=0}^{t-1} h(s+\nu-1) + c_1\Gamma(\nu). \quad (6)$$

Using the boundary condition $\Delta_{\nu-1}^{\nu-1}u(\nu+N) = 0$ in Eq. (6) we have

$$c_1\Gamma(\nu) = \sum_{s=0}^{\lceil(\nu-1)+N\rceil} h(s+\nu-1).$$

Since $\nu - 1 \leq 1$, therefore

$$c_1 = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{N+1} h(s+\nu-1). \quad (7)$$

Substituting (7) in Eq. (5) we get

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{(\nu-1)} h(s+\nu-1) + \frac{t^{(\nu-1)}}{\Gamma(\nu)} \sum_{s=0}^{N+1} h(s+\nu-1) \\ &= \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} [t^{(\nu-1)} - (t-s-1)^{(\nu-1)}] h(s+\nu-1) \\ &\quad + \frac{t^{(\nu-1)}}{\Gamma(\nu)} \sum_{s=t-\nu+1}^{N+1} h(s+\nu-1) \\ &= \sum_{s=0}^{N+1} G(t,s) h(s+\nu-1). \end{aligned}$$

□

Lemma 5 *The Green's function $G(t,s)$ in (4) satisfies the following properties:*

(i) $G(t,s) > 0$ *for* $t \in [\nu-1, \nu+N]_{\mathbb{N}_{\nu-1}}$ *and* $s \in [0, N+1]_{\mathbb{N}_0}$,
(ii) $\max_{t \in [\nu-1, \nu+N]_{\mathbb{N}_{\nu-1}}} G(t,s) = G(s+\nu, s)$ *and*
(iii) $\min_{t \in \left[\frac{N+\nu}{4}, \frac{3(N+\nu)}{4}\right]_{\mathbb{N}_{\nu-1}}} G(t,s) \geq \gamma \max_{t \in [\nu-1, \nu+N]_{\mathbb{N}_{\nu-1}}} G(t,s)$ *for some* $\gamma \in (0,1)$, $s \in [0, N+1]_{\mathbb{N}_0}$.

Proof (i) For $t - \nu < s \leq N+1$, clearly $G(t,s) > 0$. Since $t^{(q)}$ is increasing function for $0 < q < 1$, therefore $(t-s-1)^{(\nu-1)} < t^{(\nu-1)}$. Hence $G(t,s) > 0$ for $0 < s < t - \nu \leq N+1$.

(ii) For $t-\nu < s \le N+1$, obviously $\Gamma(\nu)\Delta_t G(t,s) = (\nu-1)t^{(\nu-2)} > 0$ which implies that G is increasing for $0 \le t \le s+\nu$. Furthermore $t^{(q)}$ is a decreasing function for $q \in (-1,0)$, which implies $\Delta_t G(t,s) = (\nu-1)[t^{(\nu-2)} - (t-s-1)^{(\nu-2)}] < 0$ for $s < t-\nu \le N$. Thus G is decreasing for $s+\nu < t \le \nu+N$. Hence $\max_{t\in[\nu-1,\nu+N]_{\mathbb{N}_{\nu-2}}} G(t,s) = G(s+\nu,s)$.

Now

$$\frac{G(t,s)}{G(s+\nu,s)} = \begin{cases} \frac{t^{(\nu-1)}-(t-s-1)^{(\nu-1)}}{(s+\nu)^{(\nu-1)}}, & s \le t-\nu \le N+1; \\ \frac{t^{(\nu-1)}}{(s+\nu)^{(\nu)}}, & t-\nu < s \le N+1. \end{cases} \tag{8}$$

Observe that, for $\frac{(N+\nu)}{4} \le t \le \frac{3(N+\nu)}{4}$ and $t-\nu < s \le N$, we have $t^{(\nu-1)} \ge \left(\frac{N+\nu}{4}\right)^{(\nu-1)}$ and $(s+\nu)^{(\nu-1)} \le (N+\nu)^{(\nu-1)}$. Therefore

$$\frac{G(t,s)}{G(s+\nu,s)} > \frac{\left(\frac{1}{4}(N+\nu)\right)^{(\nu-1)}}{(N+\nu)^{(\nu-1)}}.$$

