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Vector-valued interval functions and the Dedekind completion of $C(X, E)$

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Abstract We consider interval valued functions with values in a Banach lattice E. Certain notions of continuity introduced earlier for real interval valued functions are generalised to the more general case considered here. As an application, we characterise the Dedekind completion of the space of continuous, E-valued functions on a paracompact T_1-space, extending a result of Anguelov.

Keywords Banach lattice valued functions · Dedekind completion · interval functions

Mathematics Subject Classification Primary 46E40 · 46E05 · 46B42 · 46A40 · 65G99; Secondary 54C60

1 Introduction

The field of interval analysis, more accurately the analysis of interval valued functions, was initiated in the 1960s by Moore [27]. While this field is traditionally associated with numerical analysis and validated computing [27,28], it has found a number of applications in other branches of mathematics. Indeed, methods of interval analysis are widely used in approximation theory [7,30], nonsmooth and nonlinear analysis [9,13], differential equations and inclusions [3,8,12,17,21], convex analysis [13,29], nonlinear partial differential equations [4], functional analysis [1,5] and optimization and optimal control theory [13,21]. Furthermore, interval analysis, and set-valued analysis in general, provide useful methods for the design and analysis of mathematical mod-

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els in the life sciences. Markov [23] argues that biological dynamic systems typically involve uncertain data and / or parameters, numerical and / or inherent sensitivity, and structural uncertainties which necessitate model validation. Problems related to these issues of uncertainty and sensitivity, including computing enclosures for sets of solutions [23] and estimation of parameter ranges [15], essentially belong to set-valued analysis in general and are often addressed within the setting of interval analysis. We refer the reader to [23] for more details.

A recent, nontrivial application of interval analysis to functional analysis concerns the characterisation of the Dedekind completion of the Riesz space $C(X)$ of all continuous, real valued functions on a completely regular topological space X, see [1,15]. The aim of this paper is to generalise this characterisation of the Dedekind completion of $C(X)$ by showing that, for a paracompact T_1-space X and a Banach lattice E with order continuous norm, the Dedekind completion of $C(X, E)$ is represented concretely as the set $\mathbb{H}(X, E)$ of Hausdorff continuous E-interval valued functions on X. In order to do so, some of the fundamental concepts and results of interval analysis are extended to the setting of Banach lattice valued functions. In particular, central notions of continuity of real interval valued functions are considered in the setting of functions with values closed order intervals in a Banach lattice.

The paper is organised as follows. In Sect. 2, we extend a result of Ercan and Wickstead [16] on semi-continuous functions with values in a Banach lattice, and obtain some new results which are used in subsequent sections. These results are mostly simple generalisations of well known results for real valued functions. Section 3 deals with the generalisation of the so called Upper and Lower Baire Operators to the setting of vector valued functions. The basic properties of these operators are discussed, in particular their relation to semi-continuity of vector valued functions. Vector valued interval functions are considered in Sect. 4. We pay special attention to the space of Hausdorff continuous functions, and generalise some of the important properties of these functions to the vector valued context. The space $\mathbb{H}(X, E)$ is shown to be a Dedekind complete Riesz space in Sect. 5. This is nontrivial, even in the scalar valued case, as $\mathbb{H}(X, E)$ is not closed under pointwise operations on functions (addition, finite infima and suprema). The main result of this paper, namely, a characterisation of the Dedekind completion of $C(X, E)$, is presented in Sect. 6. The relationship between Hausdorff continuous interval valued functions and minimal upper semi-continuous compact valued maps is discussed in Sect. 7.

Throughout the paper, X and Y denote (nonempty) topological spaces, with $\mathcal{V}_x$ denoting the set of open neighbourhoods of $x \in X$. A set-valued map Φ from a set A into another set B is a function from A into the powerset of B, and is denoted $\Phi : \rightrightarrows B$. E is a nontrivial (real) Banach lattice. For $u \in E$ and $r > 0$, the ball centred at u with radius r is denoted by $B(u, r)$. The closure of a set A is denoted by $\overline{A}$. The reader is referred to [22,25] for terminology and notation relating to Riesz spaces and Banach lattices in particular.

2 Semi-continuous functions

Ercan and Wickstead [16] generalised the concepts of real valued lower and upper semi-continuous functions to functions taking values in a Banach lattice. The aim of this section is to generalise certain results from [16] concerning such vector valued semi-continuous functions. In particular, [16, Theorem 5.7] and some of its consequences remain valid in a more general setting than that considered by Ercan and Wickstead. Further results on vector valued semi-continuous functions that are required for later use are also included.

Let $\mathcal{A}(X, E)$ denote the set of all E-valued functions on X.

Definition 2.1 A function $\varphi \in \mathcal{A}(X, E)$ is *upper semi-continuous* if $\varphi^{-1}(U - E^+)$ is open in X whenever U is open in E.

Definition 2.2 A function $\varphi \in \mathcal{A}(X, E)$ is *lower semi-continuous* if $\varphi^{-1}(U + E^+)$ is open in X whenever U is open in E.

Denote by $\mathcal{U}(X, E)$ and $\mathcal{L}(X, E)$ the sets of all upper semi-continuous, respectively lower semi-continuous, functions in $\mathcal{A}(X, E)$. The sets $\mathcal{U}(X, E)$ and $\mathcal{L}(X, E)$ are closed under (pointwise) addition and multiplication by positive scalars; that is, for all $\varphi, \psi \in \mathcal{A}(X, E)$ and every real number $\alpha \geq 0$,

$$\varphi, \psi \in \mathcal{U}(X, E) \Rightarrow \varphi + \psi, \alpha\varphi \in \mathcal{U}(X, E) \tag{2.1}$$

and

$$\varphi, \psi \in \mathcal{L}(X, E) \Rightarrow \varphi + \psi, \alpha\varphi \in \mathcal{L}(X, E). \tag{2.2}$$

On the other hand, if $\alpha < 0$, then

$$\varphi \in \mathcal{U}(X, E) \Rightarrow \alpha\varphi \in \mathcal{L}(X, E), \;\; \psi \in \mathcal{L}(X, E) \Rightarrow \alpha\psi \in \mathcal{U}(X, E). \tag{2.3}$$

The mentioned results from [16] are based on a selection theorem of Asimow and Atkinson [6], which in turn is an application of Michael's classical Selection Theorem [26]. In order to generalise the relevant results from [16] it is sufficient to prove a correspondingly more general version of the selection theorem [6, Corollary 1.3], see also [16, Theorem 5.6].

