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Spectral theory in ordered Banach algebras

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Abstract We give a survey of the development of the spectral theory in ordered Banach algebras; from its roots in operator theory to the modern abstract context.

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1 Introduction

Let T be an $n \times n$ matrix with complex entries. One can view T as an operator of $\mathbb{C}^n$ into $\mathbb{C}^n$ in the normal way. It was discovered around the turn of the previous century that the spectrum of an $n \times n$ matrix with positive entries has certain special features. The first result was by Perron.

Theorem 1.1 ([55]) *Let $T = [t_{ij}]$ be an $n \times n$ matrix with $t_{ij} > 0$ for all i and j. Then:*

1. *T has strictly positive spectral radius $r(T)$.*
2. *$r(T)$ is a simple eigenvalue of T with strictly positive eigenvector.*
3. *T is primitive, i.e. $r(T)$ is the unique eigenvalue on the spectral circle $|\lambda| = r(T)$.*

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Since that time a slow but steady development has taken place. In 1948 Krein and Rutman considered positive operators defined on ordered Banach spaces. A Banach space X is called an ordered Banach space if it is ordered by a (not necessarily closed) cone C and an operator T on X is called positive if $TC \subseteq C$. Krein and Rutman proved that *if* T *is a positive compact operator defined on an ordered Banach space* X *with* $r(T) > 0$, *then* $r(T)$ *is an eigenvalue of* T *with positive eigenvector* x, *i.e.* $x \in C$ *and* $Tx = r(T)x$. In the early 1970s the Krein-Rutman results were generalised to the setting of positive operators defined on Banach lattices; see for instance the monograph of Schaefer [60]. For a fairly complete account of spectral theory of positive operators defined on Banach lattices, see [23].

The notion of an ordered Banach algebra appeared in the literature for the first time in 1975 in a paper [65] by White. Despite this, the theory of ordered Banach algebras remained dormant for almost two decades. In the early 1990s Raubenheimer and Rode (Mouton) [52,58] generalised the notion of a positive operator defined on a Banach lattice to positive elements in an ordered Banach algebra (OBA), where positive means that an element belongs to some subset C of a Banach algebra A. The subset C is called an algebra cone of A and it gives rise to a partial ordering on A that is compatible with the algebraic structure of A. The above two papers together with results on spectral theory of positive operators on Banach lattices created a lot of activity for future research [3,9–11,13,28–30,33,38,43–45,47,49–51].

Our paper is organised as follows: In Sect. 2 basic notions in spectral theory of Banach algebras are mentioned as well as notions concerning operators defined on Banach lattices. Section 3 contains the definition of an ordered Banach algebra (OBA) as well as the basic properties of these algebras. The main results in this paper appear in Sect. 4. Section 4.1 contains the generalisations of the Perron-Frobenius and Krein–Rutman theorems—see Theorems 4.1.1 and 4.1.4. Herzog and Schmoeger [29] proved an interesting converse of the Perron-Frobenius result—see Theorem 4.1.3.

The so-called domination problem in OBAs is discussed in Sect. 4.2. More precisely, if an element b in an OBA has certain properties (spectral or topological) and if b dominates a positive element a, does a inherit these properties of b? This problem is of course inspired by work of Dodds and Fremlin [20] and work of Aliprantis and Burkinshaw [4,5] in the context of Banach lattices and positive operators. Hitherto, the domination problem in OBAs has been investigated for radical elements [38], Riesz elements [33,58], inessential elements [9], quasi compact elements [52] and ergodic elements [51].

The work in Sect. 4.3 is motivated by the Gelfand-Hille theorems: If an element a in a Banach algebra A has unit spectrum, i.e. if $\sigma(a) = \{1\}$, does it follow that $a = 1$? Gelfand and Hille [22,31] proved that an element with unit spectrum which is doubly power bounded is the identity. Later, this problem was also studied in the context of operators defined on Banach lattices [24,62]. It was shown that if an operator T defined on a Banach lattice is positive with $\sigma(T) = \{1\}$ and if T satisfies some boundedness condition weaker than power boundedness, then $T = I$. In OBAs this problem is investigated in [13,44]. One can also study a weaker version of the Gelfand-Hille theorems: If T is a positive operator defined on a Banach lattice with $\sigma(T) = \{1\}$, does it follow that $T \geq I$? This question was investigated in [67] and in the context of OBAs in [44].

In spectral theory of Banach algebras the main questions concern the properties of the spectrum function $a \mapsto \sigma(a)$ and the spectral radius function $a \mapsto r(a)$. In particular, what continuity properties do these functions have? In Banach algebras this study was initiated by Newburgh [54] in the 1950s. Section 4.4 contains interesting and nontrivial results of Mouton where she investigates continuity properties of the spectrum and spectral radius functions in the context of OBAs [45,47]. Aupetit's scarcity theorem is a deep result in spectral theory of Banach algebras and many applications of this result can be found in [7]. It was shown by Mouton and Mouton [38] that the application of the scarcity theorem when studying the domination problem for radical elements in OBAs is crucial. Further applications of the scarcity theorem in OBAs are discussed in Sect. 4.5.

It turned out that positive irreducible operators on Banach lattices have good spectral properties [60, Chapter V.5] and [23]. Alekhno [3] introduced the notion of an irreducible element in OBAs and he illustrated that irreducible elements have useful spectral properties which can be used to obtain spectral information of positive elements in an OBA. For an account of these results see Sect. 4.6. Fredholm theory in Banach algebras plays a central role in spectral theory of Banach algebras [3,27,39–41]. Section 4.7 contains results of Mouton and Benjamin [10,11] in OBAs which illustrate the effect of positive elements in Fredholm theory.

2 Notation and preliminaries

Throughout A will denote a Banach algebra with unit 1, in which we denote the set of all invertible elements by A^{-1}. Unless otherwise stated, A will be over $\mathbb{C}$. By "ideal" we will always mean "(not necessarily closed) proper two-sided ideal". If A and B are Banach algebras, then a linear operator $T : A \to B$ is called a *homomorphism* if $T(ab) = TaTb\,(a, b \in A)$ and $T1 = 1$. The null space (kernel) of T will be indicated by $N(T)$. The spectrum of an element a in A will be denoted by $\sigma(a)$, the non-zero spectrum of a by $\sigma'(a)$, the set of isolated points of the spectrum of a by $\operatorname{iso}\sigma(a)$, the connected hull of the spectrum of a (i.e. the union of $\sigma(a)$ with the bounded components of $\mathbb{C}\backslash\sigma(a)$) by $\eta\sigma(a)$ and the spectral radius of a by $r(a)$ (or $\sigma(a, A)$, etc., when necessary to avoid confusion). The *peripheral spectrum* $\operatorname{psp}(a)$ of a is the set $\sigma(a) \cap \{\lambda \in \mathbb{C} : |\lambda| = r(a)\}$. It is a non-empty closed subset of the spectrum.

If $\lambda \in \mathbb{C}$, the element $\lambda 1$ of A will be denoted by λ. If $a \in A$ and $r(a) \in \operatorname{iso}\sigma(a)$, then the coefficient of $(\lambda - r(a))^j$ in the Laurent series of the resolvent $(\lambda - a)^{-1}$ of a around $r(a)$ will be denoted by a_j, for all integers j.

The set of all non-negative real numbers will be denoted by $\mathbb{R}^+$, the distance from $\lambda \in \mathbb{C}$ to a non-empty compact set $K \subseteq \mathbb{C}$ by $d(\lambda, K)$ and the number of elements in a set $K \subseteq \mathbb{C}$ by $\#K$. The open disk, closed disk and circle in $\mathbb{C}$ with centre λ and radius ϵ will be denoted by $D(\lambda, \epsilon)$, $\overline{D}(\lambda, \epsilon)$ and $C(\lambda, \epsilon)$, respectively.

Certain well-known examples of Banach algebras will enter throughout our discussion. If X is a Banach space, then $\mathcal{L}(X)$ will denote the Banach space of bounded linear operators on X. Under the operator norm

$$||T|| = \sup\{||Tx|| : ||x|| = 1\}, \tag{2.1}$$

with $T \in \mathcal{L}(X)$, and multiplication defined as composition of operators, $\mathcal{L}(X)$ is a Banach algebra with unit the identity operator I on X. In this Banach algebra $\mathcal{F}(X)$ will denote the ideal of all finite rank operators and $\mathcal{K}(X)$ the ideal of all compact operators on X.

Let $M_n(\mathbb{C})$ denote the vector space of all $n \times n$ matrices with complex entries. If we define multiplication as multiplication of matrices and equip $M_n(\mathbb{C})$ with the matrix norm

$$||A|| = \sup \left\{ \sum_{j=1}^{n} |a_{ij}| : i \in \{1, 2, \ldots, n\} \right\}, \tag{2.2}$$

where $A = (a_{ij}) \in M_n(\mathbb{C})$, then $M_n(\mathbb{C})$ becomes a Banach algebra.

If X is a Banach space with dim $X = n$, then we can identify $M_n(\mathbb{C})$ with $\mathcal{L}(X)$, so that $M_n(\mathbb{C})$ equipped with the norm (2.1) is then also a Banach algebra. Note that in this case the norms (2.1) and (2.2) are equivalent. The subalgebra of all upper triangular matrices in $M_n(\mathbb{C})$ will be denoted by $M_n^u(\mathbb{C})$.

$C(K)$ will denote the Banach space of all continuous complex valued functions on a compact Hausdorff space K. If multiplication of functions in $C(K)$ is defined pointwise and if $C(K)$ is equipped with the sup norm

$$||f|| = \sup\{|f(x)| : x \in K\},$$

where $f \in C(K)$, then $C(K)$ is a Banach algebra with unit the identity function $1(x) = 1$ for all $x \in K$.

If D is the open unit disk in the complex plane, then $A(D)$ denotes the Banach algebra of all continuous complex valued functions defined on the closure $\overline{D}$ of D which are analytic on D. Since $A(D) \subseteq C(D)$, $A(D)$ has the same algebraic operations as $C(D)$. The norm in $A(D)$ is the sup norm

$$||f|| = \sup\{|f(z)| : z \in \overline{D}\} = \sup\{|f(z)| : |z| = 1\},$$

where $f \in A(D)$.

If A is a Banach algebra, then $l^\infty(A)$ will indicate the collection of all bounded sequences of elements of A. If addition, scalar multiplication and multiplication are defined pointwise, then $l^\infty(A)$ becomes a Banach algebra under the norm

$$||x|| = ||(x_n)|| = \sup\{||x_n|| : n = 1, 2, \ldots\},$$

with $x = (x_n) \in l^\infty(A)$. The unit in this Banach algebra is $\mathbf{1} = (1)$ with 1 the unit in A. In the special case that $A = \mathbb{C}$, this Banach algebra will be denoted by l^∞.

In addition, $\mathcal{L}^r(E)$ will denote the Banach algebra of all regular operators on a complex Banach lattice E (where a regular operator is one that can be written as a linear combination over $\mathbb{C}$ of positive operators) with the usual operations and the r-norm

$$||T||_r = \inf\{||S|| : S \in \mathcal{L}(E),\ S \geq 0,\ |Tx| \leq S|x| \text{ for all } x \in E\},$$

with $\mathcal{K}^r(E)$ indicating the ideal of all *r-compact* operators, i.e. the closure in $\mathcal{L}^r(E)$ of $\mathcal{F}(E)$ (see [60, IV §1] or [6]). The *o-spectrum* of $T \in \mathcal{L}^r(E)$, i.e. $\sigma(T, \mathcal{L}^r(E))$, will be denoted by $\sigma_o(T)$. This concept was introduced by Schaefer in [61]. Similarly, the *o-spectral radius* and the *o-peripheral spectrum* of T will be denoted by $r_o(T)$ and $\mathrm{psp}_o(T)$, respectively.

