Positivity (2017) 21:1067–1088
DOI 10.1007/s11117-016-0453-x

Positivity

Weak sharp solutions for generalized variational inequalities

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Received: 21 December 2015 / Accepted: 31 October 2016 / Published online: 9 November 2016

Abstract We consider weak sharp solutions for the generalized variational inequality problem, in which the underlying mapping is set-valued, and not necessarily monotone. We extend the concept of weak sharpness to this more general framework, and establish some of its characterizations. We establish connections between weak sharpness and (1) gap functions for variational inequalities, and (2) global error bound. When the solution set is weak sharp, we prove finite convergence of the sequence generated by an arbitrary algorithm, for the monotone set-valued case, as well as for the case in which the underlying set-valued map is either Lipschitz continuous in the set-valued sense, for infinite dimensional spaces, or inner-semicontinuous when the space is finite dimensional.

Keywords Generalized variational inequalities · Weak sharp solutions · Paramonotone operators · Gap function · Inner-semicontinuous maps · Lipschitz continuous set-valued maps · Finite convergence

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Mathematics Subject Classification 49J40 · 90C33 · 49J53 · 47J20 · 90C25 · 47H04

1 Introduction

In 1998, Marcotte and Zhu [14] introduced the concept of weak sharp solutions of a point-to-point variational inequality. They derived the necessary and sufficient condition for a solution set to be weakly sharp, and also studied the finite convergence of iterative algorithms for solving variational inequalities whose solution set is weakly sharp. The finite convergence of an iterative algorithm means that the algorithm finds a solution in a finite number of iterates. Zhou and Wang [23] re-examined the unified treatment of finite termination of a class of iterative algorithms, and showed that some results given by Marcotte and Zhu [14] remain valid under more general assumptions. Wu and Wu [18] presented several equivalent (and sufficient) conditions for weak sharp solutions of a variational inequality in the setting of Hilbert spaces. They gave a finite convergence result for a class of algorithms for solving variational inequalities. Using the concept of a dual gap function, Zhang et al. [22] characterized the directional derivative and subdifferential of the dual gap function. Their analysis opens the way for a better understanding of the concepts of a global error bound, weak sharpness, and minimum principle sufficiency property for a variational inequality, where the operator is pseudo-monotone point-to-point. Xiu and Zhang [20] further studied finite convergence of a specific family of algorithms. They established the finite convergence of the proximal point algorithm when the solution set is weakly sharp.

If the objective function of a constrained optimization problem is neither convex nor differentiable, then the problem cannot be formulated as a point-to-point monotone variational inequality. In this case, generalized variational inequalities (i.e., with a non-monotone set-valued underlying mapping) are needed.

On the other hand, the literature addressing the case of weak sharp solutions for generalized variational inequalities (i.e., for set-valued variational inequalities) is very scarce, and, as far as we know, treats the maximally monotone case in finite dimensions, see, e.g., the recent work [19]. Under a weak-sharpness assumption, the latter paper shows finite termination for the set-valued maximally monotone case. To address this gap is the main aim of the present paper.

The paper is structured as follows: in the next section, we present the formulation of generalized variational inequality problem in which the underlying operator is a set-valued mapping. We also present some basic definitions and results from nonlinear and convex analysis, which will be used in the sequel. Section 3 deals with the notion of a gap function for generalized variational inequality problems. We investigate some properties of the gap functions and give some characterizations of the solution set of generalized variational inequality problems in terms of gap functions. In Sect. 4, on the lines of Marcotte and Zhu [14], we introduce the concept of weak sharp solution set for generalized variational inequalities. We provide some necessary and sufficient conditions in terms of a gap function for the solution set of the generalized variational inequality to be weak sharp. We give a necessary condition for a weak sharp solution set of a generalized variational inequality in terms of global error bound. In the last section, we prove finite termination of an iterative algorithm under weak sharpness of the solution set. We provide the finite termination result under two types of assumptions on the underlying set-valued map F: (1) when F is monotone, and (2) when F is continuous in the set-valued sense (see Definition 2).

The results of this paper extend and generalize corresponding results for variational inequalities studied in [14,15,23].

2 Formulations, preliminaries and basic definitions

Let H be a Hilbert space with inner product denoted by $\langle\cdot,\cdot\rangle$, and induced norm $\|\cdot\| : H \to \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of nonnegative real numbers. Given C a nonempty closed convex subset of H and $F : H \rightrightarrows H$ a set-valued mapping, the *generalized variational inequality problem* (GVIP) is stated as follows: find $\bar{x} \in C$ and $\bar{u} \in F(\bar{x})$ such that

$$\langle \bar{u}, y - \bar{x}\rangle \geq 0, \quad \text{for all} \quad y \in C. \tag{1}$$

An important particular case arises when $F \equiv \partial f$, the *subdifferential* of a proper, lower semicontinuous and convex function $f : H \to \mathbb{R} \cup \{+\infty\}$, or the *Clarke subdifferential* [8] ∂f^C of a locally Lipschitz function $f : H \to \mathbb{R} \cup \{+\infty\}$. In these cases, the GVIP (1) provides a necessary and sufficient optimality condition for solving a convex/non-convex and non-smooth optimization problem. For further details on GVIP and their generalizations with applications, we refer to [1,5,10,13,16] and the references therein.

For a given nonempty subset C of H, we denote by $\operatorname{int}C$, $\operatorname{ri}C$, and $\operatorname{cl}C$ the *interior*, the *relative interior*, and the *closure* of C, respectively. We denote by $B(x,r)$ the open ball in H with center x and radius $r > 0$ and by $B[x,r]$ the closed ball in H with center x and radius $r > 0$. The *polar cone* C° of C is defined by

$$C^\circ := \{y \in H : \langle y, x\rangle \leq 0 \quad \text{for all} \quad x \in C\}.$$

For a nonempty closed convex subset of $\mathbb{R}^n$, the *tangent cone to C at $x \in C$* is defined by

$$T_C(x) := \operatorname{cl}\left(\bigcup_{\lambda>0} \frac{C-x}{\lambda}\right).$$

By construction, $T_C(x)$ is a nonempty, closed and convex cone. The *normal cone to the set C at x* is defined by

$$N_C(x) := \begin{cases} \{y \in H : \langle y, z-x\rangle \leq 0 \quad \text{for all} \quad z \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{otherwise,} \end{cases}$$

that is, $N_C(x) := [T_C(x)]^\circ$. The *distance from $x \in H$ to C* is given by

$$d(x, C) := \inf\{\|x - c\| : c \in C\}$$

Let C be a nonempty closed convex subset of H. The *projection of* $x \in H$ *onto* C is defined by

$$P_C(x) := \arg\min_{y \in C} \|x - y\|.$$

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. The *domain of* f is denoted by $\operatorname{dom} f$ and defined as

$$\operatorname{dom} f := \{x \in H : f(x) < +\infty\}.$$

Fix $d \in H$ and $x \in \operatorname{dom} f$. Recall that the *directional derivative of* f *at* x *in the direction* d is defined by

$$f'(x; d) := \lim_{t \to 0^+} \frac{f(x + td) - f(x)}{t}.$$

Fix $x \in \operatorname{dom} f$. The *subdifferential of* f at x is defined by

$$\partial f(x) := \{\xi \in H : f(y) - f(x) \geq \langle \xi, y - x \rangle \quad \text{for all } y \in H\}.$$

Let $\varepsilon \geq 0$. The ε*-subdifferential of* f at x is defined by

$$\partial_\varepsilon f(x) := \{\xi \in H : f(y) - f(x) \geq \langle \xi, y - x \rangle - \varepsilon \quad \text{for all } y \in H\}.$$

Note that $\partial_\varepsilon f(x) \supset \partial f(x)$ for all $x \in \operatorname{dom} f$ and all $\varepsilon \geq 0$.