Theorem 2.3 *Let X be a paracompact T_1-space, F a Banach space and $\Phi\colon X \rightrightarrows F$ a set-valued map with closed, nonempty and convex values. Suppose that for every $\epsilon > 0$ there exists a lower semi-continuous map $\Phi_\epsilon\colon X \rightrightarrows F$ with closed, convex and nonempty values so that $\Phi(x) \subseteq \Phi_\epsilon(x) \subseteq \Phi(x) + B(0, \epsilon)$ for all $x \in X$. Then there exists a continuous function $\varphi\colon X \to F$ so that $\varphi(x) \in \Phi(x)$ for every $x \in X$.*

Proof Applying Michael's Selection Theorem [26, Theorem 3.2''] to $\Phi_{2^{-1}}$, there exists a continuous function $\varphi_1\colon X \to F$ so that $\varphi_1(x) \in \Phi_{2^{-1}}(x) \subseteq \Phi(x) + B(0, 2^{-1})$ for every $x \in X$. For each natural number $n \geq 1$, let

$$\Phi'_{2^{-(n+1)}}(x) = \overline{B(\varphi_n(x), 2^{-n}) \cap \Phi_{2^{-(n+1)}}(x)}, \;\; x \in X.$$

Then $\emptyset \neq \Phi'_{2^{-(n+1)}}(x) \subseteq \Phi(x) + \overline{B(0, 2^{-(n+1)})}$ is closed and convex for every $x \in X$. By [26, Propositions 2.3 & 2.5], $\Phi'_{2^{-(n+1)}}$ is lower semi-continuous. Therefore we may apply Michael's Selection Theorem to $\Phi'_{2^{-(n+1)}}$ to find a continuous function $\varphi_{n+1}\colon X \to F$ so that $\varphi_{n+1}(x) \in \Phi'_{2^{-(n+1)}}(x) \subseteq \Phi(x) + \overline{B(0, 2^{-(n+1)})}$ for every $x \in X$. Since $\Phi'_{2^{-(n+1)}}(x) \subseteq \overline{B(\varphi_n(x), 2^{-n})}$ for every $x \in X$ and $n \in \mathbb{N}$, it follows that $\|\varphi_{n+1}(x) - \varphi_n(x)\| < 2^{-n}$ for every $x \in X$ and $n \in \mathbb{N}$. Therefore the sequence (φ_n) is uniformly Cauchy so that (φ_n) converges uniformly to a continuous function $\varphi\colon X \to F$. Since $\Phi(x)$ is closed in F, and $\varphi_n(x) \in \Phi(x) + \overline{B(0, 2^{-n})}$ for every $x \in X$ and $n \in \mathbb{N}$, it follows that $\varphi(x) \in \Phi(x)$ for every $x \in X$. □

The following results now follow in exactly the same way as the corresponding results in [16].

Corollary 2.4 *Let X be a paracompact T_1-space and E a Banach lattice. If $\varphi_0 \in \mathcal{U}(X, E)$ and $\varphi_1 \in \mathcal{L}(X, E)$ and $\varphi_0 \leq \varphi_1$, then there exists a continuous function $\psi\colon X \to E$ so that $\varphi_0 \leq \psi \leq \varphi_1$.*

Corollary 2.5 *Let X be a paracompact T_1-space, E a Banach lattice with order continuous norm and $\varphi \in \mathcal{A}(X, E)$. Then the following statements are equivalent.*

(i) *φ is lower semi-continuous.*
(ii) *φ is the pointwise supremum of some family of continuous functions.*
(iii) *φ is the pointwise supremum of some upward directed family of continuous functions.*

Corollary 2.6 *Let X be a paracompact T_1-space, E a Banach lattice with order continuous norm and $\varphi \in \mathcal{A}(X, E)$. Then the following statements are equivalent.*

(i) *φ is upper semi-continuous.*
(ii) *φ is the pointwise infimum of some family of continuous functions.*
(iii) *φ is the pointwise infimum of some downward directed family of continuous functions.*

We next consider additional results on semi-continuous functions. These are all generalisations of well known properties of real valued semi-continuous functions.

Proposition 2.7 *Consider functions $\varphi, \psi \in \mathcal{A}(X, E)$. The following statements are true.*

(i) *If φ and ψ are lower semi-continuous, then so are the functions $\varphi \vee \psi\colon X \ni x \mapsto \varphi(x) \vee \psi(x) \in E$ and $\varphi \wedge \psi\colon X \ni x \mapsto \varphi(x) \wedge \psi(x) \in E$.*
(ii) *If φ and ψ are upper semi-continuous, then so are the functions $\varphi \wedge \psi\colon X \ni x \mapsto \varphi(x) \wedge \psi(x) \in E$ and $\varphi \vee \psi\colon X \ni x \mapsto \varphi(x) \vee \psi(x) \in E$.*

Proof Assume that φ and ψ are lower semi-continuous, and let $U \subseteq E$ be open. If $x \in [\varphi \vee \psi]^{-1}(U + E^+)$ then $\varphi(x) \vee \psi(x) \in U + E^+$. Due to the joint continuity of the lattice operations on E there exist open subsets U_1 and U_2 of E so that $\varphi(x) \in U_1 \subseteq U_1 + E^+$, $\psi(x) \in U_2 \subseteq U_2 + E^+$ and $\{v \vee w\colon v \in U_1, w \in U_2\} \subseteq U + E^+$. In fact, we have

$$\{v \vee w\colon\ v \in U_1 + E^+, w \in U_2 + E^+\} \subseteq U + E^+. \tag{2.4}$$

Since φ and ψ are lower semi-continuous there exists $V \in \mathcal{V}_x$ so that $V \subseteq \varphi^{-1}(U_1 + E^+) \cap \psi^{-1}(U_2 + E^+)$. It now follows from (2.4) that $V \subseteq [\varphi \vee \psi]^{-1}(U + E^+)$ so that $\varphi \vee \psi$ is lower semi-continuous.

Now suppose that $x \in [\varphi \wedge \psi]^{-1}(U + E^+)$ so that $\varphi(x) \wedge \psi(x) \in U + E^+$. It follows from the joint continuity of the lattice operations on E there exist open subsets U_1 and U_2 of E so that $\varphi(x) \in U_1 \subseteq U_1 + E^+$, $\psi(x) \in U_2 \subseteq U_2 + E^+$ and $\{v \wedge w\colon\ v \in U_1, w \in U_2\} \subseteq U + E^+$. In fact, we have

$$\{v \wedge w\colon v \in U_1 + E^+, w \in U_2 + E^+\} \subseteq U + E^+. \tag{2.5}$$

Since φ and ψ are lower semi-continuous there exists $V \in \mathcal{V}_x$ so that $V \subseteq \varphi^{-1}(U_1 + E^+) \cap \psi^{-1}(U_2 + E^+)$. It now follows from (2.5) that $V \subseteq [\varphi \wedge \psi]^{-1}(U + E^+)$ so that $\varphi \wedge \psi$ is lower semi-continuous.

