Positivity (2017) 21:897–910
DOI 10.1007/s11117-016-0441-1

Positivity

Eigenstructure and iterates for uniquely ergodic Kantorovich modifications of operators

Margareta Heilmann[1] · Ioan Raşa[2]

Received: 25 April 2016 / Accepted: 17 August 2016 / Published online: 23 August 2016

Abstract We consider Markov operators L on $C[0, 1]$ such that for a certain $c \in [0, 1)$, $\|(Lf)'\| \leq c\|f'\|$ for all $f \in C^1[0, 1]$. It is shown that L has a unique invariant probability measure ν, and then ν is used in order to characterize the limit of the iterates L^m of L. When L is a Kantorovich modification of a certain classical operator from approximation theory, the eigenstructure of this operator is used to give a precise description of the limit of L^m. This way we extend some known results; in particular, we extend the domain of convergence of the dual functionals associated with the classical Bernstein operator, which gives a partial answer to a problem raised in 2000 by Cooper and Waldron (JAT 105:133–165, 2000, Remark after Theorem 4.20).

Keywords Uniquely ergodic operator · Kantorovich modification · Iterates of operators · Dual functionals

Mathematics Subject Classification 37A30 · 41A36

✉ Margareta Heilmann
heilmann@math.uni-wuppertal.de

Ioan Raşa
Ioan.Rasa@math.utcluj.ro

1 School of Mathematics and Natural Sciences, University of Wuppertal, Gaußstraße 20, 42119 Wuppertal, Germany

2 Department of Mathematics, Technical University, Str. Memorandumului 28, 400114 Cluj-Napoca, Romania

1 Introduction

The asymptotic behavior of the iterates of a linear operator L is investigated in connection with ergodic theory, approximation theory and related fields (see, e.g., [1–4,6–8,10–13,17–19,21,22]).

In this paper we study the iterates of certain Markov operators acting on $C[0, 1]$.

General criteria under which a Markov operator L is uniquely ergodic (i.e., L admits a unique invariant probability measure ν) can be found in [2,3,11,18]. In Sect. 2 we give a short and direct proof in a special case. Then ν is used in order to characterize the limits of the iterates of L.

The results are applied in Sect. 3 to the linking operators $B_{n,\rho}$, where $B_{n,1}$ is the genuine Bernstein-Durrmeyer operator and $B_{n,\infty}$ the classical Bernstein operator. The invariant probability measure $\nu_{n,k,\rho}$ for the kth order Kantorovich type modification $B_{n,\rho}^{(k)}$ of $B_{n,\rho}$ is related to the corresponding dual functional $u_{n,k,\rho}$ which accompanies the eigenstructure of $B_{n,\rho}^{(k)}$.

A similar approach is used in Sect. 4 in connection with the kth order Kantorovich type modification of the Bernstein-Durrmeyer operators with Jacobi weights. A remarkable feature of this case is that now the eigenpolynomials are orthogonal and independent of n; consequently, the dual functionals and the limits of the iterates are independent of n.

In Sect. 5 we use the relation between $\nu_{n,k,\rho}$ and $u_{n,k,\rho}$ to extend [6, Theorem 4.20] and [12, Theorem 5.1], enlarging thus the domain of convergence of the dual functionals. This gives a partial answer to a problem raised by Cooper and Waldron in [6, p. 149].

We denote by $C[0, 1]$ the space of all real-valued, continuous functions on [0, 1], endowed with the supremum norm $\|\cdot\|$ and the usual ordering. By e_0 we denote the constant function of constant value 1. $\mathcal{P}$ will stand for the space of all polynomial functions defined on [0, 1] and $\mathcal{P}_k$ for the space of all polynomials of degree at most k, $k \in \mathbb{N}_0$. A positive linear operator $L : C[0, 1] \longrightarrow C[0, 1]$ such that $Le_0 = e_0$ will be called a Markov operator. In what follows we use the notation $a^{\overline{j}} := \prod_{l=0}^{j-1}(a+l),\ \ j \in \mathbb{N};\quad a^{\overline{0}} := 1$ for the rising factorials.

2 A uniquely ergodic operator and its iterates

The convergence of the Cesàro averages $m^{-1}\sum_{i=0}^{m-1} L^i$ is an important object of study in Ergodic Theory (see e.g. [18]). In this section we consider uniquely ergodic operators L on $C[0, 1]$ for which the unaveraged sequence $L^m f$ converges to a limit $\bar{f}$ which is a constant function (compare with [18, Proposition 5.1.3]).