Given a set-valued mapping $F : H \rightrightarrows H$, its *domain* is denoted by $D(F)$ and defined as

$$D(F) := \{x \in H : F(x) \neq \emptyset\}.$$

The *graph* of F is denoted by $G(F)$ and defined as

$$G(F) := \{(x, u) : x \in D(F) \quad \text{and} \quad u \in F(x)\}.$$

Let $C \subset H$ be nonempty. We denote by $G_C(F)$ the graph of F restricted to C, namely,

$$G_C(F) := \{(x, u) \in G(F) : x \in C\}.$$

A set-valued mapping $F : H \rightrightarrows H$ is said to be

(a) *monotone* if

$$\langle u - v, x - y \rangle \geq 0, \quad \text{for all} \quad x, y \in H \quad \text{and all} \quad u \in F(x), v \in F(y);$$

(b) *pseudomonotone* if for all $x, y \in H$,

$$\langle u, y - x\rangle \geq 0 \quad \text{for all} \quad u \in F(x) \quad \Rightarrow \quad \langle v, y - x\rangle \geq 0 \quad \text{for all} \quad v \in F(y).$$

(c) *maximally monotone* if it is monotone and the graph of F cannot be enlarged without destroying the monotonicity property. In other words, if $\tilde{F}$ is monotone and $G(F) \subset G(\tilde{F})$, then $F \equiv \tilde{F}$;

(d) *paramonotone* if it is monotone and whenever $\langle u - v, x - y\rangle = 0$ with $u \in F(x)$ and $v \in F(y)$, then this implies that $u \in F(y)$ and $v \in F(x)$;

We will consider in our analysis maps which are continuous in the point-to-set sense. The following definitions and results collect all the tools we will need. We start by defining convergence of a sequence of sets.

Definition 1 *([5, Definition 2.2.3 and Proposition 2.2.7])* Given a sequence $\{C^k\}$ of sets such that $C^k \subset H$ for all k, we define the *interior limit of the sequence* $\{C^k\}$ *of sets* as the set

$$\liminf_{k\to\infty} C^k := \left\{ z \in H : \liminf_{k\to\infty} d(z, C_k) = 0 \right\}.$$

Recall that in a Hilbert space, the strong topology is the metric topology induced by the norm.

Definition 2 *([5, Definition 2.5.1(a) (b) and Definition 2.5.3 (i)])* Let $F : H \rightrightarrows H$ and U be a subset of $D(F)$ such that F is closed-valued on U. In the definitions below, we consider the convergence with respect to the strong (i.e., the norm) topology in H. We say that F is:

(a) *inner-semicontinuous* at $x \in D(F)$ if whenever a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x we have

$$F(x) \subset \liminf_{n\to\infty} F(x_n).$$

(b) *Lipschitz continuous* on U if there exists a *Lipschitz constant* $\kappa > 0$ such that for all $x,\ x' \in U$ it holds that

$$F(x) \subset F(x') + \kappa\|x - x'\|B.$$

The next result, which will be used for establishing finite convergence under no monotonicity assumptions, is an adaptation of [5, Proposition 2.5.29(i)] to our framework. Since the statement is slightly changed from that in [5], and for convenience of the reader, we include its proof here.

Proposition 1 *Let $F : H \rightrightarrows H$ be inner-semicontinuous (with respect to the strong topology) at $\bar{x} \in \operatorname{int} D(F)$. Consider the following assumptions.*

(i) *H is infinite dimensional and $F(\bar{x})$ is compact (with respect to the strong topology in H).*

(ii) $H = \mathbb{R}^n$ *(i.e., H is finite dimensional) and* $F(\bar{x})$ *is closed.*

If either (i) *or* (ii) *hold, then for every* $\rho > 0$, $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$F(\bar{x}) \cap B[0, \rho] \subset F(x) + B[0, \varepsilon], \quad \text{for all} \quad x \in B(\bar{x}, \delta).$$

Proof Assume that (i) holds. If the statement on neighborhoods were not true, we could find ρ_0, $\varepsilon_0 > 0$, and a sequence $\{x_n\}$ converging to $\bar{x}$ such that $F(\bar{x}) \cap B[0, \rho_0] \not\subset F(x_n) + B[0, \varepsilon_0]$. Define a sequence $\{z_n\}$ such that $z_n \in F(\bar{x}) \cap B[0, \rho_0]$ and $z_n \notin F(x_n) + B[0, \varepsilon_0]$. By compactness of $F(\bar{x})$, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ converging to some $\bar{z} \in F(\bar{x}) \cap B[0, \rho_0]$. By inner-semicontinuity of F, we can find a sequence $\{y_n \in F(x_n)\}$ also converging to $\bar{z}$. Then $\lim_k z_{n_k} - y_{n_k} = 0$, so that for large enough k we have that $z_{n_k} - y_{n_k} \in B[0, \varepsilon_0]$. Altogether, we conclude that $z_{n_k} = y_{n_k} + [z_{n_k} - y_{n_k}] \in F(x_{n_k}) + B[0, \varepsilon_0]$ for large enough k, but this contradicts the definition of $\{z_n\}$. Therefore, the claim on neighborhoods must hold.

Assume now that (ii) holds. In this case, the proof follows the same steps, but with the difference that we use the compactness of the ball, which only holds in finite dimensions. Indeed, we have now that the set $F(\bar{x}) \cap B[0, \rho_0]$ is compact, because the ball is compact and $F(\bar{x})$ is closed. Hence, as in the proof for assumption (i), there is subsequence $\{z_{n_k}\}$ of $\{z_n\}$ converging to some $\bar{z} \in F(\bar{x}) \cap B[0, \rho_0]$. The proof now follows exactly as for the previous assumption. □

Remark 1 The Lipschitz continuity assumption rules out maximally monotone mappings which are not point-to-point. Hence this assumption only makes sense for non-monotone set-valued variational inequalities.

The following is a simple extension of results in the literature, and will be needed in the sequel. We provide here its simple proof for convenience of the reader.

Lemma 1 (See [15, Lemma 1] and [7, Lemma 3.1]) *Let* $C \subset H$ *be a closed convex set and* $x \in C$. *Then, for every* $u \in H$, *we have*

(a) $\left\langle P_{T_C(x)}(-u), u \right\rangle = -\left\| P_{T_C(x)}(-u) \right\|^2$;
(b) $\min\{\langle v, u\rangle : v \in T_C(x),\ \|v\| \le 1\} = -\left\| P_{T_C(x)}(-u) \right\|$.