The statement in (ii) follows from (i), (2.2) and [22, Theorem 11.5]. □

Proposition 2.8 *Assume that E has order continuous norm, and that $\mathcal{F} \subseteq \mathcal{A}(X, E)$ is order bounded in $\mathcal{A}(X, E)$. Then the following statements are true.*

(i) *If $\mathcal{F} \subset \mathcal{L}(X, E)$, then the function $\psi\colon X \ni x \mapsto \sup\{\varphi(x)\colon \varphi \in \mathcal{F}\} \in E$ is lower semi-continuous.*
(ii) *If $\mathcal{F} \subset \mathcal{U}(X, E)$, then the function $\psi\colon X \ni x \mapsto \inf\{\varphi(x)\colon \varphi \in \mathcal{F}\} \in E$ is upper semi-continuous.*

Proof Assume that $\mathcal{F} \subseteq \mathcal{L}(X, E)$. In view of Proposition 2.4, we may assume that $\mathcal{F} = \{\varphi_\lambda\colon \lambda \in \Lambda\}$ is upward directed. Let $U \subseteq E$ be open, and suppose $x_0 \in \psi^{-1}(U + E^+)$ so that $\psi(x_0) \in U + E^+$. Since the norm on E is order continuous, and $\varphi_\lambda(x_0) \uparrow \psi(x_0)$, it follows that $(\varphi_\lambda(x_0))$ converges in norm to $\psi(x_0)$. But $U + E^+$ is open, so there exists $\lambda_0 \in \Lambda$ so that $\varphi_{\lambda_0}(x_0) \in U + E^+$, hence $x_0 \in \varphi^{-1}(U + E^+)$. Since φ_{λ_0} is lower semi-continuous there exists $V \in \mathcal{V}_{x_0}$ so that $V \subseteq \varphi_{\lambda_0}^{-1}(U + E^+)$. As $\varphi_{\lambda_0} \leq \psi$ it follows that $\varphi_{\lambda_0}^{-1}(U+E^+) \subseteq \psi^{-1}(U+E^+)$ so that $V \subseteq \psi^{-1}(U+E^+)$. Hence $\psi^{-1}(U + E^+)$ is open so that ψ is lower semi-continuous.

That (ii) is true follows from (i), (2.2) and [22, Theorem 13.1 (i)]. □

Proposition 2.9 *If $\varphi \in \mathcal{L}(X, E)$, $\varphi(X) \subseteq E^+$ and $\varphi^{-1}(0)$ is dense in X, then $\varphi = 0$.*

Proof Let $U = E \setminus (-E^+)$. Since U is open and $U = U + E^+$, it follows from the lower semi-continuity of φ that $\varphi^{-1}(U)$ is open in X. But $\varphi(x) \in E^+$ for every $x \in X$ so that $\varphi^{-1}(U) = \{x \in X\colon \varphi(x) > 0\} = X \setminus \varphi^{-1}(0)$. Since $\varphi^{-1}(0)$ is dense in X, it follows that $\{x \in X\colon \varphi(x) > 0\} = \emptyset$, hence $\varphi = 0$. □

Corollary 2.10 *If $\varphi \in \mathcal{U}(X, E)$, $\varphi(X) \subseteq -E^+$ and $\varphi^{-1}(0)$ is dense in X, then $\varphi = 0$.*

Proof The result follows immediately from (2.3) and Proposition 2.9. □

3 Baire operators

For locally bounded, real valued functions, semi-continuity is characterised as

$$\varphi\colon X \to \mathbb{R} \text{ is lower semi-continuous} \Leftrightarrow I[\varphi] = \varphi$$

and

$$\varphi\colon X \to \mathbb{R} \text{ is upper semi-continuous} \Leftrightarrow S[\varphi] = \varphi,$$

where $I, S\colon \mathcal{A}(X,\mathbb{R}) \to \mathcal{A}(X,\mathbb{R})$ are the Lower and Upper Baire Operators, respectively, defined by

$$I[\varphi](x) = \liminf_{y\to x} \varphi(y),\ x \in X \tag{3.1}$$

and

$$S[\varphi](x) = \limsup_{y\to x} \varphi(y),\ x \in X, \tag{3.2}$$

see for instance [1,10,32].

For functions with values in a Banach lattice E, this characterisation is not available, as even a continuous function $\varphi\colon X \to E$ may fail to be locally order bounded. However, the Baire operators (3.1) and (3.2) may be expressed in a way which is amenable to a generalisation to Banach lattice valued functions.

Proposition 3.1 *Consider a function $\varphi \in \mathcal{A}(X,\mathbb{R})$. If there exist $\psi_0 \in \mathcal{L}(X,\mathbb{R})$ and $\psi_1 \in \mathcal{U}(X,\mathbb{R})$ so that $\psi_0 \leq \varphi \leq \psi_1$, then*

$$I[\varphi](x) = \ell[\varphi](x) := \sup\{\psi(x)\colon \psi \in \mathcal{L}(X,\mathbb{R}),\ \psi \leq \varphi\},\ x \in X$$

and

$$S[\varphi](x) = u[\varphi](x) := \inf\{\psi(x)\colon \psi \in \mathcal{U}(X,\mathbb{R}),\ \varphi \leq \psi\},\ x \in X.$$

Proof Clearly, $\ell[\varphi] \leq \varphi$ so that $I[\ell[\varphi]] \leq I[\varphi]$. But the pointwise supremum of a family of lower semi-continuous functions is again lower semi-continuous, hence $\ell[\varphi] = I[\ell[\varphi]] \leq I[\varphi]$. On the other hand, $I[\varphi]$ is lower semi-continuous, and $I[\varphi] \leq \varphi$. Hence $I[\varphi] \leq \ell[\varphi]$ so that $I[\varphi] = \ell[\varphi]$. The second identity follows in exactly the same way. □

Assume that E is Dedekind complete, and let

$$\mathcal{A}_0(X,E) = \left\{\varphi\colon X \to E \,\middle|\, \begin{array}{l} \exists\ \psi_0 \in \mathcal{L}(X,E),\ \psi_1 \in \mathcal{U}(X,E)\colon \\ \psi_0 \leq \varphi \leq \psi_1 \end{array}\right\}.$$