Let $L : C[0, 1] \to C[0, 1]$ be a Markov operator. Then (see, e.g., [18, p. 178]) there exists at least one invariant probability measure ν for L, i.e., a probability Borel measure ν on [0, 1] such that

$$\int_0^1 L(f;t)d\nu(t) = \int_0^1 f(t)d\nu(t),\ \ f \in C[0,1]. \tag{1}$$

Moreover, suppose that $L(C^1[0,1]) \subset C^1[0,1]$ and there exists $c \in [0,1)$ such that

$$\|(Lf)'\| \le c\|f'\|, \ f \in C^1[0,1]. \tag{2}$$

Theorem 1 *L admits a unique invariant probability measure ν, i.e., L is uniquely ergodic. For each $f \in C[0,1]$ one has*

$$\lim_{m\to\infty} L^m(f;x) = \int_0^1 f(t)d\nu(t), \tag{3}$$

uniformly on $[0,1]$.

Proof From (2) it follows that

$$\|(L^m f)'\| \le c^m\|f'\|, \ f \in C^1[0,1], \ m \ge 1.$$

Let $f \in C^1[0,1]$, $x \in [0,1]$ and ν be an invariant probability measure for L. Then

$$\begin{aligned}\left|L^m(f;x) - \int_0^1 L^m(f;t)d\nu(t)\right| &\le \int_0^1 \left|L^m(f;x) - L^m(f;t)\right| d\nu(t) \\ &\le \|(L^m f)'\| \int_0^1 |x-t|d\nu(t) \\ &\le c^m\|f'\|.\end{aligned}$$

Therefore,

$$\left|L^m f - \left(\int_0^1 L^m(f;t)d\nu(t)\right)e_0\right| \le c^m\|f'\|e_0. \tag{4}$$

On the other hand, (1) entails

$$\int_0^1 L^m(f;t)d\nu(t) = \int_0^1 f(t)d\nu(t),$$

and from (4) we get

$$\left|L^m f - \left(\int_0^1 f(t)d\nu(t)\right)e_0\right| \le c^m\|f'\|e_0. \tag{5}$$

This implies (3) for $f \in C^1[0,1]$. Since $C^1[0,1]$ is dense in $C[0,1]$, and $\|L^m\| = 1$, (3) holds for all $f \in C[0,1]$. The unicity of ν follows from (3). □

Remark 1 1. The above proof is a short and direct one. More general results, examples and applications can be found in [2,11] and [3, Section 1.4], where quantitative estimates are also given.

2. (5) expresses a quantitative result for $f \in C^1[0,1]$. It was proved in [11, Corollary 3.4] that

$$\left\| L^m f - \left(\int_0^1 f(t) d\nu(t) \right) e_0 \right\| \leq \widetilde{\omega}(f, c^m), \; f \in C[0,1], \tag{6}$$

where for a function g the least concave majorant of the first order modulus of continuity $\widetilde{\omega}(g, \varepsilon)$ is given by

$$\tilde{\omega}(g, \varepsilon) = \begin{cases} \sup\limits_{0 \leq x \leq \varepsilon \leq y \leq 1} \dfrac{(\varepsilon - x)\omega(g; y) + (y - \varepsilon)\omega(g, x)}{y - x}, & 0 \leq \varepsilon \leq 1 \\ \omega(g, 1), & \varepsilon > 1 \end{cases}.$$

3 Application to linking Bernstein type operators

We now apply the results of Sect. 2 to the operators $B_{n,\rho}$ which constitute a non-trivial link between the genuine Bernstein-Durrmeyer operators and the classical Bernstein operators.

Definition 1 Let $\rho \in \mathbb{R}_+$, $n \in \mathbb{N}$. For a function $f \in C[0,1]$ the operators $B_{n,\rho} : C[0,1] \longrightarrow \mathcal{P}_n$ are defined by

$$\begin{aligned} B_{n,\rho}(f; x) = & \; f(0) p_{n,0}(x) + f(1) p_{n,n}(x) \\ & + \sum_{j=1}^{n-1} p_{n,j}(x) \int_0^1 \mu_{n,j,\rho}(t) f(t) dt \end{aligned}$$

where

$$p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}, \; 0 \leq j \leq n, \; x \in [0,1]$$

denote the Bernstein basis polynomials and

$$\mu_{n,j,\rho}(t) = \frac{t^{j\rho - 1}(1-t)^{(n-j)\rho - 1}}{B(j\rho, (n-j)\rho)}, \; 1 \leq j \leq n-1, \; t \in (0,1)$$

with Euler's Beta function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \; x, y > 0.$$