Proof We denote by $\bar{u} := P_{T_C(x)}(-u) \in T_C(x)$. The properties of the projection imply that

$$\langle y - \bar{u}, -u - \bar{u}\rangle \le 0, \quad \text{for all} \quad y \in T_C(x). \tag{2}$$

Taking $y = 0 \in T_C(x)$ and $y = 2\bar{u} \in T_C(x)$ in (2), we obtain sequentially that

$$\langle -\bar{u}, -u - \bar{u}\rangle \le 0 \text{ and } \langle \bar{u}, -u - \bar{u}\rangle \le 0,$$

which readily imply (a). Assume first that $\bar{u} = 0$. From (2) and (a), we have

$$\langle y, u\rangle \ge 0, \quad \text{for all} \quad y \in T_C(x).$$

So,

$$\inf\{\langle v, u\rangle : v \in T_C(x), \quad \|v\| \le 1\} \ge 0.$$

If $\bar{u} = 0$, then the above infimum is attained when $v := \bar{u}$ and is equal to 0. Hence, (b) holds in this case. Assume now that $\bar{u} \neq 0$. The fact that $\bar{u} = P_{T_C(x)}(-u)$ means that

$$\bar{u} = \operatorname*{argmin}_{y \in T_C(x)} \|y + u\|^2.$$

Therefore, for all $y \in T_C(x)$ such that $\|y\| \leq \|\bar{u}\|$, we have

$$\|\bar{u} + u\|^2 \leq \|y + u\|^2.$$

Using (a) in this inequality and simplifying the resulting expression yields

$$\langle y, u \rangle + \frac{\|\bar{u}\|^2}{2} \geq \langle y, u \rangle + \frac{\|y\|^2}{2} \geq -\frac{\|\bar{u}\|^2}{2}, \tag{3}$$

where in the left-most inequality we used the fact that $y \in T_C(x)$ is such that $\|y\| \leq \|\bar{u}\|$. Combine the left-most and right-most expressions in (3) to obtain

$$\langle y, u \rangle \geq -\|\bar{u}\|^2,$$

for all $y \in T_C(x)$ such that $\|y\| \leq \|\bar{u}\|$. Since $\bar{u} \neq 0$, we can re-write this expression as

$$\left\langle \frac{y}{\|\bar{u}\|}, u \right\rangle \geq -\|\bar{u}\|, \tag{4}$$

for all $y \in T_C(x)$ such that $\|y\| \leq \|\bar{u}\|$. Since $\|y\| \leq \|\bar{u}\|$, $\frac{y}{\|\bar{u}\|}$ represents any $y' \in T_C(x)$ such that $\|y'\| \leq 1$. With this in mind, (4) reads

$$\langle y', u \rangle \geq -\|\bar{u}\|, \tag{5}$$

for all $y' \in T_C(x)$ such that $\|y'\| \leq 1$. This proves that

$$\min\{\langle v, u \rangle : v \in T_C(x), \quad \|v\| \leq 1\} \geq -\|\bar{u}\|.$$

To obtain the opposite inequality, take $v := \frac{\bar{u}}{\|\bar{u}\|} \in T_C(x)$, we have

$$\min\{\langle v, u \rangle : v \in T_C(x), \quad \|v\| \leq 1\} \leq \left\langle \frac{\bar{u}}{\|\bar{u}\|}, u \right\rangle = -\|\bar{u}\|,$$

where we used (a) in the right-most equality. This completes the proof of (b). □

3 Gap functions and solution set for generalized variational inequalities

3.1 Characterization of solution set for generalized variational inequalities

The GVIP can be stated in terms of the graph of F as follows:

$$\text{Find } (\bar{x}, \bar{u}) \in G_C(F) \text{ such that } \langle \bar{u}, y - \bar{x} \rangle \geq 0, \quad \text{for all} \quad y \in C. \tag{6}$$

The set of solutions of GVIP is denoted by S, that is,

$$S = \{(\bar{x}, \bar{u}) \in G_C(F) : \langle \bar{u}, y - \bar{x} \rangle \geq 0 \quad \text{for all} \quad y \in C\}.$$

We will also consider the set $\bar{S}$, the projection of S onto the first coordinate :

$$\bar{S} := \{x \in C : \exists u \in F(x) \text{ satisfying } (x, u) \in S\}. \tag{7}$$

From (6), we have that $\bar{x} \in \bar{S}$ if and only if there exists $\bar{u} \in F(\bar{x})$ such that $-\bar{u} \in N_C(\bar{x})$ or equivalently, $P_{T_C(\bar{x})}[-\bar{u}] = 0$.

Now we give a characterization of the solution set $\bar{S}$ of (1) under paramonotonicity.

Proposition 2 *Let $F : H \rightrightarrows H$ be paramonotone and $\bar{x} \in \bar{S}$. Then, $\bar{S} = \tilde{S}$, where*

$$\tilde{S} := \{z \in C : \exists \bar{u} \in F(z) \cap F(\bar{x}) \text{ such that } \langle \bar{u}, z - \bar{x} \rangle = 0\}.$$

Proof Let $z \in \bar{S}$. Then, there exists $w \in F(z)$ such that $\langle w, \bar{x} - z \rangle \geq 0$. Since $\bar{x} \in \bar{S}$, there exists $\bar{u} \in F(\bar{x})$ such that $\langle \bar{u}, z - \bar{x} \rangle \geq 0$. Combining the last two inequalities, we get

$$\langle w - \bar{u}, z - \bar{x} \rangle \leq 0. \tag{8}$$

By monotonicity of F, we have

$$\langle w - \bar{u}, z - \bar{x} \rangle \geq 0. \tag{9}$$

From (8) and (9), we get $\langle w - \bar{u}, z - \bar{x} \rangle = 0$. The paramonotonicity of F implies that $w \in F(\bar{x})$ and $\bar{u} \in F(z)$. Hence, $\bar{u} \in F(z) \cap F(\bar{x})$. Moreover,

$$0 \geq \langle w, z - \bar{x} \rangle = \langle \bar{u}, z - \bar{x} \rangle \geq 0,$$

and thus, $\langle \bar{u}, z - \bar{x} \rangle = 0$. Hence, $z \in \tilde{S}$, therefore, $\bar{S} \subseteq \tilde{S}$.