We define the Upper and Lower Baire Operators $S\colon \mathcal{A}_0(X,E) \to \mathcal{A}_0(X,E)$ and $I\colon \mathcal{A}_0(X,E) \to \mathcal{A}_0(X,E)$ by

$$S[\varphi](x) = \inf\{\psi(x)\colon \psi \in \mathcal{U}(X,E),\ \psi \geq \varphi\},\quad x \in X \tag{3.3}$$

and

$$I[\varphi](x) = \sup\{\psi(x)\colon \psi \in \mathcal{L}(X,E),\ \psi \leq \varphi\},\quad x \in X \tag{3.4}$$

respectively. It is clear that

$$I[\varphi] \leq \varphi \leq S[\varphi] \tag{3.5}$$

and

$$I \circ I[\varphi] = I[\varphi], \; S \circ S[\varphi] = S[\varphi] \tag{3.6}$$

for each $\varphi \in \mathcal{A}_0(X, E)$. Furthermore,

$$I[\varphi] \leq I[\psi], \quad S[\varphi] \leq S[\psi] \tag{3.7}$$

whenever $\varphi \leq \psi$. Combining (3.5) and (3.6) in a judicious way, we see that

$$I \circ S[I \circ S[\varphi]] = I \circ S[\varphi], \quad S \circ I[S \circ I[\varphi]] = S \circ I[\varphi] \tag{3.8}$$

for all $\varphi \in \mathcal{A}_0(X, E)$. It follows from (2.1), (2.2) and (2.3) that

$$I[\varphi + \psi] \geq I[\varphi] + I[\psi], \quad S[\varphi + \psi] \leq S[\varphi] + S[\psi] \tag{3.9}$$

and

$$I[-\varphi] = -S[\varphi], \quad S[-\varphi] = -I[\varphi] \tag{3.10}$$

for all $\varphi, \psi \in \mathcal{A}_o(X, E)$.

Clearly $I[\varphi] = \varphi$ whenever φ is lower semi-continuous, while $S[\varphi] = \varphi$ whenever φ is upper semi-continuous. The converse holds whenever E has order continuous norm.

Proposition 3.2 *Suppose that E has order continuous norm. The following statements are true for every $\varphi \in \mathcal{A}_0(X, E)$.*

(i) *If $I[\varphi] = \varphi$, then φ is lower semi-continuous.*
(ii) *If $S[\varphi] = \varphi$, then φ is upper semi-continuous.*

Proof According to Proposition 2.8, $I[\varphi]$ is lower semi-continuous and $S[\varphi]$ is upper semi-continuous for every $\varphi \in \mathcal{A}_0(X, E)$. The result follows immediately. □

Remark 3.3 Kaplan [19,20] introduced the closely related operators ℓ and u on the bidual $C(X)''$ of $C(X)$, with X a compact Hausdorff space. In particular, for $\varphi \in C(X)''$,

$$\ell(\varphi) = \sup\{\psi \leq \varphi \colon \psi \text{ is lower semi-continuous}\}$$

and

$$u(\varphi) = \inf\{\psi \geq \varphi \colon \psi \text{ is upper semi-continuous}\},$$

the supremum and infimum being formed with respect to the the natural order on $C(X)''$.

4 Interval functions

Let IE denote the set of all non-empty order intervals in E. That is,

$$IE = \{[\underline{u}, \overline{u}] \colon \underline{u}, \overline{u} \in E,\ \underline{u} \leq \overline{u}\}$$

where $[\underline{u}, \overline{u}] = \{w \in E \colon \underline{u} \leq w \leq \overline{u}\}$. Two intervals $u = [\underline{u}, \overline{u}]$ and $v = [\underline{v}, \overline{v}]$ in IE are equal if they are equal as sets. This is equivalent to

$$\underline{u} = \underline{v}, \quad \overline{u} = \overline{v}.$$

We define a partial order on IE by setting

$$u \leq v \Leftrightarrow \underline{u} \leq \underline{v}, \quad \overline{u} \leq \overline{v}. \tag{4.1}$$

It is easy to see that (4.1) does indeed define a partial order on IE. In fact, IE is a lattice with respect to the order (4.1).

Associating each $u \in E$ with the degenerate interval $[u, u] \in IE$, we may consider E as a subset of IE. Furthermore, the order on IE extends that of E in the sense that, for all $u, v \in E$,

$$u \leq v \text{ in } E \Leftrightarrow u \leq v \text{ in } IE.$$

Denote by $\mathbb{A}(X, E)$ the set of all functions from X into IE. Note that, through the identification of points in E with degenerate intervals in IE, we may view $\mathcal{A}(X, E)$ in a natural way as a subset of $\mathbb{A}(X, E)$. Conversely, we may associate with each $f \in \mathbb{A}(X, E)$ two functions $\underline{f}, \overline{f} \in \mathcal{A}(X, E)$ by setting

$$\underline{f}(x) = \inf f(x), \quad \overline{f}(x) = \sup f(x),\ x \in X.$$

By abuse of notation, we write $f = [\underline{f}, \overline{f}]$. The partial order (4.1) on IE induces an order relation on $\mathbb{A}(X, E)$, namely, the pointwise order

$$f \leq g \Leftrightarrow f(x) \leq g(x) \Leftrightarrow \underline{f}(x) \leq \underline{g}(x) \quad \text{and} \quad \overline{f}(x) \leq \overline{g}(x), \quad x \in X. \tag{4.2}$$

Clearly this ordering of $\mathbb{A}(X, E)$ extends the pointwise order on $\mathcal{A}(X, E)$. The inclusion relation on $\mathbb{A}(X, E)$ is likewise defined in a pointwise way. In particular, for $f, g \in \mathbb{A}(X, E)$,

$$f \subseteq g \Leftrightarrow f(x) \subseteq g(x), \quad x \in X.$$

Generalising the corresponding notions for real interval-valued functions, we introduce the following concepts of continuity for functions in $\mathbb{A}(X, E)$.

Definition 4.1 A function $f \colon X \ni\mapsto [\underline{f}(x), \overline{f}(x)] \in IE$ is called Sendov continuous (S-continuous) if $\underline{f}$ is lower semi-continuous and $\overline{f}$ is upper semi-continuous.

Definition 4.2 An S-continuous function $f : X \to IE$ is called Hausdorff continuous (H-continuous) if $f = g$ for every S-continuous function g such that $g \subseteq f$.

We denote by $\mathbb{H}(X, E)$ the set of H-continuous interval functions from X into E. As we show next, H-continuous functions are abundant.

Proposition 4.3 *If E has order continuous norm and $f \in \mathbb{A}(X, E)$ is S-continuous, then there exists an H-continuous function $g \in \mathbb{A}(X, E)$ so that $g \subseteq f$.*

Proof Noting that Definition 4.2 is nothing but a minimality condition on the set $\{g \subseteq f : g \text{ is S} - \text{continuous}\}$, it is clear that the result follows immediately from Proposition 2.8 and Zorn's Lemma. □

Real-valued H-continuous functions are characterised through the Baire operators [1, Theorem 1]. For E-valued functions the situation is analogous, provided that E has order continuous norm.

Proposition 4.4 *Consider the following statements, for some function $f = [\underline{f}, \overline{f}] \in \mathbb{A}(X, E)$.*

(i) *f is H-continuous.*
(ii) *$I[\overline{f}] = \underline{f}$ and $S[\underline{f}] = \overline{f}$.*

If f is S-continuous, then (ii) implies (i). If E has order continuous norm, then (i) and (ii) are equivalent.