For $k \in \mathbb{N}_0$ the kth order Kantorovich modification $B_{n,\rho}^{(k)} : C[0,1] \longrightarrow \mathcal{P}_{n-k}$ is given by

$$\begin{aligned} B_{n,\rho}^{(0)} &= B_{n,\rho} \\ B_{n,\rho}^{(k)} &= D^k \circ B_{n,\rho} \circ I_k \\ &= D^{k-l} \circ B_{n,\rho}^{(l)} \circ I_{k-l},\ 0 \le l \le k \end{aligned}$$

where D^k denotes the k-th order ordinary differential operator and

$$I_k f = f, \text{ if } k = 0, \text{ and } I_k(f,x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f(t)dt, \text{if } \quad k \in \mathbb{N}.$$

For simplicity we omit the superscript (k) in case $k = 0$ as indicated by the definition above.

In [9] it is proved that

$$\lim_{\rho \to \infty} B_{n,\rho} f = B_{n,\infty} f \text{ uniformly on } [0,1] \text{ for any } f \in C[0,1],$$

where here and in the following $B_{n,\infty}$ denotes the classical Bernstein operator.

Remark 2 For $\rho = 1$, $\rho = \infty$ we have the explicit representations for the kth order Kantorovich modifications (see [14, (3.5)] and [20, §1.4])

$$(B_{n,1}^{(k)} f)(x) = \frac{n!(n-1)!}{(n-k)!(n+k-2)!} \sum_{j=0}^{n-k} p_{n-k,j}(x) \int_0^1 p_{n+k-2,j+k-1}(t) f(t)dt,$$

$$(B_{n,\infty}^{(k)} f)(x) = \frac{n!}{(n-k)!} \sum_{j=0}^{n-k} p_{n-k,j}(x) \Delta_{\frac{1}{n}}^k \left(I_k \left(f; \frac{j}{n} \right) \right),$$

where the kth order forward difference for a function g and step h is given by

$$\Delta_h^k (g(x)) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} g(x+ih).$$

From [16, Corollary 1] we know that

$$B_{n,\rho}^{(k)}(e_0; x) = \frac{\rho^k}{(n\rho)^{\overline{k}}} \cdot n^{\underline{k}}. \tag{7}$$

For $n-k \geq 0,\ 0 \leq k+j \leq n$ the eigenvalues of $B_{n,\rho}^{(k)}$ are given by

$$\lambda_{n,j,\rho}^{(k)} = \frac{\rho^{k+j}}{(n\rho)^{\overline{k+j}}} \frac{n!}{(n-k-j)!},\ \rho \in \mathbb{R}_+, \tag{8}$$

$$\lambda_{n,j,\infty}^{(k)} := \lim_{\rho\to\infty} \lambda_{n,j,\rho}^{(k)} = \frac{1}{n^{j+k}} \frac{n!}{(n-k-j)!}. \tag{9}$$

For $k=0$, $\rho=\infty$ see [6, (1.3)], for $k=0$, $\rho=1$ [23, (1.25)], for $k\geq 2$, $\rho=1$ [15, Theorem 9], for $k=0$, $\rho\in\mathbb{R}_+$ [12, (3.4)]. It is not difficult to find the eigenvalues for an arbitrary k, using the method described, e.g., in [15, Theorem 9].

Now for $1 \leq k \leq n$ we define the operators $V_{n,\rho}^{(k)} : C[0,1] \longrightarrow \mathcal{P}_{n-k}$ by

$$V_{n,\rho}^{(k)} = \frac{1}{\lambda_{n,0,\rho}^{(k)}} B_{n,\rho}^{(k)}.$$

Then it is clear from the definition that $V_{n,\rho}^{(k)}$ is a positive linear operator with

$$V_{n,\rho}^{(k)}(e_0;x) = e_0(x), \tag{10}$$

having the property

$$f \in C^1[0,1] \Rightarrow V_{n,\rho}^{(k)} f \in C^1[0,1].$$

In view of Theorem 1 we need the following estimate.