If A is a Banach algebra and D a domain in $\mathbb{C}$, then a map $g : A \to A$ will be called *D-analytic* if $g \circ f : D \to A$ is analytic for every analytic function $f : D \to A$. It is easy to see that the maps $g(x) = a + x$ and $g(x) = a(1 + x)$ (for a fixed $a \in A$), as well as every continuous, linear map g, are D-analytic, for every domain $D \subseteq \mathbb{C}$.

The following famous and very deep result of Aupetit, which is known as the scarcity theorem, states in very general terms that if a function f is analytic on a domain D in the complex plane and with values in a Banach algebra, then either the subset of D on which the spectrum of f is finite is "very small" in some sense or it is the whole of D, in which case the spectrum of f is even uniformly finite on D:

Theorem 2.3 ([7], Theorem 3.4.25) *Let $f : D \to A$ be analytic, where D is a domain in $\mathbb{C}$ and A is a Banach algebra. Then either the set of $\lambda \in D$ such that $\sigma(f(\lambda))$ is finite is a Borel set having zero capacity, or there exist an integer $n \geq 1$ and a closed discrete subset E of D such that $\#\sigma(f(\lambda)) = n$ for all $\lambda \in D \backslash E$ and $\#\sigma(f(\lambda)) < n$ for all $\lambda \in E$.*

Here, the *capacity* of a Borel set in the complex plane (see [7, pp. 177–180]) is in some sense a measure of its size. It is a monotone function, with compact sets having zero capacity being very small. For our purposes it suffices to know that balls and line segments have non-zero capacities. (By "ball" and "line segment" we imply strictly positive radius and length, respectively.)

It also follows from the scarcity theorem that if $\sigma(f(\lambda))$ is uniformly finite on a subset of D with non-zero capacity, then it is (uniformly) finite on the whole of D with the same bound:

Corollary 2.4 *Let $f : D \to A$ be analytic, where D is a domain in $\mathbb{C}$ and A is a Banach algebra. If $n \geq 1$ is such that $\#\sigma(f(\lambda)) \leq n$ for all λ in a subset of D with non-zero capacity, then $\#\sigma(f(\lambda)) \leq n$ for all $\lambda \in D$.*

Corollary 2.4, together with [7, Corollary 3.4.18], yields the following result about quasinilpotent elements:

Corollary 2.5 ([38], Corollary 2.3) *Let f be an analytic function from a domain D in $\mathbb{C}$ into a Banach algebra A. If $\{\lambda \in D : \sigma(f(\lambda)) = \{0\}\}$ contains a ball or a line segment, then $\sigma(f(\lambda)) = \{0\}$ for all λ in D.*

A point a in a vector space X is said to be an *absorbing point* of a subset U of X if for all $x \in X$ there exists $r > 0$ such that $a + \lambda x \in U$ for all real λ with $|\lambda| \leq r$. A subset U of a vector space X is called an *absorbing set* if U contains an absorbing point. Open sets are absorbing, but not vice versa. With $\mathrm{Rad}(A)$ denoting the Jacobson radical of a Banach algebra A, we have:

Theorem 2.6 ([7], Theorem 5.4.2 and its proof) *Let A be a Banach algebra. If A contains an absorbing subset U such that*

1. *$\sigma(x)$ is finite for all $x \in U$, then* $A/\mathrm{Rad}(A)$ *is finite dimensional,*
2. *$\#\sigma(x) \leq n$ for all $x \in U$ and some fixed $n \in \mathbb{N}$, then* $\dim A/\mathrm{Rad}(A) \leq n^6$.

A Banach algebra A is said to be *semiprime* if, for all $a \in A$, we have that $a = 0$ whenever $aAa = \{0\}$. If a is a nonzero element of a semiprime Banach algebra A, then a is a *rank one element* if $aAa \subseteq \mathbb{C}a$, and a is a *finite rank element* if $a = 0$ or a is a finite sum of rank one elements—see [56]. The sets of all rank one elements and all finite rank elements are denoted by $\mathcal{F}_1$ and $\mathcal{F}$, respectively, and $\mathcal{F}$ equals the *socle* $\mathrm{Soc}(A)$ (defined as the sum of the minimal left ideals) of A (see [56, p. 659]). "Rank one element" is, of course, a generalisation of "rank one operator" and "finite rank element" is a generalisation of "finite rank operator".

A is *semisimple* if $\mathrm{Rad}(A) = \{0\}$. A semisimple Banach algebra is also semiprime (see [12, Proposition 5, p. 155]). All the examples mentioned before, except those involving upper triangular matrices, are in fact semisimple. In a semisimple Banach algebra there exist spectral characterisations of rank one and finite rank elements:

Theorem 2.7 ([8,42]) *Let A be a semisimple Banach algebra. Then*

$$\{a \in A : \text{ there exists } n \in \mathbb{N} \text{ such that } \#\sigma'(xa) \leq n \text{ for all } x \in A\}$$
$$= \mathrm{Soc}(A) = \{a \in A : \#\sigma'(xa) < \infty \text{ for all } x \in A\}$$

and

$$\mathcal{F}_1 = \{0 \neq a \in A : \#\sigma'(xa) \leq 1 \text{ for all } x \in A\}.$$

Alternative characterisations of rank one elements and of the socle are given in Theorems 2.8 and 2.9, respectively:

Theorem 2.8 ([8], Theorem 2.2 (1), ([42], Theorem 2.2) *Let A be a semisimple Banach algebra and $0 \neq a \in A$. Then the following are equivalent:*

1. *a is rank one.*
2. *$\sigma(x + s_0a) \cap \sigma(x + s_1a) \subseteq \sigma(x)$ for all $s_0, s_1 \in \mathbb{C}\backslash\{0\}$ with $s_0 \neq s_1$ and all $x \in A$.*
3. *$\eta\sigma(x + s_0a) \cap \eta\sigma(x + s_1a) \subseteq \eta\sigma(x)$ for all $s_0, s_1 \in \mathbb{C}\backslash\{0\}$ with $s_0 \neq s_1$ and all $x \in A$.*

Theorem 2.9 ([8], Theorem 2.2 (2), [42], Theorem 3.1) *Let A be a semisimple Banach algebra and $a \in A$. Then the following are equivalent:*

1. *$a \in \mathrm{Soc}(A)$.*
2. *There exists $n \in \mathbb{N}$ such that $\cap_{t \in F}\sigma(x + ta) \subseteq \sigma(x)$ for all $(n + 1)$-element subsets F of $\mathbb{C}\backslash\{0\}$ and all $x \in A$.*
3. *There exists $n \in \mathbb{N}$ such that $\cap_{t \in F}\eta\sigma(x + ta) \subseteq \eta\sigma(x)$ for all $(n + 1)$-element subsets F of $\mathbb{C}\backslash\{0\}$ and all $x \in A$.*

If I is an ideal in a Banach algebra A (not necessarily semiprime or semisimple), then an element $a \in A$ is called an *inessential element (in A relative to I)* if $a + \overline{I} \in$

$\mathrm{Rad}(A/\overline{I})$ and a *Riesz element (in A relative to I)* if $a+\overline{I} \in \mathrm{QN}(A/\overline{I})$. (Here, $\mathrm{QN}(A)$ denotes the set of all quasinilpotent elements of A, i.e. those elements a for which $\sigma(a) = \{0\}$.) The symbols $\mathrm{kh}(I)$ and $\mathcal{R}(I)$ will indicate the sets of all inessential elements relative to I and all Riesz elements relative to I, respectively. In a semiprime Banach algebra the sets mentioned above clearly satisfy the following inclusions:

$$\mathcal{F}_1 \subseteq \mathrm{Soc}(A) \subseteq \mathrm{kh}(\mathrm{Soc}(A)) \subseteq \mathcal{R}(\mathrm{Soc}(A))$$

An ideal I of A is an *inessential ideal* if the spectrum of each element of I is either finite or a sequence converging to zero (i.e. the spectrum of each $a \in I$ has at most 0 as a limit point), and if B is also a Banach algebra and $T : A \to B$ is a homomorphism, then T is said to have the *Riesz property* if $N(T)$ is an inessential ideal. Note that $\mathrm{Soc}(A)$ is an inessential ideal and $\mathrm{kh}(\mathrm{Soc}(A))$ is a closed inessential ideal (see [7, Corollaries 5.7.6 and 5.7.7]). The ideal of compact operators on a Banach space X and the ideal of r-compact operators on a Banach lattice E are closed and inessential in the Banach algebras $\mathcal{L}(X)$ and $\mathcal{L}^r(E)$, respectively. For $a \in A$, a point $\lambda \in \mathrm{iso}\,\sigma(a)$ is said to be a *Riesz point of $\sigma(a)$ (relative to I)* if the spectral idempotent $p(a,\lambda)$ is an element of I. Given any element $a \in A$, we define the (compact) set $D(a)$ (or $D(a, I)$ when necessary to avoid confusion) as follows:

$$D(a) = \sigma(a)\backslash\{\lambda \in \sigma(a) : \lambda \text{ is a Riesz point of } \sigma(a)\}$$

Aupetit obtained the following important result in 1986:

Theorem 2.10 ([7], Theorem 5.7.4) *Let I be an inessential ideal in a Banach algebra A and $a \in A$. Then $\sigma(a+\overline{I}) \subseteq D(a)$ and $\eta\sigma(a+\overline{I}) = \eta D(a)$.*

Theorem 2.10 is often called Aupetit's perturbation theorem, because it relies on certain perturbation properties of a by elements of I. It follows from this theorem that a is a Riesz element (relative to a specified inessential ideal I) if and only if $\sigma(a)$ is finite or a sequence converging to zero and every nonzero point of $\sigma(a)$ is a Riesz point of $\sigma(a)$ (relative to I) [7, Corollary 5.7.5].

It was shown in [37, Theorem 1.4] that if A is semisimple, then every inessential ideal is contained in $\mathrm{kh}(\mathrm{Soc}(A))$. This yields the important fact that if $a \in A$ and I is an inessential ideal, then every Riesz point of $\sigma(a)$ is a pole of $(\lambda - a)^{-1}$ [52, Theorem 3.11], [43, Lemma 2.1]. We make use of this observation in, for instance, Theorems 4.1.4, 4.2.13 and 4.7.4.

If an ideal I is closed and inessential, then an element $a \in A$ is *quasi inessential (in A relative to I)* if there exist $t \in I$ and $n \in \mathbb{N}$ such that $||a^n - t|| < 1$. The set of all these elements is denoted by $\mathrm{qkh}(I)$, and, by [52, Proposition 5.1], we have the following inclusions:

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I) \subseteq \mathrm{qkh}(I)$$

If X is a Banach space, then $T \in \mathcal{L}(X)$ is an *inessential operator* (*Riesz operator, quasi compact operator*) if T is an inessential element (Riesz element, quasi inessential element) in $\mathcal{L}(X)$ relative to $\mathcal{K}(X)$. Finally, if E is a Banach lattice, then $T \in \mathcal{L}^r(E)$ is an *r-inessential operator* if T is an inessential element in $\mathcal{L}^r(E)$ relative to $\mathcal{K}^r(E)$ and T is an *r-asymptotically quasi finite rank operator* if T is a Riesz element in $\mathcal{L}^r(E)$ relative to $\mathcal{K}^r(E)$.