Proof Assume that f is S-continuous and satisfies (ii). Suppose that $g \subseteq f$ is S-continuous so that $\underline{f} \leq \underline{g} \leq \overline{g} \leq \overline{f}$. Then it follows from (3.7) that

$$\underline{f} = I[\underline{f}] \leq I[\underline{g}] \leq I[\overline{f}] = \underline{f} \text{ and } \overline{f} = S[\underline{f}] \leq S[\overline{g}] \leq S[\overline{f}] = \overline{f}$$

so that $\underline{g} = I(\underline{g}) = \underline{f}$ and $\overline{g} = S(\overline{g}) = \overline{f}$. Therefore $f = g$ so that f is H-continuous.

Now assume that E has order continuous norm. Suppose that f is H-continuous. It follows from (3.5) and (3.7) that $\underline{f} \leq S[\underline{f}] \leq S[\overline{f}] = \overline{f}$. Therefore $g = [\underline{f}, S[\underline{f}]] \subseteq f$. According to Proposition 3.2, $S[\underline{f}]$ is upper semi-continuous so that g is S-continuous. Therefore $f = g$ so that $S[\underline{f}] = \overline{f}$. It follows in the same way that $I[\overline{f}] = \underline{f}$. Conversely, suppose that $I[\overline{f}] = \underline{f}$ and $S[\underline{f}] = \overline{f}$. It follows from (3.6) that

$$S[\overline{f}] = S \circ S[\underline{f}] = S[\underline{f}] = \overline{f}, \; I[\underline{f}] = I \circ I[\overline{f}] = I[\overline{f}] = \underline{f}.$$

Therefore, by Proposition 3.2, $\underline{f}$ is lower semi-continuous and $\overline{f}$ is upper semi-continuous so that f is S-continuous. Since f is assumed to satisfy (ii), it follows that f is H-continuous.

As an application of Proposition 4.4, we obtain the following, see [1, Theorem 2] for the corresponding result for real valued functions.

Corollary 4.5 *Assume that $\varphi \in \mathcal{A}_0(X, E)$ and that E has order continuous norm. Then $f = [I \circ S[\varphi], S \circ I \circ S[\varphi]]$ and $g = [I \circ S \circ I[\varphi], S \circ I[\varphi]]$ are H-continuous, and $g \leq f$.*

Proof We have $S[\underline{f}] = S \circ I \circ S[\varphi] = \overline{f}$ and, by (3.8), $I[\overline{f}] = I \circ S \circ I \circ S[\varphi] = I \circ S[\varphi] = \underline{f}$. Therefore f is H-continuous by Proposition 4.4. In the same way it follows that g is H-continuous. The inequalities (3.5) and (3.7) imply that $g \leq f$. □

As we will see in Sect. 5, S-continuous functions containing a unique H-continuous functions play an important role in defining the algebraic operations on $\mathbb{H}(X, E)$, see also [32]. We therefore end this section with two characterisations of such functions.

Proposition 4.6 *Assume that E has order continuous norm and that $f \in \mathbb{A}(X, E)$ is S-continuous. Then the following statements are equivalent.*

(i) *f contains a unique H-continuous function.*
(ii) *$I \circ S[\underline{f}] = I[\overline{f}]$ and $S \circ I[\overline{f}] = S[\underline{f}]$.*

Proof By Corollary 4.5, $g = [I \circ S[\underline{f}], S[\underline{f}]]$ and $h = [I[\overline{f}], S \circ I[\overline{f}]]$ are H-continuous. Furthermore, (3.5) and the S-continuity of f imply that $g, h \subseteq f$. Hence (i) implies (ii).

Assume that (ii) holds, and $p \subseteq f$ is H-continuous. Then by (3.7) and Proposition 4.4, $I \circ S[\underline{f}] \leq \underline{p}$ and $S[\overline{f}] = S \circ I[\underline{f}] \geq \overline{p}$. This implies that $p \subseteq g = [I \circ S[\underline{f}], S[\underline{f}]]$. Since g is H-continuous, it follows that $p = g$ so that f contains at most one H-continuous function. The result now follows by Proposition 4.3. □

An application of Proposition 4.6 yields the following.

Proposition 4.7 *Assume that E has an order continuous norm. An S-continuous function $f = [\underline{f}, \overline{f}] \in \mathbb{A}(X, E)$ contains a unique H-continuous function if and and only if $I[\overline{f} - \underline{f}] = 0$.*

Proof Assume that $I[\overline{f} - \underline{f}] = 0$. According to (3.9) and (3.10), $0 \geq I[\overline{f}] - S[\underline{f}]$ so that $I \circ S[\underline{f}] \geq I[\overline{f}]$ by (3.6) and (3.7). But (3.7) and the upper semi-continuity of $\overline{f}$ imply that $I \circ S[\underline{f}] \leq I \circ S[\overline{f}] = I[\overline{f}]$. Hence $I \circ S[\underline{f}] = I[\overline{f}]$. In the same way, $S \circ I[\overline{f}] = S[\underline{f}]$ so that f contains a unique H-continuous function by Proposition 4.6.

Conversely, suppose that f contains a unique H-continuous function. Assume, for the sake of obtaining a contradiction, that $I[\overline{f} - \underline{f}] > 0$. Then there exists a lower semi-continuous function $\varphi\colon X \to E$ so that $\varphi(x_0) > 0$ for some $x_0 \in X$, and $\varphi \leq \overline{f} - \underline{f}$. According to Proposition 2.7 (i), we may assume that $\varphi(x) \geq 0$ for all $x \in X$. Since f contains a unique H-continuous function, it follows from Proposition 4.6 that $S \circ I[\overline{f}] = S[\underline{f}]$. But, since $\underline{f} \leq \underline{f} + \varphi \leq \overline{f}$, it follows from (3.7) and the lower semi-continuity of $\underline{f}$ and φ that $S \circ I[\overline{f}] \geq S[\underline{f} + \varphi] \geq S[\underline{f}]$. Hence $S[\underline{f} + \varphi] = S[\underline{f}]$. It therefore follows from (3.3) and (3.5) that, for an upper semi-continuous function

$\psi : X \to E$, $\psi \geq \underline{f}$ if and only if $\psi \geq \underline{f} + \varphi$. Hence

$$\begin{aligned} S[\underline{f}](x_0) &= \inf\{\psi(x_0) : \psi \in \mathcal{U}(X, E),\ \psi \geq \underline{f}\} \\ &\leq \inf\{\psi(x_0) - \varphi(x_0) : \psi \in \mathcal{U}(X, E),\ \psi \geq \underline{f}\} \\ &= \inf\{\psi(x_0) : \psi \in \mathcal{U}(X, E),\ \psi \geq \underline{f}\} - \varphi(x_0) \\ &= S[\underline{f}](x_0) - \varphi(x_0). \end{aligned}$$