Lemma 1 *Let* $f \in C^1[0,1]$. *Then*

$$\|(V_{n,\rho}^{(k)}f)'\| \leq \|f'\| \begin{cases} \dfrac{\rho(n-k)}{n\rho+k}, & \rho\in\mathbb{R}_+, \\ \dfrac{n-k}{n}, & \rho\to\infty. \end{cases}$$

Proof As $B_{n,\rho}(\mathcal{P}_k) \subset \mathcal{P}_k$ for $k\in\mathbb{N}_0$, $k\leq n$, we have

$$\begin{aligned}(B_{n,\rho}^{(k)}f)' &= (D^{k+1}\circ B_{n,\rho}\circ I_k)f \\ &= (D^{k+1}\circ B_{n,\rho}\circ I_{k+1})f'.\end{aligned}$$

Thus

$$\begin{aligned}\|(V_{n,\rho}^{(k)}f)'\| &= \frac{\lambda_{n,0,\rho}^{(k+1)}}{\lambda_{n,0,\rho}^{(k)}} \|V_{n,\rho}^{(k+1)}f'\| \\ &\leq \|f'\| \begin{cases} \dfrac{\rho(n-k)}{n\rho+k}, & \rho\in\mathbb{R}_+, \\ \dfrac{n-k}{n}, & \rho\to\infty. \end{cases}\end{aligned}$$

□

From Lemma 1 we derive that $V_{n,\rho}^{(k)}$ satisfies (2) with

$$c = c_{n,\rho}^{(k)} = \begin{cases} \frac{\rho(n-k)}{n\rho+k}, & \rho \in \mathbb{R}_+, \\ \frac{n-k}{n}, & \rho \to \infty. \end{cases}$$

Thus from Theorem 1 and (6) we obtain the following result.

Theorem 2 *$V_{n,\rho}^{(k)}$ admits a unique invariant probability measure $\nu_{n,\rho,k}$. For each $f \in C[0,1]$ one has*

$$\lim_{m\to\infty} (V_{n,\rho}^{(k)})^m (f;x) = \int_0^1 f(t)d\nu_{n,\rho,k}(t), \tag{11}$$

uniformly on $[0,1]$ *and, moreover,*

$$\left\| (V_{n,\rho}^{(k)})^m f - \left(\int_0^1 f(t)d\nu_{n,\rho,k}(t) \right) e_0 \right\| \le \widetilde{\omega} \left(f, \left(c_{n,\rho}^{(k)} \right)^m \right).$$

For $f \in C[0,1]$ the operator $B_{n,\rho}$ can be represented by

$$B_{n,\rho} f = \sum_{j=0}^{n} \lambda_{n,j,\rho}^{(0)} q_{n,j,\rho} u_{n,j,\rho}(f)$$

with the eigenvalues $\lambda_{n,j,\rho}^{(0)}$, $j = 0, 1, \dots, n$, the associated monic eigenpolynomials $q_{n,j,\rho}$ and the dual functionals $u_{n,j,\rho}$ on $C[0,1]$, such that $u_{n,j,\rho}(q_{n,i,\rho}) = \delta_{i,j}$, $i, j = 0, 1, \dots, n$ (see [6, Theorem 2.3] and [12, Theorem 3.2]).

Thus

$$\begin{aligned} B_{n,\rho}^{(k)} f &= \sum_{j=k}^{n} \lambda_{n,j,\rho}^{(0)} q_{n,j,\rho}^{(k)} u_{n,j,\rho}(I_k f) \\ &= \sum_{j=0}^{n-k} \lambda_{n,j+k,\rho}^{(0)} q_{n,j+k,\rho}^{(k)} u_{n,j+k,\rho}(I_k f) \\ &= \sum_{j=0}^{n-k} \lambda_{n,j,\rho}^{(k)} q_{n,j+k,\rho}^{(k)} u_{n,j+k,\rho}(I_k f), \end{aligned}$$

i.e.,

$$V_{n,\rho}^{(k)} f = \sum_{j=0}^{n-k} \frac{\lambda_{n,j,\rho}^{(k)}}{\lambda_{n,0,\rho}^{(k)}} q_{n,j+k,\rho}^{(k)} u_{n,j+k,\rho}(I_k f),$$

which entails

$$(V_{n,\rho}^{(k)})^m f = \sum_{j=0}^{n-k} \left(\frac{\lambda_{n,j,\rho}^{(k)}}{\lambda_{n,0,\rho}^{(k)}} \right)^m q_{n,j+k,\rho}^{(k)} u_{n,j+k.\rho}(I_k f), \; f \in C[0,1], \; m \ge 1. \tag{12}$$

For $1 \leq j \leq n-k$ we have

$$\frac{\lambda_{n,j,\rho}^{(k)}}{\lambda_{n,0,\rho}^{(k)}} = \frac{(n-k)\dots(n-(j+k-1))}{(n+\frac{k}{\rho})\dots(n+\frac{j+k-1}{\rho})} < 1$$

and

$$\frac{\lambda_{n,j,\infty}^{(k)}}{\lambda_{n,0,\infty}^{(k)}} = \frac{(n-k)\dots(n-(j+k-1))}{n^j} < 1,$$

and so (12) implies the following theorem.