Let A and B be Banach algebras and let $T : A \to B$ be a homomorphism. Then, in 1982, Harte [27] defined an element $a \in A$ to be

- *Fredholm* if $Ta \in B^{-1}$,
- *Weyl* if there exist $b \in A^{-1}$ and $c \in N(T)$ such that $a = b + c$ and
- *Browder* if there exist commuting elements $b \in A^{-1}$ and $c \in N(T)$ such that $a = b + c$.

If $\mathcal{F}_T$, $\mathcal{W}_T$ and $\mathcal{B}_T$ denote the sets of Fredholm, Weyl and Browder elements, respectively, then clearly

$$A^{-1} \subseteq \mathcal{B}_T \subseteq \mathcal{W}_T \subseteq \mathcal{F}_T.$$

These sets give rise to the *Fredholm spectrum* $\sigma(Ta)$, the *Weyl spectrum* $\omega_T(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin \mathcal{W}_T\}$ and the *Browder spectrum* $\beta_T(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin \mathcal{B}_T\}$ of an element $a \in A$, respectively. These spectra are non-empty and compact, and clearly satisfy

$$\sigma(Ta) \subseteq \omega_T(a) \subseteq \beta_T(a) \subseteq \sigma(a).$$

The following is an important result:

Theorem 2.11 ([7], Theorem 5.7.4; [39], Corollaries 7.6 and 7.8; [41], Corollary 5.4) *Let A and B be Banach algebras and let $T : A \to B$ be a homomorphism with closed range satisfying the Riesz property. Then*

$$\eta\sigma(Ta) = \eta\omega_T(a) = \eta\beta_T(a) = \eta D(a, N(T)) = \eta\sigma(a + \overline{N(T)}),$$

for all $a \in A$.

Here we note that the homomorphism in Theorem 2.11 need *not* be bounded; this is due to [25, Proposition 2.1].

Another concept that will be needed is that of *ergodicity*. An element a in a Banach algebra A is said to be *ergodic* if the sequence $\left(\sum_{k=0}^{n-1} \frac{a^k}{n}\right)$ converges (in A). A bounded, linear operator T on a Banach space X is called *uniformly ergodic* if the sequence $\left(\sum_{k=0}^{n-1} \frac{T^k}{n}\right)$ converges in $\mathcal{L}(X)$, i.e. T is an ergodic element of the Banach algebra $\mathcal{L}(X)$. We call an operator $T \in \mathcal{L}^r(E)$ (with E a Banach lattice) *r-ergodic* if the sequence $\left(\sum_{k=0}^{n-1} \frac{T^k}{n}\right)$ converges in $\mathcal{L}^r(E)$, i.e. if T is an ergodic element of the Banach algebra $\mathcal{L}^r(E)$.

We will also need the following boundedness conditions, for an element a of a Banach algebra A:

- a is said to be *power bounded* if there exists a constant C such that $||a^n|| \leq C$ for all $n \in \mathbb{N}$,
- a is *Cesàro bounded* if there exists a constant C such that $||M_n(a)|| \leq C$ for all $n \in \mathbb{N}$, where $M_n(a) = \sum_{k=0}^{n-1} \frac{a^k}{n}$, and
- a is *Abel bounded* if there exists a constant C such that

$$\left\| (1-\theta) \sum_{k=0}^{\infty} \theta^k a^k \right\| \leq C$$

  for all $\theta \in (0, 1)$.

Note that the following implications hold (see [24]):
power bounded ⇒ Cesàro bounded ⇒ Abel bounded

The notion of *uniformly Abel bounded* is obtained by replacing ∞ in the definition of Abel bounded with n and requiring that C works for all n. Additional types of Abel boundedness conditions are obtained by replacing $(1-\theta)$ by $(1-\theta)^N$. An element a is said to be *doubly* bounded of one of these forms if a is invertible and both a and a^{-1} are bounded of the same form.

The *boundary spectrum* of an element a in a Banach algebra A was introduced in [46] as

$$S_\partial(a) := \{\lambda \in \mathbb{C} : \lambda - a \in \partial A^{-1}\},$$

where ∂K indicates the topological boundary of a set K in a metric space. It is easy to see [46, Proposition 2.1] that $\partial\sigma(a) \subseteq S_\partial(a) \subseteq \sigma(a)$ and that $S_\partial(a)$ is a closed set. Therefore, the boundary spectrum of a is a non-empty compact subset of the complex plane, for every $a \in A$. In general, $\partial\sigma(a) \neq S_\partial(a) \neq \sigma(a)$ (see [46, Example 2.3] and [48, Examples 2.3 and 3.8]).

3 Ordered Banach algebras

A complex unital Banach algebra A is called an *ordered Banach algebra* (OBA) [58] if A contains an *algebra cone*, i.e. a subset C (not necessarily closed) with the properties that C contains the unit 1 and is closed under (a) addition, (b) non-negative real scalar multiplication and (c) multiplication. If, instead of condition (c), C only satisfies the condition

$$a, b \in C, \quad ab = ba \quad \Rightarrow \quad ab \in C,$$

then C is called an *algebra c-cone* and A is called a *commutatively ordered Banach algebra* (COBA) [50]. A non-empty subset C of A which is only closed under addition and under non-negative real scalar multiplication is called a *space cone*. A cone C is said to be *proper* if $C \cap -C = \{0\}$.

A Banach algebra A is partially ordered by an algebra cone C as follows:

$$a \leq b \quad \text{if and only if} \quad b - a \in C$$

This ordering is reflexive and transitive, and it is antisymmetric if and only if C is proper. We often refer to the condition $a \leq b$ by saying that a is dominated by b or b dominates a. It turns out that $C = \{a \in A : a \geq 0\}$ and therefore the elements of C are called the *positive* elements.

An algebra cone C is said to be *normal* if there exists a constant $\alpha > 0$ with the property that if $0 \leq a \leq b$ (relative to C), then $||a|| \leq \alpha||b||$. Clearly, the property of normality reconciles the order structure and the topology of A. It is obvious that if C is normal, then C is proper. It is also easy to see that if C is normal (respectively, closed) relative to a norm $||\cdot||$, then C is normal (respectively, closed) relative to any norm equivalent to $||\cdot||$. We note that if C is closed and normal, then C does not contain any interior points (recall that A is a complex space); in fact, C does not even contain any absorbing points—see [49, Proposition 3.4]. We say that C is *generating* if span $C = A$, and C is *inverse-closed* if it has the property that if $a \in C$ and a is invertible, then $a^{-1} \in C$. Finally, we say that the spectral radius is *monotone* (w.r.t. C in A) if $r(a) \leq r(b)$ whenever $0 \leq a \leq b$. If C is normal, then the spectral radius is monotone [58, Theorem 4.1].

The *Neumann series* plays an important role in ordered Banach algebras. In general,

$$|\lambda| > r(a) \Rightarrow (\lambda - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}.$$

Therefore, if C is closed and $a \in C$, then $(\lambda - a)^{-1} \in C$ for all real $\lambda > r(a)$. This implies that if $a \in C$ and $r(a)$ is a pole of order k of $(\lambda - a)^{-1}$, then the coefficient a_{-k} in the Laurent series of $(\lambda - a)^{-1}$ is a positive element [52, proof of Theorem 3.2].

At this point we note that all the preceding concepts also make sense in the context of a COBA with an algebra c-cone. However, in view of [50, Proposition 3.12 and Example 3.13], COBA-statements sometimes contain commutativity assumptions that were not necessary in their OBA-counterparts.

It is well known that if a and b are commuting elements in a Banach algebra, then $r(ab) \leq r(a)r(b)$ and $r(a+b) \leq r(a) + r(b)$. In an OBA we have that if the algebra cone C is normal and $a, b \in C$ satisfy $ab \leq ba$, then $r(ab) \leq r(a)r(b)$ [58, Proposition 4.4] and $r(a+b) \leq r(a) + r(b)$ [45, Theorem 4.7].

3.1 Subalgebras, direct sums and quotients

In this section we will consider methods of obtaining new OBAs from given OBAs. First of all, although we have defined the concept of an ordered Banach algebra for a complex algebra, the definition also makes sense in the case of a real algebra, in which case we consider the complexification of the algebra, as follows: If A is a real OBA with algebra cone C, then the complexification $A_{\mathbb{C}} = A \oplus iA$ of A (with the norm given by the Minkowski functional—see [12, §1 and §13]) is a complex OBA with algebra cone C. In addition, C is closed (proper, normal, generating, inverse-closed) in A if and only if C is closed (proper, normal, generating, inverse-closed) in $A_{\mathbb{C}}$. (We restrict our attention to complex Banach algebras, unless stated otherwise.)

If C is an algebra cone of (a complex OBA) A and B is a closed subalgebra of A containing the unit of A, then $C \cap B$ is an algebra cone of B, and if C is closed (proper, normal) in A, then $C \cap B$ is closed (proper, normal) in B.

If C_i is an algebra cone of an OBA A_i for $i = 1, \ldots, n$, then $C = C_1 \oplus C_2 \oplus \cdots \oplus C_n$ is an algebra cone of $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$. Furthermore, if C_i is closed (proper, normal, generating, inverse-closed) in A_i for all $i = 1, \ldots, n$, then C is closed (proper, normal, generating, inverse-closed) in A.

Algebra c-cones in COBAs behave similarly with respect to subalgebras and direct sums.

For a Banach algebra B and a homomorphism $T : A \to B$, we have that if C is an algebra cone of A, then TC is an algebra cone of B. Unfortunately, properties of C do not in general carry over to TC. However, if T is onto and C is generating in A, then TC is generating in B. If A has algebra cone C and F is a closed ideal in A, then we often consider the canonical homomorphism $\pi : A \to A/F$, and A/F is an OBA with algebra cone πC, which is generating in A/F if C is generating in A. The spectral radius is called *weakly monotone (w.r.t. πC in A/F)* if it follows from $0 \le a \le b$ in A that $r(a + F) \le r(b + F)$. The property of weak monotonicity of the spectral radius in the quotient algebra is sufficient to obtain many results, and we will see that it is practically applicable as well. (For particular reasons we have been referring to this property in the literature just as "monotonicity" of the spectral radius in the quotient algebra, but to be completely consistent, and to avoid possible confusion, we will now rather call it "weak monotonicity".)

If C in the previous discussion is only an algebra c-cone of A, then, in general, TC is only an algebra c-cone of B if T is injective. Therefore πC is in general not an algebra c-cone of A/F. However, this deficiency can be dealt with by using either of the concepts of *algebra c'-cone* and *maximal positive commutative set*—see [50].

3.2 Examples

The most trivial example of an ordered Banach algebra is the (commutative) Banach algebra $\mathbb{C}$ of all complex numbers, which has a closed, normal, generating and inverse-closed algebra cone $\mathbb{R}^+$. Considering matrix algebras: if C is the subset of $M_n(\mathbb{C})$ (or $M_n^u(\mathbb{C})$) consisting of all matrices with only nonnegative real entries, then it is easy to check that $M_n(\mathbb{C})$ (or $M_n^u(\mathbb{C})$) with the norm (2.2) is an OBA with closed, normal and generating algebra cone C, which is, however, not inverse-closed. Since the norm (2.1) and the norm (2.2) are equivalent in $M_n(\mathbb{C})$, the algebra cone C is also closed and normal in the OBA $M_n(\mathbb{C})$ (or $M_n^u(\mathbb{C})$) relative to the norm (2.1).