Now, for $\frac{(N+\nu)}{4} \le t \le \frac{3(N+\nu)}{4}$ and $s \le t-\nu$, we have $(t-s-1)^{\nu-1} \le (\frac{3}{4}(N+\nu)-s-1)^{\nu-1} < (\frac{3}{4}(N+\nu)-1)^{\nu-1}$ and $t^{(\nu-1)} \ge (s+\nu)^{(\nu-1)}$. Consequently

$$\frac{G(t,s)}{G(s+\nu-1,s)} \ge 1 - \frac{\left(\frac{3}{4}(N+\nu)-1\right)^{(\nu-1)}}{(\frac{3}{4}(N+\nu))^{(\nu-1)}}.$$

Therefore

$$\min_{t\in\left[\frac{N+\nu}{4},\frac{3(N+\nu)}{4}\right]_{\mathbb{N}_{\nu-1}}} G(t,s) \ge \gamma \max_{t\in[\nu-1,\nu+N]_{\mathbb{N}_{\nu-1}}} G(t,s),$$

where $\gamma := \min\{\frac{(\frac{1}{4}(N+\nu))^{(\nu-1)}}{(N+\nu)^{(\nu-1)}},\ 1-\frac{(\frac{3}{4}(N+\nu)-1)^{(\nu-1)}}{(\frac{3}{4}(N+\nu))^{(\nu-1)}}\}$. Note that $\frac{(\frac{1}{4}(N+\nu))^{(\nu-1)}}{(N+\nu)^{(\nu-1)}} < 1$ and also $\frac{(\frac{3}{4}(N+\nu)-1)^{(\nu-1)}}{(\frac{3}{4}(N+\nu))^{(\nu-1)}} < 1$. Therefore $0 < \gamma < 1$. □

4 Existence of positive solutions

In recent decades, the Krasnoselskii fixed point theorem and its generalizations have been frequently applied to establish the existence of solutions in the study of boundary value problems. We shall use following well-known Guo–Krasnoselskii fixed point theorem to establish sufficient conditions for the existence of at least one positive solution for boundary value problem (1).

Definition 2 We call a function $u(t)$ a positive solution of problem (1), if $u(t) \in C[\nu-2,\nu+N]_{\mathbb{N}_{\nu-2}}$ and $u(t) \ge 0$ for $t \in [\nu-2,\nu+N]_{\mathbb{N}_{\nu-2}}$ and satisfies (1).

Theorem 1 [18] *Let B be a Banach space and $K \subseteq B$ be a cone. Assume Ω_1 and Ω_2 are open discs contained in B with $0 \in \Omega_1$, and $\bar{\Omega}_1 \subset \Omega_2$. Furthermore assume that $T : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator such that, either*

(i) $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$.

Then T has at least one fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

We define Banach space B by

$$B = \{u \in C([\nu-1,\ \nu+N]_{\mathbb{N}_{\nu-2}}) : \Delta_{\nu-1}^{\nu-2} u \in C([\nu-1,\ \nu+N]_{\mathbb{N}_{\nu-1}}),\ \nu \in (1,2]\},$$

equipped with the norm $\|u\| = \max |u(t)|,\ t \in [\nu-2,\ \nu+N]_{\mathbb{N}_{\nu-2}}$. In addition, for some $\gamma \in (0,1)$ we define

$$K := \left\{ u \in B : u(t) \geq 0, \min_{t \in \left[\frac{N+\nu}{4}, \frac{3(N+\nu)}{4}\right]_{\mathbb{N}_{\nu-2}}} u(t) \geq \gamma \|u\| \right\}. \tag{9}$$

Let $T : B \to B$ be an operator defined as

$$Tu(t) = \sum_{s=0}^{N+1} G(t,s) f(s+\nu-1, u(s+\nu-1)), \tag{10}$$

then $u \in B$ is a solution of (10) if and only if $u \in B$ is a solution of (1).

Lemma 6 *Let T be defined as in* (10) *and K in* (9). *Then $T : K \to K$ and T is completely continuous.*

Proof Let $u \in K$. Then by Lemma 5, it follows that

$$\begin{aligned}
\min_{t \in \left[\frac{N+\nu}{4}, \frac{3(N+\nu)}{4}\right]_{\mathbb{N}_{\nu-2}}} Tu(t) &\geq \min_{t \in \left[\frac{N+\nu}{4}, \frac{3(N+\nu)}{4}\right]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{N+1} G(t,s) f(s+\nu-1, u(s+\nu-1)) \\
&\geq \gamma \sum_{s=0}^{N+1} \max_{t \in [\nu-1,\ \nu+N]_{\mathbb{N}_{\nu-1}}} G(t,s) f(s+\nu-1, u(s+\nu-1)) \\
&= \gamma \max_{t \in [\nu-1,\ \nu+N]_{\mathbb{N}_{\nu-1}}} Tu(t) = \gamma \|Tu\|.
\end{aligned}$$

Therefore $T : K \to K$. One can easily prove that T is completely continuous. □

Following conditions on the growth of function f will be used in the sequel to establish some existence results.