Theorem 3 *Let $f \in C[0,1]$. Then*

$$\lim_{m\to\infty} (V_{n,\rho}^{(k)})^m f = k! u_{n,k,\rho}(I_k f) e_0. \tag{13}$$

Now from (11) and (13) we get

Corollary 1 *The invariant probability measure $\nu_{n,k,\rho}$ is characterized by*

$$\int_0^1 f(t) d\nu_{n,\rho,k}(t) = k! u_{n,k,\rho}(I_k f),\ f \in C[0,1].$$

Example 1 According to [6, Theorem 9.8], $u_{n,1,\infty}(g) = g(1) - g(0)$, $g \in C[0,1]$. Consequently, (13) yields

$$\lim_{m\to\infty} (V_{n,\infty}^{(1)})^m (f;x) = \int_0^1 f(t)dt,\ f \in C[0,1]. \tag{14}$$

Let us remark that $V_{n,\infty}^{(1)}$ coincides with the classical Kantorovich operator. For more general versions of (14), see also [2,11] and [3, Section 1.4].

4 The special case $\rho = 1$

In this section we consider the special case $\rho = 1$ with the extension to Jacobi weighted Bernstein-Durrmeyer operators. As $\rho = 1$ is fixed in this section, we omit the corresponding index in the notations.

We denote the Jacobi weights by $w(x) = x^\alpha(1-x)^\beta$, $x \in (0,1)$, $\alpha, \beta > -1$. Then the Bernstein-Durrmeyer operators with Jacobi weights (see [5, p. 27], named $V_{n-1}^{(\alpha,\beta)}$ there) are defined by

$$B_{n,w}^{(1)} = \sum_{j=0}^{n-1} p_{n-1,j}(x) \frac{\int_0^1 p_{n-1,j}(t)w(t)f(t)dt}{\int_0^1 p_{n-1,j}(t)w(t)dt}.$$

Remark that for $\alpha = \beta = 0$ we have the Bernstein-Durrmeyer operators. Define

$$\sigma_{n,j,w}^{(k)} = \frac{(n-1)!}{(n-j-k)!} \cdot \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+j+k+\alpha+\beta)}.$$

For the following results concerning the eigenstructure of $B_{n,w}^{(1)}$ see [5, p. 28]. The eigenvalues of $B_{n,w}^{(1)}$ are given by

$$\sigma_{n,j,w}^{(1)} = \frac{(n-1)!}{(n-j-1)!} \cdot \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+j+\alpha+\beta+1)}$$

and the corresponding monic eigenpolynomials are the Jacobi polynomials on [0, 1] normalized such that the leading coefficient is 1, i.e.,

$$Q_{j,w}(x) = \frac{(-1)^j \Gamma(j+\alpha+\beta+1)}{\Gamma(2j+\alpha+\beta+1)} \cdot x^{-\alpha}(1-x)^{-\beta} D^j \left[x^{j+\alpha}(1-x)^{j+\beta}\right].$$

For $\alpha = \beta = 0$ we have $\sigma_{n,j,w}^{(k)} = \lambda_{n,j,1}^{(k)}$.

The operators $B_{n,w}^{(1)}$ can be represented in terms of their eigenvalues and eigenpolynomials by

$$B_{n,w}^{(1)} f = \sum_{j=0}^{n-1} \sigma_{n,j,w}^{(1)} Q_{j,w} h_{j,w} \int_0^1 Q_{j,w}(t) w(t) f(t) dt,$$

where

$$h_{j,w}^{-1} = \int_0^1 Q_j(t)^2 w(t) dt = \frac{1}{B(j+\alpha+1, j+\beta+1)} \cdot \frac{(j+\alpha+\beta+1)^{\overline{j}}}{j!}.$$

Thus we have

$$\begin{aligned} B_{n,w}^{(k)} f &= \sum_{j=k-1}^{n-1} \sigma_{n,j,w}^{(1)} (Q_{j,w})^{(k-1)} h_{j,w} \int_0^1 Q_{j,w}(t) w(t) I_{k-1}(f;t) dt \qquad (15) \\ &= \sum_{j=0}^{n-k} \sigma_{n,j,w}^{(k)} (Q_{j+k-1,w})^{(k-1)} h_{j+k-1,w} \int_0^1 Q_{j+k-1,w}(t) w(t) I_{k-1}(f;t) dt. \end{aligned}$$