Turning to sequence algebras, let C be the subset of l^∞ consisting of all sequences with only nonnegative real entries. Then l^∞ is an OBA with closed, normal, generating and inverse-closed algebra cone C. More generally, we can consider $A = l^\infty(M_n(\mathbb{C}))$ or $A = l^\infty(M_n^u(\mathbb{C}))$. Let C be the subset of A consisting of all sequences having as entries only matrices with nonnegative real entries. Then A is an OBA with closed, normal and generating algebra cone C. Other Banach algebras than the matrices can be used in the definition of A, but then the algebra cone would not necessarily be generating.

Let C be the subset of $C(K)$ (with K a compact Hausdorff space) consisting of all functions which are real and nonnegative at every point of K. Then $C(K)$ is an OBA with closed, normal, generating and inverse-closed algebra cone C. If, instead, C denotes the subset of $A(D)$ (with D the open unit disc in $\mathbb{C}$) consisting of all functions which are real and nonnegative at every point of $\overline{D}$, then $A(D)$ is an OBA with closed, normal and inverse-closed algebra cone C.

For a Hilbert space H we let C be the subset of $\mathcal{L}(H)$ consisting of all positive operators (i.e. T such that $\langle Tx, x\rangle \geq 0$ for all $x \in H$). Then $\mathcal{L}(H)$ is a COBA with closed, normal and inverse-closed algebra c-cone C. More generally, let A be a C^*-algebra and let $C = \{a \in A : a = a^*,\ \sigma(a) \subseteq [0, \infty)\}$. Then A is a COBA with closed, normal and inverse-closed algebra c-cone C. If A is commutative, then A is an OBA. More examples of COBAs can be found in [50].

Now, let X be any complex ordered Banach space with (space) cone C and let $K = \{T \in \mathcal{L}(X) : TC \subseteq C\}$. Then $\mathcal{L}(X)$ is an OBA with algebra cone K and if C is closed in X, then K is closed in $\mathcal{L}(X)$. The other algebra cone-properties do not, in general, carry over from C to K. However, if we replace X with a complex Banach lattice E with cone $C = \{x \in E : x = |x|\}$, we can say more: if $K = \{T \in \mathcal{L}(E) : TC \subseteq C\}$, then both $\mathcal{L}(E)$ and $\mathcal{L}^r(E)$ are OBAs with closed and normal algebra cone K, and K is generating in $\mathcal{L}^r(E)$. (The normality of K follows from [59, Lemma 3]). In addition, both $\mathcal{L}(E)/\mathcal{K}(E)$ and $\mathcal{L}^r(E)/\mathcal{K}^r(E)$ are OBAs with algebra cone πK, and πK is generating in $\mathcal{L}^r(E)/\mathcal{K}^r(E)$. Although πK is, in general, not normal in either quotient algebra, we do have that if both E and E' have order continuous norm, then πK is proper in $\mathcal{L}(E)/\mathcal{K}(E)$ and in $\mathcal{L}^r(E)/\mathcal{K}^r(E)$. We will justify this remark in Sect. 4.2 (see Proposition 4.2.3 and the remark thereafter). Finally, if E is Dedekind complete, then the spectral radius in $\mathcal{L}^r(E)/\mathcal{K}^r(E)$ is weakly monotone. This very important fact was proved in 1991 by Martinez and Mázon in [35, Theorem 2.8]. This result of Martinez and Mázon was generalised to the setting of Banach lattice algebras in [33, Theorem 3.8]. We note, in particular, that πK is not necessarily normal in $\mathcal{L}^r(E)/\mathcal{K}^r(E)$—see [58, Example 4.2].

In [30] Herzog and Schmoeger illustrated that any Banach algebra B can be embedded in an OBA A. The advantage of this is that questions in B can be answered by working in the OBA A—see [30, Theorems 6 and 7].

4 Spectral theory

We will now consider different aspects of spectral theory in OBAs and present the most important results. Many of these results have counterparts in COBAs, although sometimes with additional commutativity assumptions—see [50,51]. For a general survey on results in OBAs we refer the reader to [28,50,52,58].

4.1 Fundamental results

In this section we give the general OBA-versions of the fundamental spectral-theoretical results. Note, however, that these fundamental results have been well-known in an operator context before the OBA-versions were published. In the general

OBA-case, the proofs typically rely on things like Laurent series and other elements of complex analysis, Neumann series, the spectral mapping theorem and other basic analytic properties of Banach algebras, and, in certain cases, on Aupetit's perturbation theorem (Theorem 2.10).

First we have the following generalisation of the original Perron-Frobenius theorem:

Theorem 4.1.1 ([58], Proposition 5.1) *Let A be an OBA with closed and normal algebra cone C. If $a \in C$, then $r(a) \in \sigma(a)$.*

In the 1960s, Schaefer proved this result for positive operators on Banach lattices, using operator-theoretic methods—see [59]. A few years later Schneider and Turner then extended the result to positive operators on ordered Banach spaces in [63]. Their proof can be adapted to provide the above general version of this theorem. It is important to note that the normality of the algebra cone can, in fact, be relaxed to monotonicity of the spectral radius. This was proven by de Pagter and Schep in [19, Proposition 3.3] (see also [58, Theorem 5.2]), and is of particular interest in quotient algebras:

Theorem 4.1.2 ([58], Theorem 5.3) *Let A be an OBA with closed algebra cone C and let F be a closed ideal of A such that the spectral radius in A/F is weakly monotone. If $a \in C$, then $r(a + F) \in \sigma(a + F)$.*

Herzog and Schmoeger proved that (the de Pagter-Schep version of) Theorem 4.1.1 has a converse:

Theorem 4.1.3 ([29], Theorem 2) *Let A be a Banach algebra with $a \in A$. If $r(a) \in \sigma(a)$, then there exists an algebra cone C in A such that $a \in C$ and the spectral radius is monotone w.r.t. C.*

The original Krein-Rutman theorem states that the spectral radius $r(T)$ of a positive compact operator T on an ordered Banach space is an eigenvalue of T, with a positive eigenvector (if $r(T) \neq 0$). This result was proven by Krein and Rutman in [34], with operator-theoretic techniques. Then, in 1986, de Pagter [18] proved that if T is an ideal irreducible positive compact operator, then $r(T) \neq 0$. (Irreducibility will be discussed in Sect. 4.6).

By replacing T with the left or right regular representation on a Banach algebra A, the conclusion of the Krein-Rutman theorem makes sense in A. Also, the spectral properties of a Riesz operator, or a Riesz element in a Banach algebra, are similar to those of a compact operator; in particular, in both cases nonzero points of the spectra are poles of the resolvents. Therefore it makes sense to use Riesz elements in the OBA version of the Krein-Rutman theorem:

Theorem 4.1.4 ([52], Theorem 3.7) *Let A be a semisimple OBA with closed and normal algebra cone C and let $0 \neq a \in C$ be such that $r(a) > 0$. If I is a closed inessential ideal in A such that a is a Riesz element, then there exists $0 \neq u \in C$ such that $ua = au = r(a)u$ and $aua = r(a)^2 u$.*

A possible choice for the positive element u in Theorem 4.1.4 is the coefficient a_{-k} in the Laurent series of the resolvent of a around the order k pole $r(a)$—see [52].

In the 1960s Schaefer proved that if T is a positive operator on a Banach lattice such that the spectral radius of T is a Riesz point of its spectrum, then the peripheral spectrum of T consists of Riesz points. In the general case, we have:

Theorem 4.1.5 ([43], Theorem 4.3; [52], Theorem 4.1) *Let A be an OBA with closed algebra cone C and let I be a closed inessential ideal of A such that the spectral radius in A/I is weakly monotone. If $a \in C$ is such that $r(a)$ is a Riesz point of $\sigma(a)$, then* $\mathrm{psp}(a)$ *consists of Riesz points.*

Theorem 4.1.5 relies on Aupetit's perturbation theorem. If the Banach lattice under consideration is Dedekind complete, then, as was mentioned in Sect. 3.2, the spectral radius in the quotient algebra of the regular operators modulo the r-compact operators is weakly monotone. Therefore we can apply this result to this case. We can also apply the result to C^*-algebras: since the quotient of a C^*-algebra modulo a closed ideal is again a C^*-algebra, its algebra cone is normal, and therefore the spectral radius in the quotient algebra is monotone. In particular, we can consider the operators on a Hilbert space. These observations are contained in the following corollary:

Corollary 4.1.6 ([43], Corollaries 4.9–4.10; [52], Corollary 4.2)

1. *Let T be a positive operator on a Dedekind complete Banach lattice E. If $r_o(T)$ is a Riesz point of $\sigma_o(T)$, then* $\mathrm{psp}_o(T)$ *consists of Riesz points (relative to $\mathcal{K}^r(E)$).*
2. *Let T be a positive operator on a Hilbert space. If $r(T)$ is a Riesz point of $\sigma(T)$, then* $\mathrm{psp}(T)$ *consists of Riesz points.*

Note that if A is an OBA with an algebra cone C and I is a closed ideal in A, then many results in spectral theory of OBAs rely on the assumption that the spectral radius in the quotient algebra A/I is weakly monotone—see, for instance [9, 11, 43, 51, 52, 58].

4.2 Domination properties

The following problem is often referred to as the *domination problem*:

Let A be an OBA and let $a, b \in A$ such that $0 \leq a \leq b$. Given a spectral property (P), provide conditions which will ensure that if b satisfies (P), then a satisfies (P).

The best-known domination results are undoubtedly the following results of Aliprantis and Burkinshaw, and of Dodds and Fremlin:

Theorem 4.2.1 ([4], Theorems 2.1 and 2.2) *Let S and T be operators on a Banach lattice E such that $0 \leq S \leq T$. Suppose that T is compact.*

1. *Then S^3 is compact.*
2. *If either E or E' have order continuous norm, then S^2 is compact.*

Theorem 4.2.2 ([20], Theorem 4.5) *Let S and T be operators on a Banach lattice E such that both E and E' have order continuous norm. If $0 \leq S \leq T$ and T is compact, then S is compact.*

In [58] it was observed that domination relative to an ideal is, in fact, equivalent to properness of the algebra cone in the quotient modulo the ideal:

Proposition 4.2.3 ([58], Theorem 6.1) *Let A be an OBA with algebra cone C and F a closed ideal of A. Then the following are equivalent:*

1 *If $a, b \in A$ are such that $0 \leq a \leq b$ and $b \in F$, then $a \in F$.*
2. *The algebra cone πC in A/F is proper.*

Together with Theorem 4.2.2 it can now be seen that if both E and E' have order continuous norm, $C = \{x \in E : x = |x|\}$, $K = \{T \in \mathcal{L}(E) : TC \subseteq C\}$ and $\pi : \mathcal{L}(E) \to \mathcal{L}(E)/\mathcal{K}(E)$ is the canonical homomorphism, then the algebra cone πK is proper, as was mentioned in Sect. 3.2.

Again, due to the correspondence between the spectral properties of compact operators and Riesz operators, it makes sense to replace "compact operator" with "Riesz element" in the OBA-setting. The following result then follows immediately:

Theorem 4.2.4 ([58], Theorem 6.2) *Let A be an OBA with closed algebra cone C, F a closed ideal of A such that the spectral radius in A/F is weakly monotone and $a, b \in A$ such that $0 \leq a \leq b$. If b is a Riesz element, then a is a Riesz element.*

As before, this result can be applied to the regular operators on a Dedekind complete Banach lattice and to C^*-algebras; in particular to the operators on a Hilbert space:

Corollary 4.2.5 ([58], Corollaries 6.3 and 6.6)

1. *Let E be a Dedekind complete Banach lattice and $S, T \in \mathcal{L}^r(E)$ such that $0 \leq S \leq T$. If T is r-asymptotically quasi finite rank, then S is r-asymptotically quasi finite rank.*
2. *Let H be a Hilbert space and $S, T \in \mathcal{L}(H)$ such that $0 \leq S \leq T$ and $ST = TS$. If T is a Riesz operator, then S is a Riesz operator.*

Very recently, Koumba and Raubenheimer [33] proved that the commutativity condition in (2) can be omitted, and also that S is in fact compact. This has the interesting consequence that, on a Hilbert space, there exists no positive Riesz operator which is not compact. Finally, we remark that Troitsky in [64] also investigated the domination problem for Riesz operators and provided conditions for a positive operator dominated by a Riesz operator to be a Riesz operator.