But this contradicts the fact that $\varphi(x_0) > 0$. Hence $I[\overline{f} - \underline{f}] = 0$. □

5 The Riesz space $\mathbb{H}(X, E)$

Throughout this section, we assume that E has an order continuous norm. For real intervals $\alpha = [\underline{\alpha}, \overline{\alpha}]$ and $\beta = [\underline{\beta}, \overline{\beta}]$, the sum $\alpha + \beta$ is defined by

$$\alpha + \beta = [\underline{\alpha} + \underline{\beta}, \overline{\alpha} + \overline{\beta}],$$

while for a real number γ, the product $\gamma\alpha$ is defined as

$$\gamma\alpha = \begin{cases} [\gamma\underline{\alpha}, \gamma\overline{\alpha}] & \text{if } \gamma \geq 0 \\ \left[\gamma\overline{\alpha}, \gamma\underline{\alpha}\right] & \text{if } \gamma < 0 \end{cases}$$

This is standard practice in interval analysis, see for instance [3,27,28], and we adopt the same definitions for intervals in IE.

For functions $f, g \in \mathbb{A}(X, E)$ and $\gamma \in \mathbb{R}$, we set

$$[f \oplus g](x) = f(x) + g(x) = [\underline{f}(x) + \underline{g}(x), \overline{f}(x) + \overline{g}(x)] \tag{5.1}$$

and

$$[\gamma \odot f](x) = \gamma f(x) = \begin{cases} [\gamma\underline{f}(x), \gamma\overline{f}(x)] & \text{if } \gamma \geq 0 \\ \left[\gamma\overline{f}(x), \gamma\underline{f}(x)\right] & \text{if } \gamma < 0 \end{cases} \tag{5.2}$$

for every $x \in X$. In view of (2.1), (2.2) and (2.3), $f \oplus g$ and γf are S-continuous whenever f and g are S-continuous, and $\gamma \in \mathbb{R}$. Furthermore, it is easily seen that $\mathbb{H}(X, E)$ is closed under pointwise scalar multiplication (5.2). However, even in the scalar valued case, the set of H-continuous functions on X is in general not closed under pointwise addition (5.1). This can be seen at the hand of elementary examples, see for instance [32,33]. The key to overcoming this difficulty lies in the following.

Proposition 5.1 *Assume that E has order continuous norm. If $f, g \in \mathbb{A}(X, E)$ are S-continuous and each contains a unique H-continuous function, then $f \oplus g$ contains a unique H-continuous function.*

Proof Let $h = f \oplus g$, so that $\underline{h} = \underline{f} + \underline{g}$ and $\overline{h} = \overline{f} + \overline{g}$. Assume that $\varphi : X \to E$ is lower semi-continuous, and $\varphi \leq \overline{h} - \underline{h}$. Then $\varphi + \underline{f} - \overline{f} \leq \overline{g} - \underline{g}$ so that

$$\varphi + \underline{f} - \overline{f} = I[\varphi + \underline{f} - \overline{f}] \leq I[\overline{g} - \underline{g}] = 0$$

by (2.1), (2.3), (3.7) and Proposition 4.7. Hence

$$\varphi = I[\varphi] \leq I[\overline{f} - \underline{f}] = 0,$$

again by (3.7) and Proposition 4.7. Therefore $I[\overline{h} - \underline{h}] = 0$ so that h contains a unique H-continuous function. □

In view of Proposition 5.1, we define addition in $\mathbb{H}(X, E)$ as follows. For $f, g \in \mathbb{H}(X, E)$, set

$$f + g := \text{the unique H-continuous function contained in } f \oplus g. \tag{5.3}$$

Scalar multiplication is defined by

$$\gamma f := \gamma \odot f \tag{5.4}$$

for all $f \in \mathbb{H}(X, E)$ and $\gamma \in \mathbb{R}$.

Theorem 5.2 *$\mathbb{H}(X, E)$ is a vector space over $\mathbb{R}$, with addition and scalar multiplication defined in (5.3) and (5.4).*

Proof We verify only the associative law for addition, the other axioms of a vector space following in the same way. Consider $f, g, h \in \mathbb{H}(X, E)$. It is clear that pointwise addition (5.1) of functions in $\mathbb{A}(X, E)$ satisfies the associative law, so that

$$f \oplus (g \oplus h) = (f \oplus g) \oplus h = f \oplus g \oplus h.$$

By (5.3), $g + h \subseteq g \oplus h$ and $f + (g + h) \subseteq f \oplus (g + h)$ so that $f + (g + h) \subseteq f \oplus (g \oplus h) = f \oplus g \oplus h$. In the same way, $(f + g) + h \subseteq (f \oplus g) \oplus h = f \oplus g \oplus h$. By Proposition 5.1, $f \oplus g \oplus h$ contains a unique H-continuous function, so that $f + (g + h) = (f + g) + h$. □

We now show that, with respect to the order (4.2), $\mathbb{H}(X, E)$ is a Dedekind complete Riesz space. The proof of this fact relies on the following.

Lemma 5.3 *Assume that E has an order continuous norm, and that $f, g \in \mathbb{H}(X, E)$. Then the functions $f \vee_0 g = [\underline{f} \vee \underline{g}, \overline{f} \vee \overline{g}]$ and $f \wedge_0 g = [\underline{f} \wedge \underline{g}, \overline{f} \wedge \overline{g}]$ each contain a unique H-continuous function.*

Proof By Proposition 2.7, $\underline{f} \vee \underline{g}$ is lower semi-continuous and $\overline{f} \vee \overline{g}$ is upper semi-continuous so that $f \vee_0 g$ is S-continuous. By Proposition 4.3 there exists an H-continuous function $h \subseteq f \vee_0 g$. Then $\overline{h} = S(\underline{h}) \geq S(\underline{f}) = \overline{f}$ and $\overline{h} = S(\underline{h}) \geq S(\underline{g}) = \overline{g}$ by Proposition 4.4 and (3.7). Thus $\overline{h} = \overline{f} \vee \overline{g}$ so that $\underline{h} = I[\overline{f} \vee \overline{g}]$ by Proposition 4.4. Since this holds for an arbitrary H-continuous function $h \subseteq f \vee_0 g$ it follows that $f \vee_0 g$ contains a unique H-continuous function.