We now calculate

$$\begin{aligned} B_{n,w}^{(k)} e_0 = \sum_{j=0}^{n-k} \sigma_{n,j,w}^{(k)} (Q_{j+k-1,w})^{(k-1)} h_{j+k-1,w} \\ \times \int_0^1 Q_{j+k-1,w}(t) w(t) \frac{1}{(k-1)!} t^{k-1} dt. \qquad (16) \end{aligned}$$

Due to the orthogonality properties of the Jacobi polynomials the integral on the right-hand side vanishes for $j \neq 0$. As

$$\int_0^1 Q_{k-1,w}(t)w(t)t^{k-1}dt = \int_0^1 Q_{k-1,w}(t)w(t)Q_{k-1,w}(t)dt = \frac{1}{h_{k-1,w}}$$

we derive from (16)

$$B_{n,w}^{(k)}e_0 = \sigma_{n,0,w}^{(k)}.$$

We now define

$$V_{n,w}^{(k)}f = \sum_{j=0}^{n-k} \frac{\sigma_{n,j,w}^{(k)}}{\sigma_{n,0,w}^{(k)}} Q_{j+k-1,w}^{(k-1)} h_{j+k-1,w} \int_0^1 Q_{j+k-1,w}(t)w(t)I_{k-1}(f;t)dt.$$

Thus

$$\lim_{m\to\infty}(V_{n,w}^{(k)})^m(f;x) = (k-1)!h_{k-1,w}\int_0^1 Q_{k-1,w}(t)w(t)I_{k-1}(f;t)dt.$$

Integration by parts leads to

$$\begin{aligned}
&\int_0^1 Q_{k-1,w}(t)w(t)I_{k-1}(f;t)dt\\
&= \frac{(-1)^{k-1}}{(k+\alpha+\beta)^{\overline{k-1}}}\int_0^1 \left(t^{k+\alpha-1}(1-t)^{k+\beta-1}\right)^{(k-1)} I_{k-1}(f;t)dt\\
&= \frac{1}{(k+\alpha+\beta)^{\overline{k-1}}}\int_0^1 t^{k+\alpha-1}(1-t)^{k+\beta-1}f(t)dt.
\end{aligned}$$

So

$$\lim_{m\to\infty}(V_{n,w}^{(k)})^m(f;x) = \frac{(k-1)!}{(k+\alpha+\beta)^{\overline{k-1}}}h_{k-1,w}\int_0^1 t^{k+\alpha-1}(1-t)^{k+\beta-1}f(t)dt.$$

For $f = e_0$ this yields

$$1 = \frac{(k-1)!}{(k+\alpha+\beta)^{\overline{k-1}}}h_{k-1,w}B(k+\alpha,k+\beta).$$

Thus we derive the following result.

Theorem 4 *Let* $f \in C[0,1]$. *Then*

$$\lim_{m\to\infty}(V_{n,w}^{(k)})^m(f;x) = \frac{1}{B(k+\alpha,k+\beta)}\int_0^1 t^{k+\alpha-1}(1-t)^{k+\beta-1}f(t)dt, \quad (17)$$

uniformly on [0, 1].

Remark 3 1. For $k = 1$, qualitative and quantitative versions of (17) were obtained with different methods in [2, Section 3.2] and [11, Example 4.3].

2. The preceding results can be extended to the context of spaces L^p and semigroups of operators, in the spirit of [2] and [3, Section 1.4]. This could be the subject of a forthcoming paper.
3. Both (13) and (17) express the convergence of the iterates towards operators L (for which Lf is constant). Since the dual functionals involved in the proofs are linear and bounded, an inspection of the proofs shows that in both cases we have convergence in the uniform operator norm.
4. A significant difference between the operators $V_{n,\rho}^{(k)}$, $\rho \neq 1$, and $V_{n,w}^{(k)}$ is that for the latter the eigenpolynomials are orthogonal and independent of n; consequently, the dual functionals are also independent of n. This explains the difference between the right-hand member of (13) (depending on n) and that of (17) (where n does not appear). As (24) will show, these two right-hand members are related when $w = e_0$, i.e., $\alpha = \beta = 0$.

5 Convergence of dual functionals

In this section we extend former results concerning convergence properties of dual functionals from polynomials to smooth functions.