Next we consider domination relative to Riesz points. The first result of note in this direction was given by Caselles in 1987:

Theorem 4.2.6 ([16], Theorem 4.1) *Let E be a Banach lattice and let $S, T \in \mathcal{L}(E)$ such that $0 \leq S \leq T$ and $r(S) = r(T)$. If $r(T)$ is a Riesz point of $\sigma(T)$, then $r(S)$ is a Riesz point of $\sigma(S)$.*

Theorem 4.2.6 was strengthened by Räbiger and Wolff in 1997 (see [57, Theorem 3.1]) by using a weaker form of domination which does not imply that the smaller operator is positive.

A general version of Theorem 4.2.6, relying on Aupetit's perturbation theorem, was obtained in 1997:

Theorem 4.2.7 ([52], Theorem 4.3) *Let A be an OBA with closed and normal algebra cone C, I a closed inessential ideal of A such that the spectral radius in A/I is weakly monotone and $a, b \in A$ such that $0 \leq a \leq b$ and $r(a) = r(b)$. If $r(b)$ is a Riesz point of $\sigma(b)$, then $r(a)$ is a Riesz point of $\sigma(a)$.*

The above result implies Theorem 4.2.6 for the order spectra of the regular operators on a Dedekind complete Banach lattice, and also applies to positive operators on Hilbert spaces (see [52, Corollary 4.4] and [43, Corollaries 4.9–4.10]).

Domination of quasi compact operators (see definition following Theorem 2.10) was investigated by Martinez and Mázon in 1991:

Theorem 4.2.8 ([35], Proposition 2.5) *Let E be a Banach lattice and let $S, T \in \mathcal{L}(E)$ such that $0 \leq S \leq T$ and $r(T) \leq 1$. If T is quasi compact, then S is quasi compact.*

The general OBA-version of Theorem 4.2.8 was originally given, under additional conditions, in [52], but Muzundu noticed that some of these conditions could be omitted:

Theorem 4.2.9 ([50], pp. 573–574; [52], Corollary 5.4) *Let A be an OBA with algebra cone C, I a closed inessential ideal of A such that the spectral radius in A/I is weakly monotone and $a, b \in A$ such that $0 \leq a \leq b$. If b is quasi inessential, then a is quasi inessential.*

So comparing with Theorem 4.2.8, we see that we do not need $r(b) \leq 1$ in the general version, but instead we need to assume weak monotonicity of the spectral radius in the quotient algebra.

We now turn to domination of radical elements, where we find that generating algebra cones behave well:

Theorem 4.2.10 ([38], Theorem 4.6) *Let A be an OBA with normal and generating algebra cone C and $a, b \in A$ such that $0 \leq a \leq b$. If $b \in \mathrm{Rad}(A)$, then $a \in \mathrm{Rad}(A)$.*

Note that the proof of this theorem relies on Aupetit's scarcity theorem (Theorem 2.3), via Corollary 2.5. The algebra cones in all the examples mentioned before are normal, and many of them are generating. An example of a (non-semisimple) OBA to which Theorem 4.2.10 applies is $l^\infty(M_n^u(\mathbb{C}))$, the algebra of all bounded sequences of upper triangular $n \times n$ matrices.

Normality of the algebra cone in Theorem 4.2.10 may be replaced by monotonicity of the spectral radius. Using this observation, Behrendt and Raubenheimer noticed that, as a result, the inessential elements have the following domination property:

Theorem 4.2.11 ([9], Theorem 4.1) *Let A be an OBA with generating algebra cone C, F a closed ideal of A such that the spectral radius in A/F is monotone and $a, b \in A$ such that $0 \leq a \leq b$. If b is inessential (relative to F), then a is inessential (relative to F).*

They also gave interesting complementary results to Theorem 4.2.10 about domination of radical elements, such as:

Theorem 4.2.12 ([9], Corollary 3.4) *Let A be an OBA with normal algebra cone C and $a, b \in A$ such that $0 \leq a \leq b$. If $b \in \mathrm{QN}(A)$ and $a + a^2 \in \mathrm{Rad}(A)$, then $a \in \mathrm{Rad}(A)$.*

In fact, in the above result the polynomial $a+a^2$ can be replaced by any polynomial in a which contains a as a term and has constant term zero—see [9, Theorem 3.3].

In [9] the authors further illustrate by examples that a positive element dominated by a rank one (respectively, finite rank) element is not necessarily a rank one (respectively, finite rank) element.

Finally, we consider domination of ergodic elements. In this regard, the main result is the following:

Theorem 4.2.13 ([51], Theorem 5.5) *Let A be a semisimple OBA with closed and normal algebra cone C, I a closed inessential ideal of A such that the spectral radius in A/I is weakly monotone, and $a, b \in A$ such that $0 \leq a \leq b$ and $r(b)$ is a Riesz point of $\sigma(b)$. If b is ergodic, then a is ergodic.*

In order to establish Theorem 4.2.13, the *ergodic domination theorem*, the authors first established a theorem, the *ergodic theorem* [51, Theorem 4.10], which gives necessary and sufficient conditions for an element in a Banach algebra to be ergodic. This result generalises a theorem that Dunford proved in 1943 for the bounded linear operators on a complex Banach space—see [21, Theorem 3.16]. The proof of the ergodic theorem uses many of Dunford's ideas and relies on the Riesz functional calculus and complex analysis. Besides the ergodic theorem, the proof of the ergodic domination theorem also relies on Theorem 4.2.7.

Theorem 4.2.13 has the following corollary for the regular operators:

Corollary 4.2.14 *Let E be a Dedekind complete Banach lattice and $S, T \in \mathcal{L}^r(E)$ such that $0 \leq S \leq T$ with $r(T)$ a Riesz point of $\sigma_o(T)$ (relative to $\mathcal{K}^r(E)$). If T is r-ergodic, then S is r-ergodic.*

Again, under a weaker form of domination which does not imply that S is positive, this result was proved for the bounded linear operators on a Banach lattice (with the usual norm) by Räbiger and Wolff [57, Theorem 4.5] in 1997, using operator-theoretic methods. (See also the original result by Caselles in [16, Corollary 4.6]).

We conclude this section with some open questions regarding the domination problem in ordered Banach algebras:

- Can any of the results currently relying on weak monotonicity of the spectral radius in the quotient algebra be proved without this condition? Maybe if the algebra cone in the original Banach algebra is assumed to be normal and/or generating? (This could allow these results to apply to other cases than the regular operators.)
- Can the ergodic domination theorem be extended by replacing the condition "$0 \leq a \leq b$" with the weaker condition "$\pm a \leq b$"?

4.3 Gelfand-Hille theorems

In this section we will discuss what should rightly be known as the *Gelfand-Hille/Huijsmans-De Pagter problem*:

Let A be an OBA and let $a \geq 0$. Under what conditions does it follow from $\sigma(a) = \{1\}$ that $a \geq 1$?

The original interest was in the problem of providing conditions, generally in the absence of an ordering, which would ensure that $\sigma(a) = \{1\}$ implied $a = 1$. Theorems of this nature are referred to in the literature as *Gelfand-Hille theorems*, since the first interesting results in this area were provided by Gelfand (see below) and Hille:

Theorem 4.3.1 ([22]) *Let a be a doubly power bounded element of a Banach algebra. If $\sigma(a) = \{1\}$, then $a = 1$.*

In 1944 Hille elaborated on this result in [31]. A number of authors obtained generalisations of the Gelfand theorem by replacing power boundedness with weaker boundedness conditions. Theorem 4.3.1 was generalised to doubly Cesàro bounded elements by Mbekhta and Zemánek in 1993 (see [36]), and to elements with certain Abel boundedness properties by Grobler and Huijsmans in 1995 (see [24]).

In 2009 Braatvedt et al. investigated Gelfand-Hille type theorems in an OBA-context. They showed that, for positive elements in an OBA with closed and normal algebra cone, the notions of Cesàro boundedness and Abel boundedness coincide—see [13, Theorem 2.1]; therefore, by the Mbekhta-Zemánek theorem, the following holds:

Theorem 4.3.2 ([13], Corollary 2.2) *Let A be an OBA with closed and normal algebra cone C and let $a \in C$ be a doubly Abel bounded element with $a^{-1} \in C$. If $\sigma(a) = \{1\}$, then $a = 1$.*

It is also interesting that this result is true in a Banach lattice algebra, without any assumptions of Abel boundedness. This was shown by Huijsmans in 1988—see [32].

If normality of the algebra cone is relaxed to properness, the following can be said:

Theorem 4.3.3 ([13], Theorem 2.7) *Let A be an OBA with closed and proper algebra cone C, $a \in A$ and suppose there exist $L, N \in \mathbb{N}$ such that a^L is Abel bounded and $a^N \geq 1$. If $\sigma(a) = \{1\}$, then $a = 1$.*

An important tool in the proof of Theorem 4.3.3 is the fact that any element x in a Banach algebra with $\sigma(x) = \{1\}$ has the property that x is Abel bounded if and only if x^N is Abel bounded for all $N \in \mathbb{N}$ (or for some $N \in \mathbb{N}$)—see [13, Theorems 2.4 and 2.6].

For inverse-closed algebra cones, we have the following:

Theorem 4.3.4 *([13], Theorem 4.1) Let A be an OBA with closed, proper and inverse-closed algebra cone C and $a \in A$ such that $a^N \in C$ for some $N \in \mathbb{N}$. If $\sigma(a) = \{1\}$, then $a = 1$.*

Besides boundedness conditions, other types of conditions have also been investigated in this problem. For instance, in 1978 Schaefer et al. showed that, in the case of operators on Banach lattices, we have:

Theorem 4.3.5 ([62], Corollary 2.2) *Let T be a lattice homomorphism of a Banach lattice. If $\sigma(T) = \{1\}$, then T is the identity operator I.*

Motivated by this result, Huijsmans and de Pagter (see [67]) then asked the more general question: If T is a positive operator on a Banach lattice with $\sigma(T) = \{1\}$, does it follow that $T \geq I$? If this were true, it would imply Theorem 4.3.5.

Let us begin our discussion of this more general problem by investigating the finite-dimensional case:

Theorem 4.3.6 ([44], Theorem 4.4) *If A is a semisimple finite-dimensional Banach algebra, then A is isomorphic (as an algebra) to an OBA with closed and normal algebra cone C which has the property that if $a \in C$ and $\sigma(a) = \{1\}$, then $a \geq 1$.*

This result is obtained by considering the $n \times n$ matrices, direct sums of OBAs and the Wedderburn-Artin theorem. In this regard, see also [67, Theorem 4.1].