Conversely, let $z \in \tilde{S}$. Then, $z \in C$ and there exists $\bar{u} \in F(z) \cap F(\bar{x})$ such that

$$\langle \bar{u}, z - \bar{x} \rangle = 0. \tag{10}$$

Since $\bar{x} \in \bar{S}$, there exists $\hat{u} \in F(\bar{x})$ such that

$$\langle \hat{u}, y - \bar{x} \rangle \geq 0, \quad \text{for all} \quad y \in C. \tag{11}$$

By monotonicity, and the inclusions $\bar{u} \in F(z)$ and $\hat{u} \in F(\bar{x})$, we have

$$\langle \hat{u} - \bar{u}, \bar{x} - z \rangle \geq 0. \tag{12}$$

Using (10) and (11) for $y = z$ in the inequality above yields

$$0 \leq \langle \hat{u}, z - \bar{x} \rangle \leq \langle \bar{u}, z - \bar{x} \rangle = 0,$$

so,

$$\langle \hat{u}, z - \bar{x} \rangle = 0. \tag{13}$$

This fact and (10) imply that

$$\langle \hat{u} - \bar{u}, \bar{x} - z \rangle = 0,$$

and by paramonotonicity, we conclude that

$$\hat{u} \in F(z). \tag{14}$$

For any $y \in C$, by (11), we have

$$0 \leq \langle \hat{u}, y - \bar{x} \rangle = \langle \hat{u}, y - z \rangle + \langle \hat{u}, z - \bar{x} \rangle = \langle \hat{u}, y - z \rangle,$$

where we used (13) in the last equality. Combining the above expression with (14) implies that $z \in \bar{S}$. Therefore, $\tilde{S} \subseteq \bar{S}$ and the proof is complete. □

Remark 2 Proposition 2 extends to the point-to-set framework Lemma 2 in [15]. Since the subdifferential of a proper convex lower semicontinuous function is point-to-set and paramonotone [12], Proposition 2 also extends and generalizes Lemma 1 in [23].

3.2 Gap function and solution set for generalized variational inequalities

Let C be a nonempty subset of H and $F : H \rightrightarrows H$ be a set-valued mapping with a non-empty domain.

Definition 3 A function $g : H \to \mathbb{R} \cup \{+\infty\}$ is said to be a *gap function for GVIP* if and only if the following conditions hold:

(a) $g(x) \geq 0$ for all $x \in C$;
(b) $g(\bar{x}) = 0$ if and only if $\bar{x} \in \bar{S}$.

One of the main motivations to introduce the gap functions is that they allow to convert a GVIP into an optimization problem, so that standard methods can be used to solve GVIP.

Crouzeix [9] considered the Minty type generalized variational inequality problem (MGVIP): find $\bar{x} \in C$ such that

$$\langle v, y - \bar{x} \rangle \geq 0, \quad \text{for all} \quad y \in C \quad \text{and} \quad v \in F(y). \tag{15}$$

The solution set of MGVIP is denoted by S^*. It is clear that $\bar{S} \subseteq S^*$ if F is pseudomonotone. It was proved by Crouzeix [9] that $\bar{S} = S^*$ if C is a nonempty closed convex subset of H and F is pseudomonotone and upper semicontinuous with nonempty convex compact values.

Auslender [2] (see also [3,9]) introduced the following gap function $h_{F,C} : H \to \mathbb{R} \cup \{+\infty\}$ for GVIP:

$$h_{F,C}(x) := \begin{cases} \sup\limits_{(v,y)\in G_C(F)} \langle v, x - y\rangle, & \text{if} \quad x \in C \cap D(F), \\ +\infty, & \text{if} \quad x \notin C \cap D(F). \end{cases} \tag{16}$$

The function $h_{F,C}$ is nonnegative, lower semicontinuous and convex. The latter two properties follow from the fact that it is a supremum of affine functions. From its definition, $h_{F,C}$ is proper when $C \cap D(F) \neq \emptyset$.

When F is maximally monotone, the minimizers of $h_{F,C}$ are the solutions of GVIP. This fact, established in [4], is stated next.

Proposition 3 [3,4] *Assume that F is maximally monotone. Consider the following assumptions.*

(i) *H is finite dimensional and* $\mathrm{ri}(D(F)) \cap \mathrm{ri}(C) \neq \emptyset$.
(ii) *H is infinite dimensional and* $\mathrm{int}(D(F)) \cap C \neq \emptyset$ *or* $D(F) \cap \mathrm{int}(C) \neq \emptyset$.

Under assumptions (i) *or* (ii)*, we have:*

(a) *$h_{F,C}(x) \geq 0$ for all $x \in D(F) \cap C$;*
(b) *$h_{F,C}(\bar{x}) = 0$ if and only if $\bar{x} \in \bar{S}$.*

For brevity, we write from now on h instead of $h_{F,C}$ when referring to the gap function defined by (16).

Definition 4 For a fixed $x \in C$, we define the following sets:

$$\begin{aligned} \Lambda(x) &:= \{(z, w) \in G_C(F) : \langle w, x - z\rangle = h(x)\} \\ \Lambda_1(x) &:= \{z \in C : \exists w \in F(z) \text{ with } (z, w) \in \Lambda(x)\} \\ \Lambda_2(x) &:= \{w \in H : \exists z \in C \text{ with } (z, w) \in \Lambda(x)\}. \end{aligned} \tag{17}$$

Remark 3 For h as in (16), we have that

$$\begin{aligned} \Lambda(x) &= \{(z, w) \in G_C(F) : \langle w, x - z\rangle = h(x)\} \\ &= \{(z, w) \in G_C(F) : \langle w, x - z\rangle \geq h(x)\}. \end{aligned} \tag{18}$$

Indeed, we clearly have

$$\{(z, w) \in G_C(F) : \langle w, x - z\rangle = h(x)\} \subset \{(z, w) \in G_C(F) : \langle w, x - z\rangle \geq h(x)\}.$$

The opposite inclusion follows directly from the fact that

$$h(x) \geq \langle w, x - z\rangle \geq h(x),$$

where we used the definition of h in the first inequality. Hence, under the assumptions of Proposition 3, for every $\bar{x} \in \bar{S}$ we must have $h(\bar{x}) = 0$. In this situation, (18) yields

$$\begin{aligned}\Lambda(\bar{x}) &= \{(z, w) \in G_C(F) : \langle w, \bar{x} - z\rangle = 0\} \\ &= \{(z, w) \in G_C(F) : \langle w, \bar{x} - z\rangle \geq 0\}.\end{aligned} \tag{19}$$

Remark 4 Fix $\bar{x} \in \bar{S}$ and assume that F is maximally monotone. If assumptions (i) or (ii) of Proposition 3 hold, we must have that $h(\bar{x}) = 0$. This yields

$$\{\bar{x}\} \times \{F(\bar{x})\} \subset \Lambda(\bar{x}).$$

Indeed, we may use $z := \bar{x}$ in the scalar product in the definition of $\Lambda(\bar{x})$. As a direct consequence of the inclusion above, we have that

$$\bar{x} \in \Lambda_1(\bar{x}),$$

and

$$F(\bar{x}) \subset \Lambda_2(\bar{x}). \tag{20}$$

The aim of the next lemma is to find further relations between the sets $\Lambda_2(\bar{x})$, $\partial h(\bar{x})$ and $F(\bar{x})$.

Lemma 2 *Let F be a maximally monotone point-to-set mapping and $C \subset H$ a closed and convex set such that assumptions* (i) *or* (ii) *in Proposition* 3 *hold. Let $\bar{S} \neq \emptyset$ be as defined by* (7) *and fix $\bar{x} \in \bar{S}$. Consider h as defined by* (16). *With the notation used in* (17) *and* (19), *the following properties hold.*

(a) *For all $x \in C \cap \operatorname{dom} h$ and all nonzero vector $d \in H$, we have*

$$h'(x; d) \geq \sup_{w \in \Lambda_2(x)} \langle w, d\rangle.$$

Consequently, $\Lambda_2(x) \subseteq \partial h(x)$.