(H₁) There exists $\rho > 0$ such that $f(t,u) \leq \frac{1}{2}\eta\rho$ whenever $0 \leq u \leq \rho$,

($\mathbf{H_2}$) There exists $\rho > 0$ such that $f(t,u) \geq \mu\rho$ whenever $\gamma\rho \leq u \leq \rho$.

For convenience, we introduce following notations:

$$\eta := 2\left(\sum_{s=0}^{N+1} G(s+\nu, s)\right)^{-1}, \quad \mu := 1/\varsigma, \quad \text{where}$$

$$\varsigma := \sum_{s=\lceil \frac{\nu+N}{4} - \nu \rceil}^{\lfloor \frac{3(\nu+N)}{4} - \nu \rfloor} \gamma G\left(\left\lfloor \frac{N+1}{2} \right\rfloor + \nu, s\right).$$

We now can prove the following existence result.

Theorem 2 *Suppose that there are distinct* $\rho_1, \rho_2 > 0$ *such that condition* (H_1) *holds at* $\rho = \rho_1$ *and condition* (H_2) *holds at* $\rho = \rho_2$. *Suppose also that* $f(t,u) \geq 0$ *and continuous. Then the problem* (1) *has at least one positive solution, say* u_0, *such that* $|u_0|$ *lies between* ρ_1 *and* ρ_2.

Proof We shall assume without loss of generality that $0 < \rho_1 < \rho_2$. For $u \in K$, by Lemma 6 $Tu \in K$ and T is a completely continuous operator. Now put $\Omega_1 = \{u \in K : ||u|| < \rho_1\}$. Note that for $u \in \partial\Omega_1$, we have that $||u|| = \rho_1$, so that condition (H_1) holds for all $u \in \partial\Omega_1$. So for $u \in K \cap \partial\Omega_1$, we find that

$$\begin{aligned}
||Tu|| &= \max_{t\in[\nu-1,\nu+N]_{\mathbb{N}_{\nu-1}}} \sum_{s=0}^{N+1} G(t,s) f(s+\nu-1, u(s+\nu-1)) \\
&\leq \sum_{s=0}^{N+1} G(s+\nu, s) f(s+\nu-1, u(s+\nu-1)) \\
&\leq \frac{1}{2}\eta\rho_1 \sum_{s=0}^{N+1} G(s+\nu, s) = \rho_1 = ||u||.
\end{aligned}$$

Whence we find that $||Tu|| \leq ||u||$, whenever $u \in K \cap \partial\Omega_1$. Thus we get that the operator T is a cone compression on $K \cap \Omega_1$. On the other hand, put $\Omega_2 = \{u \in K : ||u|| < \rho_2\}$. Note that for $u \in \partial\Omega_2$, we have that $||u|| = \rho_2$, so that condition (H_2) holds for all $u \in \partial\Omega_2$. Also note that $\left\{\lfloor \frac{N+1}{2} \rfloor + \nu\right\} \subset \left[\frac{N+\nu}{4}, \frac{3(N+\nu)}{4}\right]$. So, for $u \in K \cap \partial\Omega_2$, we find that

$$\begin{aligned}
Tu\left(\lfloor \frac{N+1}{2} \rfloor + \nu\right) &= \sum_{s=0}^{N+1} G\left(\left\lfloor \frac{N+1}{2} \right\rfloor + \nu, s\right) f(s+\nu-1, u(s+\nu-1)) \\
&\geq \mu\rho_2 \sum_{s=\lceil \frac{\nu+N}{4} - \nu \rceil}^{\lfloor \frac{3(\nu+N)}{4} - \nu \rfloor} \gamma G\left(\left\lfloor \frac{N+1}{2} \right\rfloor + \nu, s\right) = \rho_2.
\end{aligned}$$

Whence $||Tu|| \geq ||u||$, whenever $u \in K \cap \partial\Omega_2$. Thus we get that the operator T is a cone expansion on $K \cap \partial\Omega_2$. So, it follows from Theorem 1 that the operator T has a fixed point. This means that (1) has a positive solution, say u_0 with $\rho_1 \leq ||u_0|| \leq \rho_2$, as claimed. □

We now establish some results that yield the existence or non existence of positive solutions under the assumption that $f(t,u)$ has the special form $f(t,u) \equiv \lambda F_1(t)F_2(u)$.