The proof that $f \wedge_0 g$ contains a unique H-continuous function is similar. □

Theorem 5.4 *With addition, scalar multiplication and the partial order defined as in* (5.3)*,* (5.4) *and* (4.2)*, respectively,* $\mathbb{H}(X, E)$ *is a Dedekind complete Riesz space.*

Proof Consider $f, g, h \in \mathbb{H}(X, E)$ so that $f \leq g$ and a real number $\gamma \geq 0$. It is clear from (5.4) and (4.2) that $\gamma f \leq \gamma g$. According to (5.3), $f + h = [I \circ S[\underline{f} + \underline{h}], S[\underline{f} + \underline{h}]]$ and $g + h = [I \circ S[\underline{g} + \underline{h}], S[\underline{g} + \underline{h}]]$, see the proof of Proposition 4.6. Furthermore, it follows from (4.2) that $\underline{f} + \underline{h} \leq \underline{g} + \underline{h}$ so that $f + h \leq g + h$ by (3.7). Hence $\mathbb{H}(X, E)$ is an ordered vector space.

For $f, g \in \mathbb{H}(X, E)$, let $f \vee g$ be the unique H-continuous function contained in $f \vee_0 g$. Then

$$\underline{f} \leq \underline{f} \vee \underline{g} \leq \underline{f \vee g} \quad \text{and} \quad \underline{g} \leq \underline{f} \vee \underline{g} \leq \underline{f \vee g}.$$

It follows from (3.7) and Proposition 4.4 that

$$\overline{f} \leq \overline{f \vee g} \quad \text{and} \quad \overline{g} \leq \overline{f \vee g}$$

so that $f, g \leq f \vee g$. Now suppose that h is H-continuous and satisfies $f, g \leq h \leq f \vee g$. Then $\underline{f} \vee \underline{g} \leq \underline{h}$ and $\overline{h} \leq \overline{f \vee g} \leq \overline{f} \vee \overline{g}$ so that $h \subseteq f \vee_0 g$. By Proposition 5.3, $h = f \vee g$ so that $f \vee g$ is the least upper bound of f and g. The existence of the greatest lower bound of f and g, namely the unique H-continuous function $f \wedge g$ contained in $f \wedge_0 g$, follows in the same way. Hence $\mathbb{H}(X, E)$ is a Riesz space.

Suppose that $\mathcal{F}$ is bounded from above in $\mathbb{H}(X, E)$. Set

$$\mathcal{G} = \{g \in \mathbb{H}(X, E) \colon f \leq g, \ f \in \mathcal{F}\}.$$

For each $x \in X$, let $\underline{f}'(x) = \sup\{\underline{f}(x) \colon f \in \mathcal{F}\}$ and $\overline{f}'(x) = \inf\{\overline{g}(x) \colon g \in \mathcal{G}\}$. According to Proposition 2.7, $\underline{f'}$ is lower semi-continuous, and $\overline{f}'$ is upper semi-continuous so that $f' = [\underline{f'}, \overline{f}']$ is S-continuous. By Proposition 4.3 there exists an H-continuous function f_0 so that $f_0 \subseteq f'$. Then, for every $f \in \mathcal{F}$, $\underline{f} \leq \underline{f'} \leq \underline{f}_0$ and, by (3.7) and Proposition 4.4, $\overline{f} = S[\underline{f}] \leq S[\underline{f}_0] = \overline{f}_0$. Therefore $f \leq f_0$ for every $f \in \mathcal{F}$. Now suppose that $g \in \mathcal{G}$. Then $\underline{f} \leq \underline{g}$ and $\overline{f} \leq \overline{g}$ for every $f \in \mathcal{F}$. Thus $\underline{f'} \leq \underline{g}$ and $\overline{f}' \leq \overline{g}$ so that $\overline{f}_0 \leq \overline{f}' \leq \overline{g}$ and so, by (3.7) and Proposition 4.4, $\underline{f}_0 = I[\overline{f}_0] \leq I[\overline{g}] = \underline{g}$. Therefore $f_0 \leq g$ so that $f_0 = \sup \mathcal{F}$. Hence $\mathbb{H}(X, E)$ is Dedekind complete. □

6 The Dedekind completion of $C(X, E)$

In general, $C(X, E)$ is not Dedekind complete, even in the scalar case. Indeed, for a compact Hausdorff space X, $C(X)$ is Dedekind complete if and only if X is extremally disconnected, see for instance [25, Proposition 2.1.4]. Ercan and Wickstead [16, Theorem 3.3] showed that, for X a compact Hausdorff space, $C(X, E)$ is Dedekind complete if and only if either X is discrete and E is Dedekind complete, or $C(X)$ is Dedekind complete and E has compact order intervals.

A number of concrete characterisations have been given of the Dedekind completion $C(X)^{\delta}$ of $C(X)$, starting with Dilworth [14], who considered the case when X is compact and Hausdorff. Dilworth's result was extended to completely regular spaces by Horn [18]. Maxey [24], see also [20], obtained a characterisation of $C(X)^{\delta}$ as a quotient of $C(X)''$. Recently, Anguelov [1] gave a characterisation of $C(X)^{\delta}$, for X completely regular, in terms of the set $\mathbb{H}(X)$ of H-continuous functions on X, see also [15]. As an application of the theory developed in the preceding sections, we give a generalisation of Anguelov's result. In particular, we show that for X a paracompact T_1-space and E a Banach lattice with order continuous norm, the Dedekind completion of $C(X, E)$ is $\mathbb{H}(X, E)$.

Theorem 6.1 *If X is a paracompact T_1-space, and E has order continuous norm, then $\mathbb{H}(X, E)$ is the Dedekind completion of $C(X, E)$.*

Proof We first prove that there exists a Riesz isomorphism from $C(X, E)$ into $\mathbb{H}(X, E)$. In this regard, for each $\varphi \in C(X, E)$, let $\mathcal{E}[\varphi]$ denote the interval valued function defined as

$$\mathcal{E}[\varphi](x) = [\varphi(x), \varphi(x)],\ x \in X.$$

Since φ is continuous, it is both upper semi-continuous and lower semi-continuous so that $\mathcal{E}[\varphi]$ is S-continuous. Furthermore, since $\mathcal{E}[\varphi](x)$ is a degenerate interval for every $x \in X$, it follows that $f = \mathcal{E}[\varphi]$ for every S-continuous function f satisfying $f \subseteq \mathcal{E}[\varphi]$. Therefore $\mathcal{E}[C(X, E)] \subseteq \mathbb{H}(X, E)$. Clearly $\mathcal{E}\colon C(X, E) \to \mathbb{H}(X, E)$ is linear, injective and monotone. It therefore remains to show that $\mathcal{E}[\varphi \vee \psi] = \mathcal{E}[\varphi] \vee \mathcal{E}[\psi]$ and $\mathcal{E}[\varphi \wedge \psi] = \mathcal{E}[\varphi] \wedge \mathcal{E}[\psi]$ for all $\varphi, \psi \in C(X, E)$. Due to the monotonicity of $\mathcal{E}$, $\mathcal{E}[\varphi] \vee \mathcal{E}[\psi] \leq \mathcal{E}[\varphi \vee \psi]$. If $\mathcal{E}[\varphi], \mathcal{E}[\psi] \leq f$ for some $f \in \mathbb{H}(X, E)$, then $\varphi = \underline{\mathcal{E}[\varphi]} \leq \underline{f}$ and $\psi = \underline{\mathcal{E}[\psi]} \leq \underline{f}$ so that $\underline{\mathcal{E}[\varphi \vee \psi]} = \varphi \vee \psi \leq \underline{f}$. In the same way, $\overline{\mathcal{E}[\varphi \vee \psi]} \leq \overline{f}$ so that $\mathcal{E}[\varphi \vee \psi] \leq f$. Hence $\mathcal{E}[\varphi \vee \psi] = \mathcal{E}[\varphi] \vee \mathcal{E}[\psi]$. Similarly, $\mathcal{E}[\varphi \wedge \psi] = \mathcal{E}[\varphi] \wedge \mathcal{E}[\psi]$ so that $\mathcal{E}$ is a Riesz isomorphism into $\mathbb{H}(X, E)$.