The Huijsmans-de Pagter version of the problem has originally been investigated by Zhang in [66,67] for the bounded linear operators on a Banach lattice:

Theorem 4.3.7 ([67], Theorem 5.3) *Let T be a positive operator on a Banach lattice. If $\sigma(T) = \{1\}$ and 1 is a pole of the resolvent of T, then $T \geq I$.*

In 2003, this problem was investigated in the general OBA-context, using properties of Neumann and Laurent series:

Theorem 4.3.8 ([44], Corollary 4.9) *Let A be an OBA with closed algebra cone C and let $a \in C$. If $\sigma(a) = \{1\}$ and 1 is a pole of the resolvent of a of order 1 or 2, then $a \geq 1$. More generally, if 1 is a pole of the resolvent of a of order $k + 1$, then $(a - 1)^k \in C$.*

In the case of inverse-closed algebra cones, the following can be said:

Theorem 4.3.9 ([44], Theorem 4.23) *Let A be an OBA with closed, proper and inverse-closed algebra cone C and let $a \in C$.*

1. *If $d(0, \sigma(a)) \geq 1$, then $a \geq 1$.*
2. *If $\sigma(a) \subseteq C(0, 1)$, then $a = 1$.*
3. *In particular, if $\sigma(a) = \{1\}$, then $a = 1$.*

Note that Theorem 4.3.9 (3) was strengthened by Braatvedt, Brits and Raubenheimer in Theorem 4.3.4.

Finally, the following are some open questions regarding the Gelfand-Hille theorems:

- Can any of the relevant results be improved if the algebra cone is assumed to be normal?
- In particular, in the case where 1 is assumed to be a pole of the resolvent of a, can the first part of Theorem 4.3.8 be proved for orders higher than 2?

4.4 Spectral continuity

The subject of spectral continuity had been initiated by Newburgh [54] in 1951, and he showed that the spectrum and spectral radius functions are

– upper semi-continuous,
– continuous at points with totally disconnected spectra and
– uniformly continuous on commutative Banach algebras.

By the "spectrum function" we mean the function from a Banach algebra into the set of all compact subsets of the complex plane, with the Hausdorff metric, which maps an element onto its spectrum. The spectral radius function is just the function from a Banach algebra into the positive real numbers mapping an element onto its spectral radius. Continuity of the spectrum function would imply continuity of the spectral radius function, but not vice versa.

Since 1951 this subject has been studied extensively, and several authors have made contributions. Most of the important earlier results were presented in [7], and in her survey paper [14] of 1994 Burlando gave an extensive account of these and of subsequent results up to that time, supplying several useful references. A more recent paper on the subject is [15].

However, spectral continuity in the context of an ordered Banach algebra has only been studied in recent years. A well-known example by Kakutani illustrates that the spectrum and spectral radius functions are, in particular, not continuous on the set of positive elements in an ordered Banach algebra:

Example 4.4.1 ([7], Example p. 49) Let $l^2(\mathbb{N})$ be the complex Banach lattice of all square-summable sequences. Then there exist a sequence of positive operators $(T_n) \in \mathcal{L}(l^2(\mathbb{N}))$ and a positive operator $T \in \mathcal{L}(l^2(\mathbb{N}))$ such that the spectrum of T_n is zero for all $n \in \mathbb{N}$ and $T_n \to T$ as $n \to \infty$, but the spectrum of T is not zero.

Motivated by this fact, it seems indicated to investigate results of the following types:

I. If a is a positive element, then there exists a set $C(a)$ such that $a \in C(a)$, contained in the set of all positive elements and with $C(0) = C$, such that the restriction of the spectral radius function to $C(a)$ is continuous at a.
II. If a is a positive element satisfying certain properties, then the restriction of the spectral radius function to the set of all positive elements is continuous at a.

Let A be an OBA with algebra cone C. For $a \in C$, define [45] the subsets $A(a)$ and $C(a)$ of A by

$$A(a) = \{x \in A : a \leq x, \quad (ax \leq xa \text{ or } xa \leq ax) \quad \text{and}$$

$$d(r(x), \sigma(a)) \geq d(\lambda, \sigma(a)) \quad \text{for all} \quad \lambda \in \sigma(x)\}$$

and

$$C(a) = \{x \in A : a \leq x \quad \text{and} \quad (ax \leq xa \quad \text{or} \quad xa \leq ax)\}.$$

It is clear that $a \in A(a) \subseteq C(a) \subseteq C$ and that $A(0) = C = C(0)$. Both $A(a)$ and $C(a)$ can, informally, be considered as generalisations of the commutant of a.

Now, from Newburgh's early results we know the following:

Theorem 4.4.2 ([7], Theorem 3.4.1; [54]) *If a is an element of a Banach algebra A, then*

$$\sigma(x) \subseteq \sigma(a) + r(a - x)$$

for all x in the commutant of a.

The meaning of the displayed formula in the above theorem is that $d(\lambda, \sigma(a)) \le r(a - x)$ for all $\lambda \in \sigma(x)$.

Analogously, it is possible to show that, in an OBA, we have the following property:

Theorem 4.4.3 ([45], Theorem 4.2) *Let A be an OBA with closed and normal algebra cone C. If $a \in C$, then*

$$\sigma(x) \subseteq \sigma(a) + r(a - x)$$

for all $x \in A(a)$.

Analogous results are available for a number of sets defined similarly as $A(a)$, but with the condition "$d(r(x), \sigma(a)) \ge d(\lambda, \sigma(a))$ for all $\lambda \in \sigma(x)$" replaced by other spectral properties — see [45].

Similarly, in general and in an OBA, respectively, we have the following results:

Theorem 4.4.4 ([7], Theorem 3.4.1; [54]) *If a is an element of a Banach algebra A and $\{a\}^c$ denotes the commutant of a, then $r_{|_{\{a\}^c}}$ is continuous at a.*

Theorem 4.4.5 ([45], Corollary 4.9) *Let A be an OBA with a normal algebra cone C. If $a \in C$, then $r_{|_{C(a)}}$ is continuous at a.*

It is now apparent that Theorem 4.4.5 is a spectral continuity result of Type I. Theorems 4.4.3 and 4.4.5 were proved using a number of fundamental properties of positive elements. In view of Theorems 4.4.2 and 4.4.4, it should be checked that the sets $A(a)$ and $C(a)$ contain elements which do not commute with a. In this sense, Theorems 4.4.3 and 4.4.5 can be applied to $A = \mathcal{L}(l^p(\mathbb{N}))$, where $1 \le p \le \infty$ and $l^p(\mathbb{N})$ is the complex Banach lattice of all p-summable sequences, as well as to $A = l^\infty(M_2^u(\mathbb{C}))$ (see [45, Examples 5.1 and 5.2]).

Our next result is a spectral continuity result of Type II:

Theorem 4.4.6 ([43], Theorem 4.5) *Let A be an OBA with closed algebra cone C and I a closed inessential ideal of A such that the spectral radius in A/I is weakly monotone. If $a \in C$ is such that $r(a)$ is a Riesz point of $\sigma(a)$, then $r_{|_C}$ is continuous at a.*

The proof of Theorem 4.4.6 relies on the upper semi-continuity of spectrum, another one of Newburgh's theorems and Aupetit's perturbation theorem. The weak monotonicity condition implies that this result has applications to sequences of positive operators on a Dedekind complete Banach lattice which converge in the r-norm, as well as to uniformly convergent sequences of positive operators on a Hilbert space—see [43, Corollaries 4.9–4.10].

In [47] the concept of boundary spectrum was utilised to obtain the following spectral continuity result:

Theorem 4.4.7 ([47], Theorem 4.6) *Let A be an OBA with closed and normal algebra cone C. If $a \in C$ such that $S_\partial(a) \cap \mathbb{R}^+ = \{r(a)\}$, then $r_{|C}$ is continuous at a.*

The proof of Theorem 4.4.7, which is a spectral continuity result of Type II, relies on the upper semicontinuity of the map $a \mapsto T(a)$ [47, Theorem 4.5], where

$$T(a) := \{\lambda \in \mathbb{C} : |\lambda| \in S_\partial(a)\}.$$

For each $a \in A$ the set $T(a)$ is a compact subset of the complex plane. If a is a positive element, then $T(a)$ is not empty and $T(a)$ contains the spectral radius of a. However, if a is not positive, then $T(a)$ may be empty and even if $T(a)$ is not empty, it does not necessarily contain the spectral radius of a [47, Lemma 4.3 and Example 4.4].

In order to illustrate applications of Theorem 4.4.7, we start with the following:

Example 4.4.8 ([26], Problem 84; [47], Example 4.7) Let $l^2(\mathbb{Z})$ be the complex Banach lattice of all bilateral square-summable sequences, and consider the OBA $A = \mathcal{L}(l^2(\mathbb{Z}))$, with algebra cone

$$K = \{T \in \mathcal{L}(l^2(\mathbb{Z})) : TC \subseteq C\},$$

where

$$C = \{x \in l^2(\mathbb{Z}) : x = |x|\}.$$

If $W : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is the bilateral shift

$$W(\ldots, \xi_{-2}, \xi_{-1}, (\xi_0), \xi_1, \ldots) = (\ldots, \xi_{-2}, (\xi_{-1}), \xi_0, \xi_1, \ldots)$$

(where the term in round brackets indicates the one corresponding to index zero), then $W \in K$ and $\sigma(W) = C(0, 1)$. Hence

$$S_\partial(W) \cap \mathbb{R}^+ = \{r(W)\}.$$

However, since $\sigma(W) \subseteq C(0, r(W))$, the continuity of the spectral radius function at W follows already from the upper semicontinuity of the spectrum. Therefore we consider the following lemma, which will provide us with additional examples to which the theorem can be applied.

Lemma 4.4.9 ([47], Lemma 4.8) *Let A be an OBA with closed and normal algebra cone C, and let $a \in C$ be such that $\sigma(a) = C(0, 1)$. If $0 < \lambda < 1$ and $b_\lambda = a + \lambda$, then $b_\lambda \in C$ and $S_\partial(b_\lambda) \cap \mathbb{R}^+ = \{r(b_\lambda)\}$.*

We see that $\sigma(b_\lambda) \nsubseteq C(0, r(b_\lambda))$, so that the conclusion of Theorem 4.4.7 is not trivial in this case.

In particular, we note that if

$$\alpha(a) = \sup\{\inf\{|\mu| : \mu \in \omega\} : \omega \text{ is a component of } \sigma(a)\},$$

then $\alpha(b_\lambda) < r(b_\lambda)$, so that b_λ does not satisfy the sufficient condition $\alpha(b_\lambda) = r(b_\lambda)$ for continuity of r at b_λ given in 1981 by Murphy in [53, Proposition 1]. On the other hand, b_λ does satisfy the sufficient condition $S_\partial(b_\lambda) \cap \mathbb{R}^+ = \{r(b_\lambda)\}$ for continuity of the restriction of r to C at b_λ given in Theorem 4.4.7.

In addition, let, in Lemma 4.4.9, $A = \mathcal{L}(l^2(\mathbb{Z}))$ and let C be the positive operators on $l^2(\mathbb{Z})$. Then Conway and Morrel showed in 1979 (see [17, Theorem 2.6]) that the spectral radius r is not continuous at b_λ. However, by Theorem 4.4.7 the restriction of r to C is continuous at b_λ.

4.5 The scarcity theorem

This section is devoted to the role of Aupetit's scarcity theorem (Theorem 2.3) in ordered Banach algebras. As mentioned in Sect. 4.2, a corollary of the scarcity theorem (Corollary 2.5) was employed in [38] to solve the domination problem for radical elements in ordered Banach algebras—see Theorem 4.2.10. In the same paper a characterisation of the radical in OBAs with generating algebra cones is obtained (see [49, Theorem 4.10] for a slightly stronger version):

Theorem 4.5.1 ([38], Theorem 4.17) *Let A be an OBA with generating algebra cone C. Then* $\mathrm{Rad}(A) = \{a \in A : aC \subseteq \mathrm{QN}(A)\}$.