(**H₃**) We assume that $\lambda > 0$, and that F_1, F_2 are nonnegative. Let

$$F_0 = \lim_{u\to 0}\frac{F_2(u)}{u}, \qquad F_\infty = \lim_{u\to\infty}\frac{F_2(u)}{u}.$$

Theorem 3 *Assume that condition rm (H_3) holds.*

(**H₄**) *If $F_0 = 0$ and $F_\infty = \infty$, then for all $\lambda > 0$ problem* (1) *has a positive solution.*
(**H₅**) *If $F_0 = \infty$ and $F_\infty = 0$, then for all $\lambda > 0$ problem* (1) *has a positive solution.*

Proof In order to prove that boundary value problem (1) has a positive solution, it is sufficient to show that operator $T : K \to K$ defined by Eq. (10) has a fixed point. The operator $T : K \to K$ is completely continuous. Define $K_\rho = \{u \in K : \|u\| < \rho\}$ and $\partial K_\rho = \{u \in K : \|u\| = \rho\}$, then the operator $T : K_\rho \to K$ defined as $Tu(t) := \lambda\sum_{s=0}^{N+1} G(t,s)F_1(s)F_2(u(s))$ is completely continuous. Now, $F_0 = 0$ implies, for $\epsilon \in (\lambda\sum_{s=0}^{N+1} G(s+\nu,s)F_1(s))^{-1}$, there exists positive $\rho_1 > 0$ such that $F_2(u) < \epsilon|u|$, whenever $0 < |u| < \rho_1$. Let $\Omega_{\rho_1} = \{u \in K : \|u\| < \rho_1\}$. For $u \in \partial\Omega_{\rho_1} \cap K$, we have

$$\begin{aligned}|Tu(t)| &\leq \lambda\epsilon\sum_{s=0}^{N+1} G(s+\nu,s)F_1(s)|u(s)|\\ &\leq \rho_1\lambda\epsilon\sum_{s=0}^{N+1} G(s+\nu,s)F_1(s) < \rho_1,\end{aligned}$$

which imply that $\|Tu(t)\| \leq \rho_1 = \|u\|$, for $u \in \partial\Omega_{\rho_1} \cap K$. The assumption $F_\infty = \infty$ implies, for $M > (\lambda\gamma^2\sum_{s=\lceil\frac{N+\nu}{4}\rceil}^{\lfloor\frac{3(N+\nu)}{4}\rfloor} G(s+\nu,s)F_1(s))^{-1}$, there exists $N_0 > 0$ such that $F_2(u) > M|u|$, whenever $|u| > N_0$. Take $\rho_2 > \{\rho_1, \frac{N_0}{\gamma}\}$, and let $\Omega_{\rho_2} = \{u \in K : \|u\| < \rho_2\}$. Then for $u \in \partial\Omega_{\rho_2} \cap K$, we have $u(t) \geq \gamma\|u\| = \gamma\rho_2 > N_0$, for $\frac{N+\nu}{4} \leq t \leq \frac{3(N+\nu)}{4}$. Hence

$$Tu(t) \geq \lambda\sum_{s=\lceil\frac{N+\nu}{4}\rceil}^{\lfloor\frac{3(N+\nu)}{4}\rfloor} G(t,s)F_1(s)F_2(u(s))$$

$$
\begin{aligned}
&\geq \lambda\gamma M \sum_{s=\lceil \frac{N+\nu}{4} \rceil}^{\lfloor \frac{3(N+\nu)}{4} \rfloor} G(s+\nu,s)F_1(s)|u(s)| \\
&\geq \lambda\gamma^2 M \sum_{s=\lceil \frac{N+\nu}{4} \rceil}^{\lfloor \frac{3(N+\nu)}{4} \rfloor} G(s+\nu,s)F_1(s)||u|| > ||u||,
\end{aligned}
$$

which implies that $||Tu|| \geq ||u||$, for $u \in \partial\Omega_{\rho_2} \cap K$. From Theorem 1(i), it follows that T has at least one fixed point in $K \cap (\bar{\Omega}_{\rho_2}/\Omega_{\rho_1})$. From Lemma 4, the fixed point $u \in K \cap (\bar{\Omega}_{\rho_2}/\Omega_{\rho_1})$ is the positive solution of (1).