(b) *If F is paramonotone and $\bar{x} \in \bar{S}$, then $\Lambda_2(\bar{x}) = F(\bar{x}) \subset \partial h(\bar{x})$.*

Proof

(a) Given $d \in H$, use the definition of h and $\Lambda(x)$, to write

$$\begin{aligned} h(x + td) &= \sup\{\langle w, (x + td) - y\rangle : (y, w) \in G_C(F)\} \\ &\geq \sup\{\langle w, x - y\rangle + t\langle w, d\rangle : (y, w) \in \Lambda(x)\} \\ &= h(x) + t \sup\{\langle w, d\rangle : w \in \Lambda_2(x)\}, \end{aligned}$$

where we also used the fact that $\Lambda(x) \subset G_C(F)$ in the first inequality. The above expression yields,

$$\frac{h(x+td)-h(x)}{t} \geq \sup_{w\in\Lambda_2(x)} \langle w, d\rangle \geq \langle u, d\rangle, \quad \text{for all} \quad u \in \Lambda_2(x).$$

Thus, for all $d \neq 0$ we have that $h'(x; d) \geq \langle u, d\rangle$ for all $u \in \Lambda_2(x)$. By [17, Proposition 4.1.6], we deduce that $u \in \partial h(x)$, and hence, $\Lambda_2(x) \subseteq \partial h(x)$.

(b) Let $w \in \Lambda_2(\bar{x})$. By (19), there exists $y \in C \cap F^{-1}(w)$ such that $\langle w, \bar{x} - y\rangle = 0$. Since $\bar{x} \in \bar{S}$, there exists $\bar{w} \in F(\bar{x})$ such that

$$\langle \bar{w}, y - \bar{x}\rangle \geq 0, \quad \text{for all} \quad y \in C. \tag{21}$$

By monotonicity of F, for all $w \in F(y)$, we have

$$0 \leq \langle w - \bar{w}, y - \bar{x}\rangle = \langle w, y - \bar{x}\rangle + \langle \bar{w}, \bar{x} - y\rangle \leq 0,$$

where we used (21) and the fact that $\langle w, y - \bar{x}\rangle = 0$. Altogether, we have

$$\langle w - \bar{w}, y - \bar{x}\rangle = 0 \quad \text{with} \quad w \in F(y) \quad \text{and} \quad \bar{w} \in F(\bar{x}).$$

Since F is paramonotone, we have $\bar{w} \in F(y)$ and $w \in F(\bar{x})$. Hence, for all $w \in \Lambda_2(\bar{x})$, we have $w \in F(\bar{x})$. Therefore, $\Lambda_2(\bar{x}) \subseteq F(\bar{x})$. The opposite inclusion follows from (20). The inclusion involving the subgradient of h follows directly from (a) for $x = \bar{x}$.

□

Remark 5 For $\bar{x} \in \bar{S}$ and $\varepsilon \geq 0$, we can define enlargements of the set $\Lambda_2(\bar{x})$ given in (19) as follows.

$$\Lambda_2^{\varepsilon}(x) := \{w \in R(F) : \exists z \in C \cap F^{-1}(w) \quad \text{with} \quad \langle w, x - z\rangle \geq -\varepsilon\}. \tag{22}$$

With straightforward modifications, the proof of Lemma 2 (a) can be adapted to show that $\Lambda_2^{\varepsilon}(x) \subseteq \partial_{\varepsilon} h(x)$.

4 Weak sharp solutions for generalized variational inequalities

In this section, we consider GVIP with solution set $\bar{S}$. Unless specifically stated, F is a set-valued mapping which is not necessarily maximally monotone.

Definition 5 We say that $\bar{S}$ is *weak sharp* if there exists $\alpha > 0$ such that

$$\alpha B \subseteq F(\bar{x}) + (T_C(\bar{x}) \cap N_{\bar{S}}(\bar{x}))^{\circ}, \quad \text{for all} \quad \bar{x} \in \bar{S}, \tag{23}$$

where B denotes the unit ball with center at origin in H. In this situation, we say that $\bar{S}$ is *weak sharp with parameter* α.

We will consider in our analysis the following alternative concept of weak sharp solution, introduced in [6,24] in the context of error bounds.

Definition 6 Let g be a gap function for GVIP (as given in Definition 3) and assume that g is convex, proper and lower semicontinuous. We say that $\bar{S}$ is *weak sharp with respect to g* if there exists $\alpha > 0$ such that

$$\alpha B \subseteq \partial g(\bar{x}) + (T_C(\bar{x}) \cap N_{\bar{S}}(\bar{x}))^{\circ}, \quad \text{for all} \quad \bar{x} \in \bar{S}. \tag{24}$$

In this situation, we say that $\bar{S}$ is *g-weak sharp with parameter α*.

The following proposition gives the characterization of a weak sharp solution set, and will be used in the next section for establishing finite convergence.

Proposition 4 *The solution set $\bar{S}$ of GVIP is weak sharp with parameter α if and only if for every $\bar{x} \in \bar{S}$ there exists $\bar{u} \in F(\bar{x})$ such that*

$$\langle \bar{u}, v \rangle \geq \alpha \|v\|, \tag{25}$$

for all $v \in V(\bar{x}) = T_C(\bar{x}) \cap N_{\bar{S}}(\bar{x})$.

Proof Assume that the solution set $\bar{S}$ of GVIP is weak sharp. Then, for all $x \in B$ and every $\bar{x} \in \bar{S}$, we have

$$\alpha x \in F(\bar{x}) + (V(\bar{x}))^{\circ}.$$

That is, there exist $\bar{u} \in F(\bar{x})$ and $z \in (V(\bar{x}))^{\circ}$ such that $\alpha x - \bar{u} = z \in (V(\bar{x}))^{\circ}$. Hence, for all $v \in V(\bar{x})$ with $v \neq 0$, we have $\langle \alpha x - \bar{u}, v \rangle \leq 0$, equivalently, $\alpha \langle x, v \rangle \leq \langle \bar{u}, v \rangle$. Take $x = \frac{v}{\|v\|}$, then $\alpha \left\langle \frac{v}{\|v\|}, v \right\rangle \leq \langle \bar{u}, v \rangle$. That is, there exists $\bar{u} \in F(\bar{x})$ such that

$$\alpha \|v\| \leq \langle \bar{u}, v \rangle, \quad \text{for all} \quad v \in V(\bar{x}).$$

Conversely, assume that (25) holds and let $x \in B$. For $v \in V(\bar{x})$ and $\bar{u} \in F(\bar{x})$ as in (25), we can write

$$\begin{aligned} \langle \alpha x - \bar{u}, v \rangle &= \alpha \langle x, v \rangle - \langle \bar{u}, v \rangle \\ &\leq \alpha \langle x, v \rangle - \alpha \|v\| \\ &= \alpha(\langle x, v \rangle - \|v\|) \leq 0, \end{aligned}$$

where we used (25) in the first inequality and the fact that $x \in B$ in the second one. This implies that for all $x \in B$ there exists $\bar{u} \in F(\bar{x})$ such that

$$\alpha x - \bar{u} \in (V(\bar{x}))^{\circ}.$$

Equivalently, $\alpha x \in F(\bar{x}) + (V(\bar{x}))^{\circ}$ for all $x \in B$. Hence, (23) holds. □

The next proposition gives the necessary condition for a weak sharp solution set of a GVIP in terms of global error bound.