In [49] this line of thought was expanded to obtain stronger versions of many spectral theoretical results in ordered Banach algebras in which the algebra cone is suitably well-behaved.

Relying heavily on the scarcity theorem, the main result Theorem 4.5.2 shows that several spectral properties extend from certain subsets of the algebra cone C of an ordered Banach algebra to the linear spans of larger subsets of C:

Theorem 4.5.2 ([49], Lemma 4.1 and Theorem 4.2) *Let A be an OBA with algebra cone C, G a subset of A and B a subset of C which is a space cone of A containing a point which is absorbing in G. Also, let $g : A \to A$ be a $\mathbb{C}$-analytic map.*

1. *If $\#\sigma(g(c)) < \infty$ for all $c \in B \cap G$, then there exists $m \in \mathbb{N}$ such that $\#\sigma(g(x)) \leq m$ for all $x \in \mathrm{span}(B)$.*
2. *If $n \in \mathbb{N}$ and $\#\sigma(g(c)) \leq n$ for all $c \in B \cap G$, then $\#\sigma(g(x)) \leq n$ for all $x \in \mathrm{span}(B)$.*
3. *If $\sigma(g(c)) = \{0\}$ for all $c \in B \cap G$, then $\sigma(g(x)) = \{0\}$ for all $x \in \mathrm{span}(B)$.*

The above theorem applies, for instance, to $G = A$ and $B = B_1 \cap C$ for any vector subspace B_1 of A. The most important case is when $G = A$ and $B = C$, which shows that, when the algebra cone C is generating, then certain properties extend from C to all of A. It is also useful in cases where the algebra cone generates other subsets of the algebra, such as the set of quasinilpotent elements—see [49].

A typical application of Theorem 4.5.2 follows by considering characterisations of finite dimensional Banach algebras. Theorem 2.6 says that, for A a general Banach algebra, $A/\mathrm{Rad}(A)$ is finite dimensional provided that the spectrum is finite on a very "small" part of A, namely on an absorbing set. Theorem 4.5.3 below shows that if A is an ordered Banach algebra with a generating algebra cone C, then in order for $A/\mathrm{Rad}(A)$ to be finite-dimensional, it is sufficient that the spectrum is finite on an even "smaller" part of A: for any subset G of A which contains a point of C which is absorbing in G, the spectrum only has to be finite at all positive elements of G.

Theorem 4.5.3 ([49], Theorem 4.4) *Let A be an OBA with generating algebra cone C, and let G be any subset of A which contains a point of C which is absorbing in G.*

1. *If $\#\sigma(c) < \infty$ for all $c \in C \cap G$, then* $\dim A/\mathrm{Rad}(A) < \infty$.
2. *If $n \in \mathbb{N}$ and $\#\sigma(c) \leq n$ for all $c \in C \cap G$, then* $\dim A/\mathrm{Rad}(A) \leq n^6$.
3. *If $\#\sigma(c) = 1$ for all $c \in C \cap G$, then* $A/\mathrm{Rad}(A) \cong \mathbb{C}$.

It is well known that the spectrum of each element in a finite-dimensional Banach algebra is a finite set, and therefore, by taking $G = A^{-1}$ in Theorem 4.5.3, we have the following property:

Corollary 4.5.4 *Let A be a semisimple OBA with generating algebra cone C. Then:*

1. $\dim A < \infty$ *if and only if the spectrum of each positive invertible element in A is finite.*
2. $A \cong \mathbb{C}$ *if and only if the spectrum of each positive invertible element in A consists of one element only.*

More applications of Theorem 4.5.2 are obtained when the sets of rank one and finite rank elements are investigated. In analogy with Theorem 2.7, we have the following characterisations of these sets in semisimple OBAs with generating algebra cones:

Theorem 4.5.5 ([49], Theorems 4.8 and 4.9) *Let A be a semisimple OBA with generating algebra cone C, and let G be any subset of A which contains a point of C which is absorbing in G. Then*

$$\{a \in A: \text{ there exists } n \in \mathbb{N} \text{ such that } \#\sigma'(ca) \leq n \text{ for all } c \in C \cap G\}$$

$$= \mathrm{Soc}(A) = \{a \in A : \#\sigma'(ca) < \infty \text{ for all } c \in C \cap G\},$$

and if $\dim A = \infty$ *and* $0 \neq a \in A$*, then a is rank one if and only if $\#\sigma'(ca) \leq 1$ for all $c \in C \cap G$.*

In addition, [49, Theorems 4.18 and 4.19] illustrate that, in a semisimple OBA with closed and generating algebra cone, sharper characterisations of the sets of rank one and finite rank elements, analogous to those in Theorems 2.8 and 2.9, can be obtained by replacing the Banach algebra A in (2) and (3) by the smaller set of all positive invertible elements.

4.6 Irreducibility

In 2012, Alekhno [3] established and investigated the concept of irreducible elements in OBAs. His main ideas can, very informally, be described as follows:

Let A be an OBA. Then under natural conditions:

- *The spectrum of a positive element is determined by the spectra of certain irreducible elements.*
- *A (positive) irreducible element has useful spectral properties.*

In order to expand on these ideas, we need to recall a number of new OBA-concepts introduced in [3]. For all unexplained terminology about Banach lattices and positive operators, we refer the reader to [5].

A subset B of an OBA A is said to be *order-bounded above* if there exists $a \in A$ such that $b \leq a$ for all $b \in B$, and A is said to be *Dedekind complete* if every non-empty order-bounded above set in A has a supremum. The OBAs $\mathbb{C}$, $M_n(\mathbb{C})$ and l^∞ are Dedekind complete, but that is not in general the case for the continuous functions $C(K)$ (with K a compact Hausdorff space). If E is a Dedekind complete Banach lattice, then $\mathcal{L}^r(E)$ is a Dedekind complete OBA and if E is an AM-space with unit or an AL-space, then $\mathcal{L}^r(E) = \mathcal{L}(E)$ is a Dedekind complete OBA [5, §4.4].

Let A be an OBA with algebra cone C. An idempotent $p \in A$ is called an *order idempotent* if $0 \leq p \leq 1$. The set of all order idempotents of A is denoted by $OI(A)$. It can be shown that if A is Dedekind complete and C is proper, then $OI(A)$ is Dedekind complete [3, Corollary 2.2]. If $A = \mathcal{L}(E)$, with E a Banach lattice, then $T \in OI(A)$ if and only if T is an order projection on E [5, Theorem 1.44].

If $a \in A$, then an order idempotent p is called *a-invariant* if $(1 - p)ap = 0$. If $a \in C$ and $0 < p \in OI(A)$, then a is said to be *irreducible w.r.t. p* if there exists no $q \in OI(A)$ with $0 < q < p$ such that $(p - q)aq = 0$, and a is *irreducible* if a is irreducible w.r.t. 1, i.e. if a has no non-trivial invariant order idempotents. If a Banach lattice E is Dedekind complete, then a positive operator T on E is an irreducible element in $\mathcal{L}(E)$ if and only if T is a band irreducible operator on E [3, p. 144].

Let $a \in A$ and $p_0, \ldots, p_n \in OI(A)$. If p_i is a-invariant for each $i \in \{0, \ldots, n\}$ and $p_n \geq \cdots \geq p_0$, then the totally ordered set $\{p_n, \ldots, p_0\}$ is called an *a-invariant chain*. For $T \in \mathcal{L}(E)$ (with E any Banach lattice) a *T-invariant chain* is of the form $\{B_n, \ldots, B_0\}$ where each B_i is a T-invariant projection band and $B_n \supseteq \cdots \supseteq B_0$.

For $p \in OI(A)$ and $a \in A$, let $p^d = 1 - p$ and $a_p = pap$. An element $b \in A$ is called a *block* of a positive element a if there exists an a-invariant chain $\{p_2, p_1\}$, with $p_2 > p_1$, such that $b = a_{p_2 p_1^d}$. Clearly, $0 \leq b \leq a$. If C is proper, then $r(b) \leq r(a)$ [3, Lemma 2.6], and b is said to be a *spectral block* of a if b is a block of a with $r(b) = r(a)$.

An element $a \in C$ is said to be *order continuous* if $p_\alpha a \downarrow 0$ and $ap_\alpha \downarrow 0$ whenever $p_\alpha \in OI(A)$ with $p_\alpha \downarrow 0$ in $OI(A)$ (where $p_\alpha \downarrow 0$ means that the net (p_α) is decreasing and $\inf_\alpha p_\alpha$ exists and is zero). If the algebra cone C is proper, then an order continuous element $a \in C$ is said to be *spectrally order continuous* if every spectral block b of a satisfies the property that if $r(b)$ is a pole of the resolvent $(\lambda - b)^{-1}$ of b of order m, then the coefficient b_{-m} in the Laurent series of $(\lambda - b)^{-1}$ is order continuous. In OBAs such as $\mathbb{C}$, $M_n^u(\mathbb{C})$ and l^∞ every positive element is spectrally order continuous. If a Banach lattice E is Dedekind complete, then a positive operator T on E is a spectrally order continuous element in $\mathcal{L}(E)$ if and only if T is a spectrally order continuous operator on E [3, Example 2.9(a)].

If $a \in C$, then $r(a)$ is said to be an *f-pole* of the resolvent of a if $0 \le b \le a$ implies $r(b) \le r(a)$, and $r(b)$ is a pole of the resolvent of b whenever $r(b) = r(a)$. We have the following relationship between Riesz points and f-poles:

Proposition 4.6.1 ([11], Proposition 5.19) *Let A be a semisimple OBA with closed and normal algebra cone C and I a closed inessential ideal such that the spectral radius in A/I is weakly monotone. If $b \in C$ is such that $r(b)$ is a Riesz point of $\sigma(b)$, then $r(b)$ is an f-pole of the resolvent of b.*

It can be shown that if $a \in A$ and $\{1 = p_n, p_{n-1}, \dots, p_1, p_0 = 0\}$ is an a-invariant chain, then $\sigma(a) \subseteq \cup_{i=1}^n \sigma(a_{p_i p_{i-1}^d})$ [3, Lemma 2.4]. Therefore, if C is proper and $a \in C$, then $r(a_{p_i p_{i-1}^d}) \le r(a)$ for all $i \in \{1, \dots, n\}$ and there exists $i_0 \in \{1, \dots, n\}$ such that $r(a) = r(a_{p_{i_0} p_{i_0-1}^d})$.

If $a \in C$, then *a has a Frobenius normal form* (or *its Frobenius normal form exists*) if there exists an a-invariant chain $\{1 = p_n, p_{n-1}, \dots, p_1, p_0 = 0\}$ such that, for any $i \in \{1, \dots, n\}$, the element $a_{p_i p_{i-1}^d}$ is irreducible w.r.t. $p_i p_{i-1}^d$ whenever $r(a_{p_i p_{i-1}^d}) = r(a)$. It follows that a positive operator T on a Dedekind complete Banach lattice E has a Frobenius normal form if and only if there exists a T-invariant chain $\{E = B_n, B_{n-1}, \dots, B_1, B_0 = \{0\}\}$ such that the restriction of $P_{Q_i} T P_{Q_i}$ to Q_i is a band irreducible operator whenever $r(P_{Q_i} T P_{Q_i}) = r(T)$, where $Q_i = B_i \cap B_{i-1}^d$—see [11, Remark 5.17].