The hypothesis $F_0 = \infty$ implies that, for $M > (\lambda\gamma^2 \sum_{s=\lceil \frac{N+\nu}{4} \rceil}^{\lfloor \frac{3(N+\nu)}{4} \rfloor} G(s+\nu,s)F_1(s))^{-1}$, there exists a positive $\rho_1 > 0$ such that $F_2(u) > M|u|$, whenever $0 < |u| < \rho_1$. Let $\Omega_{\rho_1} = \{u \in K : \|u\| < \rho_1\}$. For $u \in \partial\Omega_{\rho_1} \cap K$, we have

$$
\begin{aligned}
Tu(t) &\geq \lambda \sum_{s=\lceil \frac{N+\nu}{4} \rceil}^{\lfloor \frac{3(N+\nu)}{4} \rfloor} G(t,s)F_1(s)F_2(u(s)) \\
&\geq \lambda\gamma M \sum_{s=\lceil \frac{N+\nu}{4} \rceil}^{\lfloor \frac{3(N+\nu)}{4} \rfloor} G(s+\nu,s)F_1(s)|u(s)| \\
&\geq \lambda\gamma^2 M \sum_{s=\lceil \frac{N+\nu}{4} \rceil}^{\lfloor \frac{3(N+\nu)}{4} \rfloor} G(s+\nu,s)F_1(s)||u|| > ||u||,
\end{aligned}
$$

which implies that $||Tu|| \geq ||u||$, for $u \in \partial\Omega_{\rho_1} \cap K$.

$F_\infty = 0$ implies, for $\epsilon \in \left(2\lambda \sum_{s=0}^{N+1} G(s+\nu,s)F_1(s)\right)^{-1}$, there exists positive $N_0 > 0$ such that $F_2(u) < \epsilon|u|$, for $|u| > N_0$. Therefore, $F_2(u) \leq \epsilon|u| + \Lambda$, for $u \in [0,+\infty)$, where $\Lambda = \max_{0 \leq u \leq N_0} F_2(u) + 1$. Let $\Omega_{\rho_2} = \{u \in K : ||u|| < \rho_2\}$, where $\rho_2 > \{\rho_1, 2\Lambda\lambda \sum_{s=0}^{N+1} G(s+\nu,s)F_1(s)\}$, then, for $u \in \partial\Omega_{\rho_2} \cap K$, we have

$$
\begin{aligned}
|Tu(t)| &\leq \lambda\epsilon \sum_{s=0}^{N+1} G(s+\nu,s)F_1(s)|u(s)| + \lambda\Lambda \sum_{s=0}^{N+1} G(s+\nu,s)F_1(s) \\
&\leq \rho_2\lambda\epsilon \sum_{s=0}^{N+1} G(s+\nu,s)F_1(s) + \lambda\Lambda \sum_{s=0}^{N+1} G(s+\nu,s)F_1(s) \\
&\leq \frac{\rho_2}{2} + \frac{\rho_2}{2} = \rho_2,
\end{aligned}
$$

which implies $||Tu|| \leq ||u||$, for $u \in \partial\Omega_{\rho_2} \cap K$. It follows from Theorem 1 (ii) that T has at least one fixed point u in $K \cap (\bar{\Omega}_{\rho_2}/\Omega_{\rho_1})$. From Lemma 4, the fixed point $u \in K \cap (\bar{\Omega}_{\rho_2}/\Omega_{\rho_1})$ is the positive solution of (1). □

Theorem 4 *Assume $F_0 = 0$ or $F_\infty = 0$ and condition* (H$_3$) *holds. Then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ problem* (1) *has a positive solution.*

Proof Choose $\rho > 0$ and define $K_\rho = \{u \in K : ||u|| < \rho\}$, $\partial K_\rho = \{u \in K : ||u|| = \rho\}$. Then, the operator $T : K_\rho \to K$ is completely continuous. For fixed $\rho_1 > 0$, we define $\lambda_0 := \rho_1 \Big(\gamma m_{\rho_1} \sum_{s=\lceil \frac{N+\nu}{4} \rceil}^{\lfloor \frac{3(N+\nu)}{4} \rfloor} G(s+\nu, s) F_1(s)\Big)^{-1}$ and $\Omega_{\rho_1} := \{u \in K : ||u|| < \rho_1\}$, where $m_{\rho_1} := \min_{\gamma\rho_1 \le u \le \rho_1} F_2(u)$. By (H_3), $m_{\rho_1 > 0}$. Then for $u \in K \cap \partial\Omega_{\rho_1}$, by the Lemma 5, we have