Proposition 5 *If the solution set $\bar{S}$ of the GVIP is weak sharp with parameter α, then*

$$h(x) \geq \alpha d(x, \bar{S}), \quad \text{for all } x \in C. \tag{26}$$

Proof Given a fixed $x \in C$, set $\bar{x} = P_{\bar{S}}(x)$. Then,

$$x - \bar{x} \in T_C(\bar{x}) \cap N_{\bar{S}}(\bar{x}) := V(\bar{x}).$$

By Proposition 4, there exists $\bar{u} \in F(\bar{x})$ such that

$$\langle \bar{u}, v \rangle \geq \alpha \|v\|, \quad \text{for all } v \in V(\bar{x}).$$

In particular, $\langle \bar{u}, x - \bar{x} \rangle \geq \|x - \bar{x}\|$. The definition of h and the inequality above yield

$$h(x) \geq \langle \bar{u}, x - \bar{x} \rangle \geq \alpha \|x - \bar{x}\| = \alpha d(x, \bar{S}).$$

Hence, (26) holds. □

The following propositions give some nice properties in terms of directional derivatives or subdifferential of a function if the global error bound condition (26) is satisfied.

Proposition 6 *Let $\bar{x} \in \bar{S}$. If*

$$h(x) \geq \alpha d(x, \bar{S}), \quad \text{for all} \quad x \in C \quad \text{and} \quad \alpha > 0, \tag{27}$$

then $h'(\bar{x}; v) \geq \alpha \|v\|$ for all $v \in V(\bar{x}) = T_C(\bar{x}) \cap N_{\bar{S}}(\bar{x})$.

Proof Assume that (27) holds. If $v \in V(\bar{x})$ with $v = 0$, then trivially we are done. So, we assume that $v \in V(\bar{x})$ but $v \neq 0$. In this case,

$$\langle v, v \rangle = \|v\|^2 > 0. \tag{28}$$

On the other hand, the fact that $v \in V(\bar{x})$ implies that, in particular, $v \in N_{\bar{S}}(\bar{x})$, and hence

$$\langle v, \bar{y} - \bar{x} \rangle \leq 0, \quad \text{for all} \quad \bar{y} \in \bar{S}. \tag{29}$$

This means that

$$\bar{S} \subset H_v^- := \{x : \langle v, x - \bar{x} \rangle \leq 0\}. \tag{30}$$

Since $v \in T_C(\bar{x})$, we have

$$v = \lim_{k \to \infty} v_k, \tag{31}$$

with $\bar{x} + t_k v_k \in C$ for all k and $t_k \downarrow 0$. Combining (31) and (28), we deduce that for all $k \geq k_0$, $\langle v_k, v \rangle > 0$, that is,

$$\langle (\bar{x} + t_k v_k) - \bar{x}, v \rangle = t_k \langle v_k, v \rangle > 0, \quad \text{for all} \quad k \geq k_0.$$

Equivalently, $(\bar{x} + t_k v_k) \notin H_v^-$ for all $k \geq k_0$. This implies that their projection onto the half-space H_v^- will belong to the boundary of H_v^-, which is

$$H_v := \{x : \langle v, x - \bar{x} \rangle = 0\}.$$

Denote by $\bar{z}_k := P_{H_v}(\bar{x} + t_k v_k)$ for all $k \geq k_0$. Using also the inclusion in (30), we can write

$$\begin{aligned} d(\bar{x} + t_k v_k, \bar{S}) &\geq d(\bar{x} + t_k v_k, H_v^-) \\ &= d(\bar{x} + t_k v_k, H_v) \\ &= \|\bar{x} + t_k v_k - P_{H_v}(\bar{x} + t_k v_k)\| \\ &= \|\bar{x} + t_k v_k - \bar{z}_k\|. \end{aligned} \tag{32}$$

By construction, the vector $(\bar{x} + t_k v_k - \bar{z}_k)$ is a positive multiple of v. Namely, there is a positive scalar λ_k such that

$$\bar{x} + t_k v_k - \bar{z}_k = \lambda_k v.$$

Let us compute λ_k. Since $\langle v, \bar{z}_k - \bar{x} \rangle = 0$, we have

$$0 = \langle v, \bar{z}_k - \bar{x} \rangle = \langle v, t_k v_k - \lambda_k v \rangle,$$

which readily implies that $\lambda_k = \dfrac{t_k \langle v_k, v \rangle}{\|v\|^2}$. This gives

$$\bar{z}_k = (\bar{x} + t_k v_k) - \frac{t_k \langle v_k, v \rangle}{\|v\|^2} v.$$

Therefore,

$$\|\bar{z}_k - (\bar{x} + t_k v_k)\| = \frac{t_k \langle v_k, v \rangle}{\|v\|^2} \|v\| = \frac{t_k \langle v_k, v \rangle}{\|v\|}.$$

Then from (32) and (27), we have

$$h(\bar{x} + t_k v_k) \geq \alpha d(\bar{x} + t_k v_k, \bar{S}) \geq \alpha \frac{t_k \langle v_k, v \rangle}{\|v\|}.$$

Since $h(\bar{x}) = 0$, we have

$$\frac{h(\bar{x} + t_k v_k) - h(\bar{x})}{t_k} \geq \alpha \frac{\langle v_k, v \rangle}{\|v\|}.$$

Taking limit for $k \to \infty$ and recalling the fact that $t_k \downarrow 0$, we obtain

$$h'(\bar{x}, v) \geq \alpha \|v\|,$$

as claimed. □

Proposition 7 *Let $\bar{x} \in \bar{S}$ and assume that (27) holds. If $\bar{x} \in \operatorname{int}(\operatorname{dom} h)$ and h is continuous at $\bar{x}$, then there exists $\bar{u} \in \partial h(\bar{x})$ such that $\langle \bar{u}, v\rangle \geq \alpha\|v\|$ for all $v \in V(\bar{x})$ and $\alpha > 0$.*

Proof From (27), we have

$$h'(\bar{x}, v) \geq \alpha\|v\|, \quad \text{for all} \quad v \in V(\bar{x}) \quad \text{and} \quad \alpha > 0. \tag{33}$$

Since $\bar{x} \in \operatorname{int}(\operatorname{dom} h)$ and h is continuous at $\bar{x}$, we can use [17, Proposition 4.1.6 (b)] to deduce that

$$h'(\bar{x}, v) = \max_{z \in \partial h(\bar{x})} \langle v, z\rangle.$$

That is, there exists $\bar{u} \in \partial h(\bar{x})$ such that $h'(\bar{x}, v) = \langle v, \bar{u}\rangle$. From (33), we have $\langle \bar{u}, v\rangle \geq \alpha\|v\|$. □

Remark 6 In several papers, condition (26) is used as a definition of weak sharp solutions of variational inequalities or optimization problems, see, for example [6,23, 24] and the references therein. In Propositions 6 and 7, we proved that if condition (26) is satisfied then we have some nice properties in terms of directional derivatives or subdifferential of a function. However, we could not establish the sufficient condition for a weak sharp solution set of GVIP in terms of global error bound. This remains an open problem, and the topic of our future research.

