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A model with uncountable set of spin values on a Cayley tree: phase transitions

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Abstract In this paper we consider a model with nearest-neighbor interactions and with the set [0,1] of spin values, on a Cayley tree of order two. This model depends on two parameters $n \in \mathbb{N}$ and $\theta \in [0, 1)$. We prove that if $0 \leq \theta \leq \frac{2n+3}{2(2n+1)}$, then for the model there exists a unique translational-invariant Gibbs measure; If $\frac{2n+3}{2(2n+1)} < \theta < 1$, then there are three translational-invariant Gibbs measures (i.e. phase transition occurs).

Keywords Cayley tree · Configuration · Gibbs measures · Phase transitions

Mathematics Subject Classification Primary 82B05 · 82B20; Secondary 60K35

1 Introduction

The notion of a Gibbs measure was first introduced by R.L.Dabrushin as well as Lanford and Ruelle [1,2], makes use of systems of compatible conditional probabilities with respect to the outside of finite subsets, when the outside is fixed in a boundary condition, to reach thereafter infinite-volume quantities. A central problem in the theory of Gibbs measures is to describe infinite (or limiting) Gibbs measures corresponding to a given Hamiltonian (see [5–8]).

In [3] the Potts model with countable set Φ of spin values on Z^d was considered and it was proved that with respect to Poisson distribution on Φ the set of limiting

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Gibbs measure is not empty. In [4] the Potts model with a nearest neighbor interaction and countable set of spin values on a Cayley tree is studied.

It is well known, the XY model is an example with an uncountable single-spin space $\Omega = \{x \in R^2 : \|x\|_2 = 1\}$. In [5–8] several models (Hamiltonians) with-nearest-neighbor interactions and with the set [0, 1] of spin values, on a Cayley tree were considered. In these papers translation-invariant Gibbs measures are studied via a non-linear functional (integral) equation.

In the present note we continue the investigation from [5] and consider a model with nearest-neighbor interactions and local state space given by the uncountable set [0, 1] on a Cayley tree. Note that, in [5] it was proved that the considered model has at least two translational-invariant Gibbs measure. Here we prove that our model has exactly three translation-invariant Gibbs measures on a Cayley tree of order two.

Let us give basic definitions.

The *Cayley tree* Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L the set of edges. Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them and we denote $l = \langle x, y\rangle$. A collection of nearest neighbor pairs $\langle x, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_{d-1}, y\rangle$ is called a *path* from x to y. The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x and y.

For a fixed $x^0 \in V$, called the root, we set

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m$$

and denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n,$$

the set of direct successors of x.

Consider models where the spin takes values in the set [0, 1], and is assigned to the vertexes of the tree. For $A \subset V$ a configuration σ_A on A is an arbitrary function $\sigma_A : A \mapsto [0, 1]$. Denote $\Omega_A = [0, 1]^A$ the set of all configurations on A. A configuration σ on V is then defined as a function $x \in V \mapsto \sigma(x) \in [0, 1]$; the set of all configurations is $[0, 1]^V$.

The (formal) Hamiltonian of the model is:

$$H(\sigma) = -J \sum_{<x,y>\in L} \xi_{\sigma(x),\sigma(y)}, \tag{1.1}$$

where $J \in R\setminus\{0\}$ and $\xi : (u, v) \in [0, 1]^2 \mapsto \xi_{u,v} \in R$ is a given bounded, measurable function.

Let λ be the Lebesgue measure on [0, 1]. On the set of all configurations on A the a priori measure λ_A is introduced as the $|A|$ fold product of the measure λ. Here and further on $|A|$ denotes the cardinality of A. We consider a standard sigma-algebra $\mathcal{B}$

of subsets of $\Omega = [0, 1]^V$ generated by the measurable cylinder subsets. A probability measure μ on $(\Omega, \mathcal{B})$ is called a *Gibbs measure* (with Hamiltonian H) if it satisfies the DLR equation, namely for any $n = 1, 2, \ldots$ and $\sigma_n \in \Omega_{V_n}$:

$$\mu(\{\sigma \in \Omega : \sigma|_{V_n} = \sigma_n\}) = \int_{\Omega} \mu(d\omega)\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n),$$

where $\nu^{V_n}_{\omega|_{W_{n+1}}}$ is the conditional Gibbs density

$$\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n) = \frac{1}{Z_n(\omega|_{W_{n+1}})} \exp(\beta H(\sigma_n \mid |\omega|_{W_{n+1}})),$$

and $\beta \geq 0$ is a free parameter proportional to the inverse temperature.

2 Non-uniqueness of Gibbs measures

Let

$$C^+[0, 1] = \{f \in C[0, 1] : f(x) \geq 0\}.$$

For every $k \in \mathbb{N}$ we consider an integral operator H_k acting in the cone $C^+[0, 1]$ as

$$(H_k f)(t) = \int_0^1 K(t, u) f^k(u)du, \quad k \in \mathbb{N}.$$

The operator H_k is called Hammerstein's integral operator of order k. This operator is well known to generate ill-posed problems. Clearly, if $k \geq 2$ then H_k is a nonlinear operator.