$$
\begin{aligned}
\min_{t \in [\frac{N+\nu}{4}, \frac{3(N+\nu)}{4}]} Tu(t) &\ge \lambda\gamma \sum_{s=\lceil \frac{N+\nu}{4} \rceil}^{\lfloor \frac{3(N+\nu)}{4} \rfloor} G(s+\nu, s) F_1(s) F_2(u(s)) \\
&> \lambda_0 \gamma m_{\rho_1} \sum_{s=\lceil \frac{N+\nu}{4} \rceil}^{\lfloor \frac{3(N+\nu)}{4} \rfloor} G(s+\nu, s) F_1(s) = \rho_1 = ||u||,
\end{aligned}
$$

which implies that

$$
||Tu|| > ||u||, \quad \text{for } u \in K \cap \partial\Omega_{\rho_1}, \lambda > \lambda_0. \tag{11}
$$

Now, $F_0 = 0$ implies, for $\epsilon \in \Big(0, (\lambda \sum_{s=0}^{N+1} G(s+\nu, s) F_1(s))^{-1}\Big)$, there exists positive $\overline{\rho}_2 > 0$ such that $F_2(u) < \epsilon|u|$, whenever $0 < |u| < \overline{\rho}_2$. Let $\Omega_{\rho_2} = \{u \in K : ||u|| < \rho_2\}$, $0 < \rho_2 < \big\{\rho_1, \overline{\rho}_2\big\}$. For $u \in K \cap \partial\Omega_{\rho_2}$, we have

$$
|Tu(t)| \le \lambda\epsilon \sum_{s=0}^{N+1} G(s+\nu, s) F_1(s) |u(s)| \le \rho_2 \lambda\epsilon \sum_{s=0}^{N+1} G(s+\nu, s) F_1(s) < \rho_2,
$$

which imply that $||Tu|| \le \rho_2 = ||u||$, for $u \in K \cap \partial\Omega\rho_2$. $F_\infty = 0$ implies, for $\epsilon \in \Big(0,\ (2\lambda \sum_{s=0}^{N+1} G(s+\nu, s) F_1(s))^{-1}\Big)$, there exists positive $N_0 > 0$ such that $F_2(u) < \epsilon|u|$, for $|u| > N_0$. Thus, $F_2(u) \le \epsilon|u| + \Lambda$, for $u \in [0, +\infty)$, where $\Lambda = \max_{0 \le u \le N_0} F_2(u) + 1$. Let $\Omega_{\rho_3} = \{u \in K : ||u|| < \rho_3\}$, where $\rho_3 > \{\rho_1, 2\lambda\Lambda \sum_{s=0}^{N+1} G(s+\nu, s) F_1(s)\}$, then, for $u \in K \cap \partial\Omega_{\rho_3}$, it follows that

$$
\begin{aligned}
|Tu(t)| &\le \rho_3 \lambda\epsilon \sum_{s=0}^{N+1} G(s+\nu, s) F_1(s) + \lambda\Lambda \sum_{s=0}^{N+1} G(s+\nu, s) F_1(s) \\
&< \frac{\rho_3}{2} + \frac{\rho_3}{2} = \rho_3,
\end{aligned}
$$

which imply that $||Tu|| \le \rho_3 = ||u||$, for $u \in K \cap \partial\Omega_{\rho_3}$. It follows from Theorem 5 that T has a fixed point in $u \in K \cap \bar{\Omega}_{\rho_2}/\Omega\rho_1$ or $u \in K \cap \bar{\Omega}_{\rho_3}/\Omega\rho_1$ according to $F_0 = 0$ or $F_\infty = 0$, respectively. Consequently, problem (1) has a positive solution for $\lambda > \lambda_0$. □

Theorem 5 *Assume $F_0 = F_\infty = 0$, and* (H$_3$) *holds. Then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ problem* (1) *has two positive solutions.*

Proof Let $\rho_3, \rho_4 > 0$ such that $\rho_3 < \rho_4$. By the same argument as for inequality (11), there exists $\lambda_0 > 0$ such that

$$||Tu|| > ||u||, \qquad \text{for} \qquad u \in K \cap \partial\Omega_{\rho_i} \ (i = 3, 4), \ \lambda > \lambda_0.$$