For each $h \in \mathbb{H}(X, E)$, let $\mathcal{L}_h = \mathcal{E}[\{\varphi \in C(X, E)\colon \mathcal{E}[\varphi] \leq h\}]$ and $\mathcal{U}_h = \mathcal{E}[\{\psi \in C(X, E)\colon h \leq \varepsilon[\psi]\}]$. By Corollaries 2.5 and 2.6, both these sets are nonempty and

$$\underline{h}(x) = \sup\{\varphi(x)\colon \varphi \in \mathcal{L}_h\},\ \overline{h}(x) = \inf\{\psi(x)\colon \psi \in \mathcal{U}_h\},\ x \in X.$$

If $h_0 \in \mathbb{H}(X, E)$ satisfies

$$\mathcal{E}[\varphi] \leq h_0 \leq h,\ \varphi \in \mathcal{L}_h$$

then $\underline{h_0} = \underline{h}$ so that $\overline{h_0} = S(\underline{h_0}) = S(\underline{h}) = \overline{h}$ by Proposition 4.4. Thus $h = h_0$ so that $h = \sup \mathcal{L}_h$. In the same way it follows that $h = \inf \mathcal{U}_h$ so that $\mathbb{H}(X, E)$ is the Dedekind completion of $C(X, E)$. □

7 Relation with minimal usco maps

Denote by $\mathcal{K}(Y)$ the set of nonempty, compact subsets of Y. A map $f: X \to \mathcal{K}(Y)$ is called upper semi-continuous (usco) if for every $x \in X$ and every open set $U \supseteq f(x)$ there exists an open set $V \in \mathcal{V}_x$ so that $f(y) \subseteq U$ for every $y \in V$. An usco map $f: X \to \mathcal{K}(Y)$ is called minimal (musco) if

$$\langle f \rangle := \{g: X \to \mathcal{K}(Y): g \text{ is usco, } g(x) \subseteq f(x), x \in X\} = \{f\}.$$

We say that f is quasi-minimal if $\langle f \rangle$ is a singleton. Note that $\langle f \rangle \neq \emptyset$ for every usco map f, see for instance [11]. Denote by $\mathcal{M}(X, Y)$ the set of musco maps $f: X \to \mathcal{K}(Y)$.

According to [31, Theorems 5.1 & 6.3], $\mathcal{M}(X, E)$ is the Dedekind completion of $C(X, E)$ whenever X is a compact Hausdorff space, and E is an AM-space with compact order intervals. In fact, due to Corollary 2.5, the result holds under the milder assumption that X is a paracompact T_1-space. It therefore follows from Theorem 6.1 that there exists a unique Riesz isomorphism

$$T: \mathcal{M}(X, E) \to \mathbb{H}(X, E)$$

that leaves each $f \in C(X, E)$ invariant, see [22, Section 32]. The aim of this section is to obtain a description of this isomorphism.

Theorem 7.1 *Let X be a paracompact T_1-space, and E an AM-space with compact order intervals. Then the unique Riesz isomorphism $T: \mathcal{M}(X, E) \to \mathbb{H}(X, E)$ leaving each $f \in C(X, E)$ invariant is given by*

$$T[f](x) = [\inf f(x), \sup f(x)], \ x \in X.$$

Proof For $f \in \mathcal{M}(X, E)$, let

$$\mathcal{A} = \{g \in C(X, E): g \leq f\}, \ \mathcal{B} = \{g \in C(X, E): g \leq T[f]\}.$$

According to Corollary 2.5, $\underline{T[f]}(x) = \sup\{g(x): g \in \mathcal{B}\}, \ x \in X$. It is shown in the proof of [31, Theorem 6.3] that $\inf f(x) = \sup\{g(x): g \in \mathcal{A}\}, \ x \in X$. But T is a Riesz isomorphism leaving each member of $C(X, E)$ invariant so that $T[\mathcal{A}] = \mathcal{B}$. Hence $\underline{T[f]}(x) = \inf f(x)$ for every $x \in X$. In exactly the same way, it follows that $\overline{T[f]}(x) = \sup f(x)$ for every $x \in X$. □

As a consequence of Theorem 7.1, we obtain an extension of the following results of Anguelov and Kalenda [2]: if $f \in \mathbb{H}(X, \mathbb{R})$, then f is quasi-minimal usco. Conversely, if $f: X \to \mathcal{K}(Y)$ is musco, then

$$X \ni x \mapsto [\inf f(x), \sup f(x)] \in I\mathbb{R}$$

is H-continuous.

Corollary 7.2 *Assume that X is a paracompact T_1-space, and E is an AM-space with compact order intervals. Then the following statements are true.*

(i) *If $f \in \mathbb{H}(X, E)$, then f is a quasi-minimal usco map.*
(ii) *If $f : X \to \mathcal{K}(E)$ is musco, then $X \ni x \mapsto [\inf f(x), \sup f(x)] \in IE$ is H-continuous.*

Proof If $f \in \mathbb{H}(X, E)$, then f is an usco map by [31, Proposition 4.3 (ii)]. Suppose that $g, h \in \langle f \rangle$. Then, by Theorem 7.1, $T[g], T[h] \subseteq f$ so that $T[g] = T[h] = f$. Hence $g = h$ so that f is quasi-minimal, proving (i). That (ii) is true follows immediately from Theorem 7.1. □

8 Conclusions

It has been shown that the Dedekind completion of $C(X, E)$, with X a paracompact T_1-space and E a Banach lattice with order continuous norm, can be characterised as the set of H-continuous E-valued interval functions on X, extending a result of Anguelov [1]. According to Ercan and Wickstead [16], spaces of Banach lattice valued continuous functions provide useful examples of Banach lattices in which features of different classes of Banach lattices occur simultaneously. Seen in this context, our result may be of broader significance.

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