It is known that the set of translation invariant Gibbs measures of the model (1.1) is described by the fixed points of the Hammerstein's operator (see [6]).

In this paper we take $k = 2$ for the model (1.1) and we take concrete ξ in the following form

$$\xi_{t,u} = \xi_{t,u}(\theta, \beta) = \frac{1}{J\beta} \ln\left(1 + \theta \sqrt[2n+1]{4\left(t - \frac{1}{2}\right)\left(u - \frac{1}{2}\right)}\right), \quad t, u \in [0, 1] \quad (2.1)$$

where $0 \leq \theta < 1$. Then for the Kernel $K(t, u)$ of the Hammerstein's operator H_2 we have

$$K(t, u) = 1 + \theta \sqrt[2n+1]{4\left(t - \frac{1}{2}\right)\left(u - \frac{1}{2}\right)}.$$

We defined the operator $V_2 : (x, y) \in R^2 \to (x', y') \in R^2$ by

$$V_2 : \begin{cases} x' = x^2 + \frac{2n+1}{2n+3} \sqrt[2n+1]{4}\theta^2 y^2; \\ y' = 2 \cdot \frac{2n+1}{2n+3}\theta xy. \end{cases} \quad (2.2)$$

Proposition 2.1 *A function $\varphi \in C[0, 1]$ is a solution of the Hammerstein's equation*

$$(H_2 f)(t) = f(t) \quad (2.3)$$

iff $\varphi(t)$ has the following form

$$\varphi(t) = C_1 + C_2\theta \sqrt[2n+1]{4\left(t - \frac{1}{2}\right)},$$

where $(C_1, C_2) \in R^2$ is a fixed point of the operator V_2 (2.2).

Proof Necessariness Assume $\varphi \in C[0, 1]$ be a solution of the Eq. (2.3). Then we have

$$\varphi(t) = C_1 + C_2\theta \sqrt[2n+1]{4\left(t - \frac{1}{2}\right)}, \quad (2.4)$$

where

$$C_1 = \int_0^1 \varphi^2(u)\mathrm{d}u, \quad (2.5)$$

$$C_2 = \int_0^1 \sqrt[2n+1]{u - \frac{1}{2}} \cdot \varphi^2(u)\mathrm{d}u. \quad (2.6)$$

Substituting the function $\varphi(t)$ (2.4) into (2.5) we get

$$C_1 = C_1^2 + \frac{2n+1}{2n+3} \sqrt[2n+1]{4}\theta^2 C_2^2,$$

and substituting the function $\varphi(t)$ into (2.6) we get

$$C_2 = 2 \cdot \frac{2n+1}{2n+3}\theta C_1 C_2.$$

Thus, the point $(C_1, C_2) \in R^2$ is a fixed point of the operator V_2 (2.2).
Sufficiency Assume that, a point $(C_1, C_2) \in R^2$ is a fixed point of the operator V_2 define the function $\varphi(t) \in C[0, 1]$ by the equality

$$\varphi(t) = C_1 + C_2\theta \sqrt[2n+1]{4\left(t - \frac{1}{2}\right)}.$$

Then

$$(H_2\varphi)(t) = \int_0^1 \left(1 + \sqrt[2n+1]{4}\theta \sqrt[2n+1]{\left(t-\frac{1}{2}\right)\left(u-\frac{1}{2}\right)}\right)\varphi^2(u)\mathrm{d}u = \int_0^1 \varphi^2(u)\mathrm{d}u$$

$$+ \sqrt[2n+1]{4}\theta \sqrt[2n+1]{t-\frac{1}{2}} \int_0^1 \sqrt[2n+1]{u-\frac{1}{2}}\varphi^2(u)\mathrm{d}u = \int_0^1 \left(C_1 + C_2\theta \sqrt[2n+1]{4\left(u-\frac{1}{2}\right)}\right)^2 \mathrm{d}u$$

$$+ \sqrt[2n+1]{4}\theta \sqrt[2n+1]{t-\frac{1}{2}} \int_0^1 \sqrt[2n+1]{u-\frac{1}{2}}\left(C_1 + C_2\theta \sqrt[2n+1]{4\left(u-\frac{1}{2}\right)}\right)^2 \mathrm{d}u$$

$$= C_1^2 \int_0^1 \mathrm{d}u + 2C_1C_2\theta \int_0^1 \sqrt[2n+1]{4\left(u-\frac{1}{2}\right)}\mathrm{d}u + \theta^2 C_2^2 \int_0^1 \left(\sqrt[2n+1]{4\left(u-\frac{1}{2}\right)}\right)^2 \mathrm{d}u$$

$$+ \sqrt[2n+1]{4}\theta \sqrt[2n+1]{t-\frac{1}{2}} \cdot (C_1^2 \int_0^1 \sqrt[2n+1]{u-\frac{1}{2}}\mathrm{d}u + 2C_1C_2\theta \sqrt[2n+1]{4} \int_0^1 \sqrt[2n+1]{\left(u-\frac{1}{2}\right)^2}\mathrm{d}u$$

$$+ \sqrt[2n+1]{16}\theta^2 C_2^2 \int_0^1 \sqrt[2n+1]{\left(u-\frac{1}{2}\right)^3}\mathrm{d}u). \tag{2.7}$$