It was proved in [12, Theorem 5.1] that

$$\lim_{n\to\infty} u_{n,k,\rho}(p) = \mu_k^*(p), \ p \in \mathcal{P}, \tag{18}$$

where (see [6, (4.16)]

$$\mu_k^*(f) = \frac{(-1)^{k-1}}{2(k-1)!}\binom{2k}{k}\int_0^1 \left(t^{k-1}(1-t)^{k-1}\right)^{(k-1)} f'(t)dt, \ f \in C^1[0,1].$$

Let $g \in C^k[0, 1]$. Since $k \geq 1$, we have $g \in C^1[0, 1]$ and so

$$\begin{aligned}\mu_k^*(g) &= \frac{(-1)^{k-1}}{2(k-1)!}\binom{2k}{k}\int_0^1 \left(t^{k-1}(1-t)^{k-1}\right)^{(k-1)} g'(t)dt \\ &= \frac{1}{2(k-1)!}\binom{2k}{k}\int_0^1 t^{k-1}(1-t)^{k-1}g^{(k)}(t)dt,\end{aligned}$$

i.e.,

$$\mu_k^*(g) = \frac{1}{k!}\cdot\frac{1}{B(k,k)}\int_0^1 t^{k-1}(1-t)^{k-1}g^{(k)}(t)dt, \ g \in C^k[0,1]. \tag{19}$$

From (18) and (19) it follows that

$$\lim_{n\to\infty} u_{n,k,\rho}(p) = \frac{1}{k!}\cdot\frac{1}{B(k,k)}\int_0^1 t^{k-1}(1-t)^{k-1}p^{(k)}(t)dt,\ p\in\mathcal{P}. \tag{20}$$

Consider the operators $P_{n,k,\rho} : C[0,1] \longrightarrow \mathcal{P}_0$, $P_{n,k,\rho}f = \lim_{m\to\infty}\left(V_{n,\rho}^{(k)}\right)^m f$, and $L_k : C[0,1] \longrightarrow \mathcal{P}_0$, $L_k f = \frac{1}{B(k,k)}\left(\int_0^1 t^{k-1}(1-t)^{k-1}f(t)dt\right)e_0$.

Both of them are positive linear operators of norm 1. Moreover, (13) shows that

$$P_{n,k,\rho}f = k!u_{n,k,\rho}(I_k f)e_0,\ f\in C[0,1]. \tag{21}$$

Let $p\in\mathcal{P}$. From (20) and (21) we get

$$\begin{aligned}\lim_{n\to\infty} P_{n,k,\rho}p &= \lim_{n\to\infty} k!u_{n,k,\rho}(I_k p)e_0\\ &= \left(\frac{1}{B(k,k)}\int_0^1 t^{k-1}(1-t)^{k-1}p(t)dt\right)e_0\\ &= L_k p,\end{aligned}$$

i.e.

$$\lim_{n\to\infty} P_{n,k,\rho}p = L_k p,\ p\in\mathcal{P}. \tag{22}$$

$\mathcal{P}$ is dense in $C[0,1]$ and $P_{n,k,\rho}$ and L_k are bounded operators of norm 1; thus (22) implies

$$\lim_{n\to\infty} P_{n,k,\rho}f = L_k f,\ f\in C[0,1]. \tag{23}$$

Now (21) and (23) yield

$$\lim_{n\to\infty} u_{n,k,\rho}(I_k f) = \frac{1}{k!}\cdot\frac{1}{B(k,k)}\int_0^1 t^{k-1}(1-t)^{k-1}f(t)dt,\ f\in C[0,1]. \tag{24}$$

Combining this with (19) we get

Theorem 5 *Let $g\in C^k[0,1]$. Then*

$$\lim_{n\to\infty} u_{n,k,\rho}(g) = \mu_k^*(g). \tag{25}$$

Let us remark that (25) extends (18) from $\mathcal{P}$ to $C^k[0,1]$; in particular, this extends [6, Theorem 4.20] from $\mathcal{P}$ to $C^k[0,1]$. (See also [6, Remark on p. 149]). Moreover, from (21) we see that

$$|u_{n,k,\rho}(g)| = \frac{1}{k!}\left\| P_{n,k,\rho}\left(g^{(k)}\right)\right\|,$$

i.e.

$$|u_{n,k,\rho}(g)| \leq \frac{1}{k!}\|g^{(k)}\|, \ g \in C^k[0,1].$$

Also from (21), $P_{n,k,\rho}$ being positive:

$$u_{n,k,\rho}(g) \geq 0 \text{ for all } g \in C^k[0,1], \ g^{(k)} \geq 0.$$

As far as we know, the validity of (25) for all $g \in C[0,1]$ is still an open problem.