5 Finite convergence analysis

We say that an algorithm has finite convergence whenever all iterates belong to the solution set after a finite number of iterations. In previous sections, we extended the notion of weak sharpness to the framework of set-valued mappings. In the present section, we show that our definition of weak sharpness ensures finite convergence in this more general framework.

Moreover, at the end of this section, we observe that finite convergence can also be obtained for paramonotone problems under the relaxed assumption that the solution set is g-weak sharp.

The following theorem establishes a necessary condition for finite convergence. Its proof is standard, we present it here for completeness.

Theorem 1 *Let $F : H \rightrightarrows H$ be a set-valued mapping with nonempty domain. Let $\{x_k\} \subseteq C \cap D(F)$ be a sequence generated by an algorithm with finite termination. In other words, there exists $k_0 \in \mathbb{N}$ such that $x_k \in \bar{S}$ for all $k \geq k_0$. In this situation,*

$$0 \in \liminf_{k\to\infty} P_{T_C(x_k)}(-F(x^k)). \tag{34}$$

Proof Assume that there exists $k_0 \in \mathbb{N}$ such that $x_k \in \bar{S}$ for all $k \geq k_0$. Then, by definition of $\bar{S}$, there exists $u_k \in F(x_k)$ such that

$$\langle u_k, y - x_k \rangle \geq 0, \quad \text{for all} \quad y \in C,$$

that is, $-u_k \in N_C(x_k)$. Hence, $P_{T_C(x_k)}(-u_k) = 0$, and thus, there exists $z^k = P_{T_C(x_k)}(-u_k) \subset P_{T_C(x_k)}(-F(x^k))$ such that $0 = \lim_{k\to\infty} z^k$. Hence, (34) holds. □

The following theorem establishes conditions under which (34) is also sufficient for a monotone mapping.

Theorem 2 *Let $F : H \rightrightarrows H$ be a monotone set-valued mapping with nonempty domain. Assume that $\bar{S}$ is closed and convex, and weak sharp with parameter α. Let $\{x_k\} \subseteq C \cap D(F)$ be a sequence generated by an algorithm such that (34) holds. Then, there exists $k_0 \in \mathbb{N}$ such that $x_k \in \bar{S}$ for all $k \geq k_0$.*

Proof Assume that the conclusion is not true. Then, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \notin \bar{S}$ for all $j \in \mathbb{N}$. Without loss of generality, we may rename $\{x_{k_j}\}$ as $\{y_j\}$ for simplicity. Then, $y_j \notin \bar{S}$ for all $j \in \mathbb{N}$. Let $z_j = P_{\bar{S}}(y_j)$. Since $y_j \notin \bar{S}$ and $z_j \in \bar{S}$, we have $\|z_j - y_j\| > 0$ for all j. By construction, we have that

$$v_j := y_j - z_j \in T_C(z_j) \cap N_{\bar{S}}(z_j) \text{ and } z_j - y_j \in T_C(y_j).$$

By weak sharpness and Proposition 4 applied to $\bar{x} = z_j$ and to $v_j = y_j - z_j \in T_C(z_j) \cap N_{\bar{S}}(z_j)$, there exists $\hat{u}_j \in F(z_j)$ such that

$$\langle \hat{u}_j, y_j - z_j \rangle \geq \alpha \|y_j - z_j\|, \quad \text{for all} \quad j,$$

which yields, for all j

$$\alpha \leq \left\langle -\hat{u}_j, \frac{z_j - y_j}{\|y_j - z_j\|} \right\rangle \tag{35}$$

By (34), for all k there exists $u_j \in F(y^j)$ such that

$$\lim_{k\to\infty} P_{T_C(y_j)}(-u_j) = 0. \tag{36}$$

By Lemma 1, monotonicity of F, and (35), we have

$$\begin{aligned} \alpha &\leq \left\langle -\hat{u}_j, \frac{z_j - y_j}{\|y_j - z_j\|} \right\rangle \leq \left\langle -u_j, \frac{z_j - y_j}{\|y_j - z_j\|} \right\rangle \\ &\leq \max\{\langle -u_j, \eta \rangle \ : \ \eta \in T_C(y_j), \ \|\eta\| \leq 1\} \\ &= \|P_{T_C(y_j)}(-u_j)\|, \end{aligned} \tag{37}$$

where we used monotonicity in the second inequality, and Lemma 1(b) in the last equality. The above expression contradicts (36). Therefore, we must have finite convergence. □

By combining Theorems 1 and 2, we have the following result.

Theorem 3 *Let $F : H \rightrightarrows H$ be a monotone set-valued mapping with nonempty domain. Assume that $\bar{S}$ is closed and convex, and weak sharp with parameter α. Let $\{x_k\} \subseteq C \cap D(F)$ be a sequence generated by an algorithm. Then, $x_k \in \bar{S}$ for sufficiently large k if and only if* (34) *holds.*

Remark 7 Theorem 3 can be seen as a generalization and refinement of Theorem 5.2 in [14] and Theorem 3.2 in [20].

Remark 8 The assumption of monotonicity in Theorem 3 is standard in the literature of weak-sharp minima (see, e.g., Theorem 2 in [14] and Theorem 4.2 in [19]). This assumption is used for proximal-like methods and its variants, such as those studied in [4]. The weak sharpness assumption has been mainly investigated for the point-to-point case, and its validity in this case is equivalent to an error bound condition for the gap function of the variational inequality (see, e.g., Theorem 3.1 in [11]). As for the point-to-set case, examples of weak sharpness of the solution set can be found in [19] for variational inequalities arising from nonsmooth constrained optimization problems.

To establish finite convergence without monotonicity, we need to strengthen condition (34) (see condition (ii) below). We also need F to have a continuity property. On the other hand, we do not assume that the sequence is bounded, which is a standard assumption used in the literature.

Theorem 4 *Let $F : H \rightrightarrows H$ be a set-valued mapping with nonempty domain. Assume that the solution set $\bar{S}$ is closed and convex, and weak sharp with parameter α. Let $\{x_k\} \subseteq C \cap D(F)$ be a sequence generated by an algorithm that verifies the following properties.*

(i) $\lim_k d(x^k, \bar{S}) = 0$, *and*
(ii)
$$\{0\} = \liminf_{k\to\infty} P_{T_C(x_k)}(-F(x^k)). \tag{38}$$

Consider the following two assumptions.