Now, we use the following equalities

$$\int_0^1 \sqrt[2n+1]{u-\frac{1}{2}}\mathrm{d}u = 0;$$

$$\int_0^1 \sqrt[2n+1]{(u-\frac{1}{2})^2}\mathrm{d}u = \frac{2n+1}{2n+3} \cdot \frac{1}{\sqrt[2n+1]{4}};$$

$$\int_0^1 \sqrt[2n+1]{(u-\frac{1}{2})^3}\mathrm{d}u = 0;$$

Then from (2.7) we get

$$= C_1^2 + \frac{2n+1}{2n+3}\sqrt[2n+1]{4}\theta^2 C_2^2 + 2 \cdot \frac{2n+1}{2n+3}\theta C_1 C_2 \cdot \theta \sqrt[2n+1]{4\left(t-\frac{1}{2}\right)}$$

$$= C_1 + C_2\theta \sqrt[2n+1]{4\left(t-\frac{1}{2}\right)} = \varphi(t),$$

i.e. the function $\varphi(t)$ is a solution of the Eq. (2.3). □

Proposition 2.2 i) *If* $0 \le \theta \le \frac{2n+3}{2(2n+1)}$, *then the Hammerstein's operator* H_2 *has unique* (*nontrivial*) *positive fixed point in the* $C[0,1]$*;*

ii) *If* $\frac{2n+3}{2(2n+1)} < \theta < 1$, *then there are exactly three positive fixed points in* $C[0,1]$ *of the Hammerstein's operator.*

Proof It is easy to see, if $\theta = 0$ the Hammerstein's operator H_2 has unique nontrivial positive fixed points $\varphi(t) \equiv 1$.

Let $\theta \ne 0$. We consider the system of equations for a fixed point of the operator V_2:

$$\begin{cases} x^2 + \frac{2n+1}{2n+3} \sqrt[2n+1]{4}\theta^2 y^2 = x, \\ 2 \cdot \frac{2n+1}{2n+3}\theta x y = y. \end{cases} \tag{2.8}$$

Case $y = 0$. We get two solutions (0,0) and (1,0) in the (2.8). By Proposition 3.2 functions

$$\varphi(t) = \varphi_0(t) \equiv 0, \quad \varphi(t) = \varphi_0(t) \equiv 1$$

are solutions of the equation (2.8).

Case $y \ne 0$. Then from (2.8) we obtain $x = \frac{2n+3}{2(2n+1)\theta}$. Hence, from the first equation of (2.8) we get

$$y^2 = \frac{(2n+3)^2}{2(2n+1)^2 \sqrt[2n+1]{4}\theta^3} \cdot \left(1 - \frac{2n+3}{2(2n+1)\theta}\right). \tag{2.9}$$

Therefore, for $\theta \ge \frac{2n+3}{2(2n+1)}$ from (2.9) we obtain

$$y = y_1^{\pm} = \pm \frac{2n+3}{2(2n+1) \sqrt[2n+1]{2}\theta^2} \cdot \sqrt{\frac{2(2n+1)\theta - (2n+3)}{2n+1}}. \tag{2.10}$$

Consequently, in the case $0 \le \theta \le \frac{2n+3}{2(2n+1)}$ operator V_2 has two fixed points: (0,0), (1,0) and in the case $\frac{2n+3}{2(2n+1)} < \theta < 1$ the operator V_2 has four fixed points: (0,0), (1,0), (x_1, y_1^+) and (x_1, y_1^-), with $x_1 = \frac{2n+3}{2(2n+1)\theta}$.

Note that, there is no any other fixed point of V_2. □

Consequently,

$$\varphi_1(t) \equiv 1,$$

$$\varphi_2(t) = \frac{2n+3}{2(2n+1)\cdot\theta}\left(1 + \sqrt{\frac{2(2n+1)\cdot\theta - (2n+3)}{2n+1}} \cdot \sqrt[2n+1]{2(t - \frac{1}{2})}\right),$$

$$\varphi_3(t) = \frac{2n+3}{2(2n+1)\cdot\theta}\left(1 - \sqrt{\frac{2(2n+1)\cdot\theta - (2n+3)}{2n+1}} \cdot \sqrt[2n+1]{2\left(t - \frac{1}{2}\right)}\right)$$

are non trivial fixed points of the Hammerstein's operator H_2. Note that $\varphi_i(t) > 0$, for $i = 1, 2, 3$ and $t \in [0, 1]$. Thus we have proved the following

Theorem 2.3 i) *If* $0 \leq \theta \leq \frac{2n+3}{2(2n+1)}$, *then for the model* (1.1) *on Cayley tree* Γ^2 *there exists a unique translation-invariant Gibbs measure;*

ii) *If* $\frac{2n+3}{2(2n+1)} < \theta < 1$, *then for the model* (1.1) *on Cayley tree* Γ^2 *there are three translation-invariant Gibbs measures.*

Remark Note that, in [7] the case $n = 1$ of (2.1), is considered. In the case $n = 1$ from Theorem 2.3 we get Theorem 4.2 of [7].

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