Since $F_0 = F_\infty = 0$, then, it follows from the proof of Theorem 4 that we can choose $0 < \rho_1 < \rho_3$ and $\rho_2 > \rho_4$ such that $||Tu|| < ||u||$ for $u \in K \cap \partial\Omega_{\rho_i}(i = 1, 2)$. Hence, by Theorem 1 the operator T has two fixed points $u_1 \in K \cap \bar{\Omega}_{\rho_3}/\Omega\rho_1$ and $u_2 \in K \cap \bar{\Omega}_{\rho_2}/\Omega\rho_4$, which are the distinct positive solutions of problem (1). □

Theorem 6 *Assume that condition* (H$_3$) *holds. If $F_0 < \infty$ and $F_\infty < \infty$, then there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ problem* (1) *has no positive solution.*

Proof Since $F_0 < \infty$ and $F_\infty < \infty$, then for arbitrary $\epsilon_1, \epsilon_2 > 0$ there exist $0 < \rho_1 < \rho_2$ such that,

$$F_2(u) \leq \epsilon_1|u|, \text{ whenever } \quad |u| \leq \rho_1, \text{ and } F_2(u) \leq \epsilon_2|u|, \text{ whenever } |u| \geq \rho_2.$$

Let

$$\epsilon = \max\left\{\epsilon_1, \epsilon_2, \max\left\{\frac{F_2(u)}{u} : \rho_1 \leq u \leq \rho_2\right\}\right\}.$$

Then we have

$$F_2(u) \leq \epsilon|u|, \quad \text{whenever} \quad \rho_1 \leq |u| \leq \rho_2.$$

On contrary, assume that $w(t)$ is a positive solution of problem (1), then, for $0 < \lambda < \lambda_0$, where $\lambda_0 = (\sigma\epsilon \sum_{s=0}^{N+1} G(s+\nu, s)F_1(s))^{-1}$,for some $0 < \sigma < 1$. Then, we have

$$\begin{aligned}\|w\| = \|Tw\| &\leq \lambda\epsilon \sum_{s=0}^{N+1} G(s+\nu, s)F_1(s)\|w\| \\ &\leq \|w\|\lambda_0\epsilon \sum_{s=0}^{N+1} G(s+\nu, s)F_1(s) = \frac{\|w\|}{\sigma}.\end{aligned}$$

That is, $\|w\| \leq \frac{\|w\|}{\sigma} < \|w\|$, which is a contradiction. Hence, problem (1) has no positive solution. □

Proofs of the following theorems are similar to Theorems 4–6.

Theorem 7 *Assume that condition* (H$_3$) *holds and $F_0 = \infty$ or $F_\infty = \infty$. Then there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ problem* (1) *has a positive solution.*

Theorem 8 *Assume that condition* (H$_3$) *holds and $F_0 = F_\infty = \infty$. Then there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ problem* (1) *has two positive solutions.*

Theorem 9 *Assume that condition* (H$_3$) *holds and* $F_0 > 0$ *or* $F_\infty > 0$. *Then there exists* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ *problem* (1) *has no positive solution.*

Example 1 Consider the discrete boundary value problem

$$\begin{aligned} -\Delta_{\nu-2}^{\nu} u(t) = & \frac{1+(t+\nu-1)^2 e^{(-1/2)(t+\nu-1)}}{1+(t+\nu-1)^2} \\ & + \frac{(t+\nu-1)^2[1+\sin^2 u(t+\nu-1)]}{1+u^2(t+\nu-1)}, \ t \in [0, N+1]_{\mathbb{N}_0}, \\ u(\nu-2) = 0, \ & \Delta_{\nu-1}^{\nu-1} u(\nu+N) = 0, \end{aligned} \tag{12}$$

where $\nu = \frac{3}{2}$ and $N = 5$. Let $f(t, u(t)) := \frac{1+t^2 e^{-1/2t}}{1+t^2} + \frac{t^2[1+\sin^2 u(t)]}{1+u^2(t)}, \ t \in [\frac{1}{2}, \frac{13}{2}]_{\mathbb{N}_{\frac{1}{2}}}$. By computations, we have the following estimate:

$$\begin{aligned} f(t, u(t)) &= \frac{1+t^2 e^{-1/2t}}{1+t^2} + \frac{t^2[1+\sin^2 u(t)]}{1+u^2(t)} \\ &\le \frac{169 e^{-1/4} + 4}{5} + 84.5 \approx 111.623. \end{aligned}$$