(A1) *The map F is Lipschitz continuous on $C \cap D(F)$.*
(A2) *The space H is finite dimensional, the set $F(\bar{S})$ is bounded, and F is closed valued and inner-semicontinuous on $\bar{S}$.*

Under either of the assumptions (A1) *or* (A2)*, there exists $k_0 \in \mathbb{N}$ such that $x_k \in \bar{S}$ for all $k \geq k_0$.*

Proof Consider first assumption (A1). If the conclusion is not true, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \notin \bar{S}$ for all $j \in \mathbb{N}$. Without loss of generality, we denote this subsequence by $\{x_j\}$. As in the proof of the previous theorem, let $z_j = P_{\bar{S}}(x_j)$. By assumption, we have that

$$\lim_k d(x_j, \bar{S}) = \lim_k \|z_j - x_j\| = 0. \tag{39}$$

By weak sharpness and Proposition 4 applied to $\bar{x} = z_j$ and to $v_j = x_j - z_j \in T_C(z_j) \cap N_{\bar{S}}(z_j)$, there exists $\hat{u}_j \in F(z_j)$ such that

$$\begin{aligned} \langle \hat{u}_j, x_j - z_j \rangle &\geq \alpha \|x_j - z_j\|, \\ \alpha &\leq \left\langle -\hat{u}_j, \frac{z_j - x_j}{\|x_j - z_j\|} \right\rangle. \end{aligned} \tag{40}$$

Since F is Lipschitz continuous on $C \cap D(F)$, there exists a constant $\lambda > 0$ such that

$$F(z_j) \subset F(x_j) + \lambda \|z_j - x_j\| B. \tag{41}$$

By (39), we can take j_0 such that, for $j \geq j_0$ we have that $\lambda \|z_j - x_j\| < \alpha/2$. Hence, for $j \geq j_0$, (41) yields

$$F(z_j) \subset F(x_j) + (\alpha/2) B. \tag{42}$$

For $j \geq j_0$, take $\hat{u}_j \in F(z_j)$ as in (40). By (42), there exist $\eta_j \in F(x_j)$ and $w_j \in B$ such that

$$\hat{u}_j = \eta_j + (\alpha/2) w_j, \quad \text{for all } j \geq j_0. \tag{43}$$

On the other hand, (38) implies that for every sequence $\theta_j \in P_{T_C(x_j)}(-F(x_j))$, we must have

$$\lim_j \|\theta_j\| = 0.$$

Using this fact for $\theta_j := P_{T_C(x_j)}(-\eta_j) \in P_{T_C(x_j)}(-F(x_j))$, there exists $j_1 \geq j_0$ such that

$$\|P_{T_C(x_j)}(-\eta_j)\| < \alpha/4, \quad \text{for all} \quad j \geq j_1.$$

Altogether, (40) and (43) yield, for $j \geq j_1$

$$\begin{aligned}
\alpha &\leq \left\langle \eta_j + (\alpha/2) w_j, \frac{x_j - z_j}{\|x_j - z_j\|} \right\rangle \\
&= \left\langle \eta_j, \frac{x_j - z_j}{\|x_j - z_j\|} \right\rangle + (\alpha/2) \left\langle w_j, \frac{x_j - z_j}{\|x_j - z_j\|} \right\rangle \\
&\leq (\alpha/2) + \left\langle -\eta_j, \frac{z_j - x_j}{\|z_j - x_j\|} \right\rangle \\
&\leq (\alpha/2) + \max\{\langle -\eta_j, u\rangle \ : \ u \in T_C(x_j), \ \|u\| \leq 1\} \\
&= (\alpha/2) + \|P_{T_C(x_j)}(-\eta_j)\| \\
&\leq (\alpha/2) + (\alpha/4) = (3/4)\alpha,
\end{aligned}$$

where we used the fact that $w_j \in B$ in the second inequality, Lemma 1 (b) in the last equality, and the assumption on j_1 in the last inequality. The above expression entails a contradiction, which implies that the algorithm must have finite convergence as stated.

Consider now assumption (A2). The proof follows the same steps until equation (40). Since $F(\bar{S})$ is bounded, there exists $\rho > 0$ such that $\|\hat{u}_j\| \leq \rho$. Take this ρ, and set $\varepsilon := \alpha/2$ in Proposition 1 (ii) to find a $\delta > 0$ such that

$$F(z_j) \cap B[0, \rho] \subset F(x) + (\alpha/2)B, \ \forall\, x \in B(z_j, \delta).$$

Take j_0 such that $x_j \in B(z_j, \delta)$ whenever $j \geq j_0$, so we deduce that

$$F(z_j) \cap B[0, \rho] \subset F(x_j) + (\alpha/2)B, \ \forall\, j \geq j_0.$$

Since $\hat{u}_j \in F(z_j) \cap B[0, \rho]$ we have $\hat{u}_j \in F(x_j) + (\alpha/2)B, \ \forall\, j \geq j_0$. Hence, we are in the situation of (43). The rest of the proof follows the same steps. □

Remark 9 Most algorithms for the variational inequality problem have some kind of monotonicity assumption on the underlying operator. In the absence of any monotonicity assumption, the usual setting is the one that has a point-to-point and continuous operator (in the classical sense), see the recent work [21]. For a point-to-point continuous map, all assumptions in (A2) of Theorem 4 hold, as long as $F(\bar{S})$ is a bounded set and $\bar{S}$ is weak sharp. The Lipschitz assumption in Theorem 4 (A1) can be relaxed by the uniform continuity assumption on F. More precisely, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *uniformly continuous on a set* $W \subset \mathbb{R}^n$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every pair of points $x, x' \in W$ we have

$$\|x - x'\| < \delta \quad \Rightarrow \quad G(x) \subset G(x') + \varepsilon B.$$

The proof under this less restrictive assumption follows the same steps as those in Theorem 4 under assumption (A1).

Remark 10 When F is paramonotone, we have seen in Lemma 2 (b) that

$$F(\bar{x}) \subset \partial h(\bar{x}) \quad \text{for all} \quad \bar{x} \in \bar{S}.$$

In this situation, whenever $\bar{S}$ is weak sharp with parameter α, then it is h-weak sharp with parameter α. This implies that, for a paramonotone F, h-weak sharpness is less restrictive than weak sharpness.

Remark 11 Under assumption (34) and h-weak sharpness of $\bar{S}$, Zhou and Wang established in [24, Theorem 3.1] finite convergence to $\bar{S}$. This implies that, when F is paramonotone, the assumption of weak-sharpness may be relaxed by h-weak sharpness, and finite convergence holds automatically under assumption (34). In fact, paramonotonicity is only used to show that h-weak sharpness is a less restrictive assumption than sharpness. Otherwise, the proof of the following theorem follows directly from [24, Theorem 3.1]. We state this result in our framework.

Theorem 5 *Let F be paramonotone and assume $\bar{S}$ is h-weak sharp. Let $\{x_k\} \subseteq C \cap D(F)$ be a sequence generated by an algorithm such that (34) holds. Then, there exists $k_0 \in \mathbb{N}$ such that $x_k \in \bar{S}$ for all $k \geq k_0$.*

Acknowledgements This research was partially done during the visit of second author to the School of Information Technology and Mathematical Sciences at University of South Australia, and King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia. This research was funded by the National Plan for Science, Technology and Innovation (MAARIFAH)—King Abdulaziz City for Science and Technology—through the Science and Technology Unit at King Fahd University of Petroleum and Minerals (KFUPM)—the Kingdom of Saudi Arabia, award number 12-MAT3023-24.

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