References

1. Altomare, F., Raşa, I.: On some classes of diffusion equations and related approximation problems, trends and applications in constructive approximation. In: de Bruin, M.G., Mache, D.H., Szabados, J. (eds.) International Series of Numerical Mathematics, vol. 151, pp. 13–26. Birkhäuser, Basel (2005)
2. Altomare, F., Raşa, I.: Lipschitz contractions, unique ergodicity and asymptotics of Markov semigroups. Bolletino U. M. I. **9**, 1–17 (2012)
3. Altomare, F., Cappelletti Montano, M., Leonessa, V., Raşa, I.: Markov Operators, Positive Semigroups and Approximation Processes. Walter de Gruyter, Berlin, Munich, Boston (2014)
4. Attalienti, A., Raşa, I.: Overiterated linear operators and asymptotic behaviour of semigroups. Mediterr. J. Math. **5**, 315–324 (2008)
5. Berens, H., Xu, Y.: On Bernstein-Durrmeyer Polynomials with Jacobi Weights, Approximation Theory and Functional Analysis (College Station, TX, 1990), pp. 25–46. Academic, Boston (1991)
6. Cooper, S., Waldron, S.: The eigenstructure of the Bernstein operator. JAT **105**, 133–165 (2000)
7. Gavrea, I., Ivan, M.: On the iterates of positive linear operators preserving the affine functions. J. Math. Anal. Appl. **372**, 366–368 (2010)
8. Gonska, H., Heilmann, M., Raşa, I.: Kantorovich operators of order k. Numer. Funct. Anal. Optimiz. **32**, 717–738 (2011)
9. Gonska, H., Păltănea, R.: Simultaneous approximation by a class of Bernstein-Durrmeyer operators preserving linear functions. Czechoslovak Math. J. **60**(135), 783–799 (2010)
10. Gonska, H., Raşa, I.: On infinite products of positive linear operators reproducing linear functions. Positivity **17**(1), 67–79 (2011)
11. Gonska, H., Raşa, I., Rusu, M.-D.: Applications of an Ostrowski-type inequality. J. Comput. Anal. Appl. **14**(1), 19–31 (2012)
12. Gonska, H., Raşa, I., Stanila, E.-D.: The eigenstructure of operators linking the Bernstein and the genuine Bernstein-Durrmeyer operators. Mediterr. J. Math. **11**, 561–576 (2014)
13. Gonska, H., Raşa, I.,. Stanila, E.-D: Beta operators with Jacobi weights. In: Ivanov, K., Nikolov, G., Uluchev, R. (eds.) Constructive Theory of Functions, Sozopol 2013 , pp. 99–112. Prof. Marin Drinov Academic Publishing House, Sofia (2014)
14. Heilmann, M.: Erhöhung der Konvergenzgeschwindigkeit bei der Approximation von Funktionen mit Hilfe von Linearkombinationen spezieller positiver linearer Operatoren, Habilitationschrift Universität Dortmund (1992)
15. Heilmann, M.: Commutativity and spectral properties of genuine Baskakov-Durrmeyer type operators and their kth order Kantorovich modification. J. Numer. Anal. Approx. Theory **44**(4), 166–179 (2015)
16. Heilmann, M., Raşa, I.: k-th order Kantorovich type modification of the operators U_n^{ρ}. J. Appl. Funct. Anal. **9**(3–4), 320–334 (2014)
17. Kelisky, R.P., Rivlin, T.J.: Iterates of Bernstein polynomials. Pac. J. Math. **21**(3), 511–520 (1967)
18. Krengel, U.: Ergodic Theorems. Walter de Gruyter, Berlin, New York (1985)
19. Mahmudov, N.I.: Korovkin type theorems for iterates of certain positive linear operators. arXiv:1103.2918v1 [math.FA]
20. Lorentz, G.G.: Bernstein Polynomials. Chelsea Publishing Company, New York (1986)
21. Raşa, I.: Asymptotic behaviour of iterates of positive linear operators. Jaen J. Approx. **1**(2), 195–204 (2009)

22. Raşa, I.: C_0 - semigroups and iterates of positive linear operators: asymptotic behaviour. Rend. Circ. Mat. Palermo **2**(Suppl 82), 123–142 (2010)
23. Wagner, M.: Quasi-Interpolanten zu genuinen Baskakov-Durrmeyer-Typ Operatoren, Disssertation Universität Wuppertal (2013)