Also, note that

$$f(t, u(t)) \ge \frac{1}{173}(e^{-13/4} + 4) \approx 0.0233455.$$

Now we compute the value of γ as

$$\gamma = \min\left\{ \frac{\left(\frac{13}{8}\right)^{(1/2)}}{\left(\frac{13}{2}\right)^{(1/2)}}, \ 1 - \frac{\left(\frac{31}{8}\right)^{(1/2)}}{\left(\frac{39}{8}\right)^{(1/2)}} \right\} = \min\{0.52912, 0.102564\} \approx 0.102564.$$

The Green's function for boundary value problem (12) is given by

$$G(t, s) = \begin{cases} \frac{2\left(t^{(1/2)} - (t-s-1)^{(1/2)}\right)}{\sqrt{\pi}}, & s \le t - 3/2 \le 6; \\ \frac{2t^{(1/2)}}{\sqrt{\pi}}, & t - 3/2 < s \le 6. \end{cases} \tag{13}$$

The value of η and μ are computed as:

$$\eta = 2\left(\sum_{s=0}^{N+1} G(s+\nu, s)\right)^{-1} \approx 0.20397.$$

Since

$$\varsigma = \sum_{s=\lceil \frac{\nu+N}{4}-\nu \rceil}^{\lfloor \frac{3(\nu+N)}{4}-\nu \rfloor} \gamma G\left(\left\lfloor \frac{N+1}{2} \right\rfloor + \nu, s\right) = \sum_{s=\lceil \frac{1}{8} \rceil}^{\lfloor \frac{27}{2} \rfloor} \gamma G\,(9/2, s) \approx 0.30849.$$

Then, $\mu = 1/\varsigma \approx 3.241557911$. Choose $\rho_1 = 1140$. Then we have $f(t, u) \leq 111.623 < \frac{1}{2}\eta\rho_1 = \frac{1}{2}(0.20397)(1095) \approx 116.263$. Now, choose $\rho_2 = 0.001$. Then $f(t, u(t)) \geq 0.0233455 > \mu\rho_2 = 0.00324156$.

All conditions of the Theorem 2 are satisfied. Therefore, boundary value problem (12) has a positive solution satisfying $0.001 \leq |u| \leq 1140$.

Example 2 Consider the following discrete boundary value problem

$$\begin{aligned} -\Delta_{\nu-2}^{\nu} u(t) &= \lambda e^{-(t+\nu-1)}\sqrt{(t+\nu-1)^2+1}\left[\frac{(u(t+\nu-1))^2}{\pi+(u(t+\nu-1))^3}\right], \quad t \in \mathbb{N}_0, \\ u(\nu-2) &= 0, \quad \Delta_{\nu-1}^{\nu-1} u(\nu+N) = 0, \end{aligned} \tag{14}$$

where $1 < \nu \leq 2$ and $N > 3$ is an integer. Let $\nu = 3/2$ and $N = 4$. Taking $f(t, u(t)) = \lambda e^{-t}\sqrt{t^2+1}[\frac{u^2(t)}{\pi+u^3(t)}]$, for $t \in [\nu-2, \nu+N]_{\mathbb{N}_{\nu-2}}$. Obviously $f(t, u(t)) = \lambda F_1(t)F_2(u(t))$, where $F_1(t) := \sqrt{t^2+1}e^{-t}$ and $F_2(u(t)) := \frac{u^2(t)}{\pi+u^3(t)}$ satisfy assumption (H_3). By computations $m_{\rho_1} = \min_{\gamma\rho_1 \leq u \leq \rho_1} F_2(u) = \min_{0.9696 \leq u \leq 8}(\frac{u^2(t)}{\pi+u^3(t)}) \approx 0.23195$, for $\rho_1 = 8$. Furthermore

$$\begin{aligned} \lambda_0 &= \rho_1 \left(\gamma m_{\rho_1} \sum_{s=\lceil \frac{N+\nu}{4} \rceil}^{\lfloor \frac{3(N+\nu)}{4} \rfloor} G(s+\nu, s)F_1(s)\right)^{-1} \\ &= 8\left((0.02811234)\sum_{s=2}^{4}\sqrt{s^2+1}e^{-s}G(s+\frac{3}{2}, s)\right)^{-1} \approx 396.186. \end{aligned}$$

Moreover, $F_0 = 0$ and $F_\infty = 0$. All assumptions of the Theorem 5 are satisfied. Consequently, the boundary value problem (14) has two positive solution.

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