Theorem 4.6.2 ([3], Theorem 2.11) *Let A be a Dedekind complete OBA with closed and normal algebra cone C and let $a \in C$ be a spectrally order continuous element such that $r(a) > 0$ and $r(a)$ is an f-pole of the resolvent of a. Then a has a Frobenius normal form.*

Theorem 4.6.2 illustrates that, under certain natural conditions, the spectrum of a positive element in an OBA is determined by the spectra of certain associated irreducible elements.

An OBA A is said to have a *disjunctive product* if for any order continuous elements a and b with $ab = 0$ there exists $p \in OI(A)$ such that $ap = 0 = (1 - p)b$. The OBAs $\mathbb{C}$, $M_n^u(\mathbb{C})$ and l^∞ all have disjunctive products and if E is a Dedekind complete Banach lattice, then $\mathcal{L}(E)$ and $\mathcal{L}^r(E)$ have disjunctive products [3, Example 3.3(a)].

Irreducible elements in an OBA with a disjunctive product have useful spectral properties, similar to those of irreducible operators. In particular, we have:

Theorem 4.6.3 ([3], Lemma 5.2) *Let A be an OBA with a disjunctive product and such that $OI(A)$ is Dedekind complete, with proper and closed algebra cone. Let $0 < p \in OI(A)$ and $a \in A$ such that a_p is a non-zero order continuous element irreducible w.r.t. p, with $r(a_p)$ a pole of the resolvent of a_p of order m and the coefficient $(a_p)_{-m}$ order continuous. Then $r(a_p) > 0$ and $r(a_p)$ is a simple pole of the resolvent of a_p.*

Using the above concepts, and in particular Theorems 4.6.2 and 4.6.3, Alekhno proved the following application:

Theorem 4.6.4 ([3], Theorem 5.5) *Let A be a Dedekind complete OBA with a disjunctive product and with closed and normal algebra cone C. Let $a \in C$ be a spectrally order continuous element such that $r(a)$ is an f-pole of the resolvent of a and $|a_{-1}|$ exists. Then $r(a) \notin \sigma(a + \lambda|a_{-1}|)$, for all $0 \neq \lambda \in \mathbb{C}$.*

In the operator case the above result has implications regarding Fredholm and Weyl theory. We will elaborate on this in the next section.

4.7 Fredholm theory

The Fredholm theory has been thoroughly investigated, both in the operator case and in the situation of general Banach algebras (see e.g. [27,39–41]). In [1,10] the element of positivity was introduced in this context. For the purposes of our discussion in this section, let E be a Banach lattice and let $\pi : \mathcal{L}(E) \to \mathcal{L}(E)/\mathcal{K}(E)$, $\phi : \mathcal{L}(E) \to \mathcal{L}(E)/\overline{\mathcal{F}(E)}$ and $\pi_r : \mathcal{L}^r(E) \to \mathcal{L}^r(E)/\mathcal{K}^r(E)$ indicate the relevant canonical homomorphisms. In addition, let A be a (general) OBA with algebra cone C, B a Banach algebra and $T : A \to B$ any homomorphism. In [10] an element $a \in A$ is defined to be

- *upper Weyl* if there exist $b \in A^{-1}$ and $c \in C \cap N(T)$ such that $a = b + c$ and
- *upper Browder* if there exist commuting elements $b \in A^{-1}$ and $c \in C \cap N(T)$ such that $a = b + c$.

If $\mathcal{W}_T^+$ and $\mathcal{B}_T^+$ denote the sets of upper Weyl and upper Browder elements, respectively, then clearly:

$$\begin{array}{ccccccccc} & & & & \mathcal{W}_T^+ & & & & \\ & & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & & \\ A^{-1} & \subseteq & \mathcal{B}_T^+ & & & & \mathcal{W}_T & \subseteq & \mathcal{F}_T \\ & & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & & \\ & & & & \mathcal{B}_T & & & & \end{array}$$

The associated (non-empty, compact) spectra are the *upper Weyl spectrum*

$$\omega_T^+(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin \mathcal{W}_T^+\} = \cap\{\sigma(a + c) : c \in C \cap N(T)\}$$

and the *upper Browder spectrum*

$$\beta_T^+(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin \mathcal{B}_T^+\} = \cap\{\sigma(a + c) : c \in C \cap N(T), \;\; ac = ca\},$$

respectively, of $a \in A$, which obviously satisfy

$$\begin{array}{ccccccccc} & & & & \omega_T^+(a) & & & & \\ & & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & & \\ \sigma(Ta) & \subseteq & \omega_T(a) & & & & \beta_T^+(a) & \subseteq & \sigma(a) \\ & & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & & \\ & & & & \beta_T(a) & & & & \end{array}$$

for all $a \in A$. If T has the Riesz property then, remarkably, $\mathcal{W}_T^+$ is closed under multiplication with *all* non-zero scalars [10, Lemma 3.2.8], implying that, for $a \in A$, $\lambda \in \omega_T^+(a)$ if and only if $\lambda - a \notin \mathcal{W}_T^+$ if and only if $a - \lambda \notin \mathcal{W}_T^+$ (and similarly for $\mathcal{B}_T^+$).

The study of these new spectra was motivated by the concept of the upper Weyl spectrum $\omega_\pi^+(T) = \cap\{\sigma(T + K) : 0 \leq K \in \mathcal{K}(E)\}$ of a positive operator T on a Banach lattice E, which was introduced by Alekhno in [1] (although the terminology was only introduced in [10]).

In [2, Theorem 3] Alekhno showed that, in fact, $\mathcal{W}_\pi^+ = \mathcal{W}_\pi$. In addition, $\mathcal{W}_{\pi_r}^+ = \mathcal{W}_{\pi_r}$ (see [10, Example 3.1.2]). However, the inclusion $\mathcal{W}_T^+ \subseteq \mathcal{W}_T$ is in general proper, as can be seen by investigating the OBA $C(K)$ with $K = [0, 1]$—see [10, Example 3.1.4].

The proof of [2, Theorem 3] shows that $\overline{\mathcal{F}(E)} = \overline{\text{span}\,(K \cap \overline{\mathcal{F}(E)})}$, i.e. $\overline{N(\phi)} = \overline{\text{span}\,(K \cap N(\phi))}$, with K the algebra cone of all positive operators on E. However, if T in the general case satisfies the Riesz property, then, whether or not the condition $\overline{N(T)} = \overline{\text{span}\,(C \cap N(T))}$ is assumed, we only obtain $\overline{\mathcal{W}_T^+} = \overline{\mathcal{W}_T}$ (see [10, p. 14] and [41, Corollary 3.8]), while the condition $N(T) = \text{span}\,(C \cap N(T))$ implies that $\mathcal{W}_T^+ = \mathcal{W}_T$ [10, Theorem 3.3.5]. It is the particular properties of a finite rank operator, and not just the fact that $\mathcal{F}(E)$ is an inessential ideal, that makes it possible to say more in the operator case.

We have the following relationship between the connected hulls of the Weyl and upper Weyl spectra:

Theorem 4.7.1 ([10], Theorem 4.3.4) *Let A be an OBA with algebra cone C, B a Banach algebra and $T : A \to B$ a homomorphism with closed range satisfying the Riesz property. If $a \in A$ is such that*

$$p(a, \lambda) \in \text{span}(C \cap N(T)) \textit{ for all } \lambda \in (\textit{iso}\,\sigma(a))\backslash\sigma(Ta), \tag{4.1}$$

then $\eta\omega_T(a) = \eta\omega_T^+(a)$.

It follows from [41, Theorem 2.4] (see also [10, Lemma 2.0.6]) that $p(a, \lambda) \in N(T)$ for all $\lambda \in (\text{iso}\,\sigma(a))\backslash\sigma(Ta)$. Condition (4.1) in Theorem 4.7.1, although stronger, is a fairly natural condition; it is, for instance, satisfied by all elements of $\mathcal{L}(E)$ as well as by all elements of $\mathcal{L}^r(E)$ [10, Lemma 4.3.3]. (Of course, in those cases the Weyl and upper Weyl spectra themselves are equal.)

It is clear from Theorem 2.11 that, in general,

$$r(a) \notin \sigma(Ta) \quad \Rightarrow \quad r(a) \notin \beta_T(a) \quad \Rightarrow \quad r(a) \notin \omega_T(a)$$

for all $a \in A$. Together with Theorem 4.7.1 we have that, if $a \in A$ satisfies (4.1), then

$$\eta\sigma(Ta) = \eta\omega_T(a) = \eta\omega_T^+(a) = \eta\beta_T(a), \tag{4.2}$$

and therefore the implication

$$r(a) \notin \sigma(Ta) \quad \Rightarrow \quad r(a) \notin \omega_T^+(a)$$

holds. However, [10, Example 4.3.1] shows that $\eta\beta_T^+(a)$ cannot be added to the list in (4.2). This motivates the study of the implication

$$r(a) \notin \sigma(Ta) \quad \Rightarrow \quad r(a) \notin \beta_T^+(a). \tag{4.3}$$

In view of Theorem 4.1.1 it seems reasonable to restrict the discussion to positive elements, so in [11] a positive element a of an OBA is defined to have the *upper Browder spectrum property* if a satisfies (4.3).

In view of Theorem 2.11, Proposition 4.6.1 and Alekhno's Theorem 4.6.4 we have the following result (with a_{-1} the coefficient of $(\lambda - r(a))^{-1}$ in the Laurent expansion of the resolvent of a around $r(a)$):

Theorem 4.7.2 ([11], Corollary 5.22) *Let A be a Dedekind complete semisimple OBA with a disjunctive product and with closed and normal algebra cone C. Also suppose that B is a Banach algebra and $T : A \to B$ is a homomorphism with closed range satisfying the Riesz property such that the spectral radius in $A/\overline{N(T)}$ is weakly monotone. Let $a \in C$ be a spectrally order continuous element such that $r(a) \notin \sigma(Ta)$ and $|a_{-1}|$ exists. Then $r(a) \notin \sigma(a + \lambda|a_{-1}|)$, for all $0 \neq \lambda \in \mathbb{C}$.*

However, unlike in the operator case (where the modulus of a finite rank operator automatically exists and is compact), in general $|a_{-1}|$ does not necessarily exist, and even if $|a_{-1}|$ exists, the fact that $a_{-1} = p(a, r(a)) \in N(T)$ does not imply that $|a_{-1}| \in N(T)$, so that Theorem 4.7.2 does not propose that $r(a) \notin \omega_T^+(a)$ (and, therefore, neither that $r(a) \notin \beta_T^+(a)$). Therefore it would be interesting to obtain additional results where the existence of a modulus is not required. Building on Alekhno's work, the following partial result can be established, where we note that a Frobenius normal form $\{1 = p_n, p_{n-1}, \ldots, p_1, p_0 = 0\}$ of a exists by Theorems 2.11 and 4.6.2 and Proposition 4.6.1:

Theorem 4.7.3 ([11], Theorem 5.24) *Let A be a Dedekind complete semisimple OBA with a disjunctive product and with closed and normal algebra cone C. Also suppose that B is a Banach algebra and $T : A \to B$ is a homomorphism with closed range satisfying the Riesz property such that the spectral radius in $A/\overline{N(T)}$ is weakly monotone. Let $a \in C$ be a spectrally order continuous element. If $r(a) \notin \sigma(Ta)$, then $r(a) \notin \cup_{i=1}^{n} \beta_T^+(a_{q_i})$, where $q_i = p_i p_{i-1}^d$.*

The above result can be applied to the regular operators—see [11, Corollaries 5.25–5.27].

A finite-dimensional semisimple OBA is algebraically isomorphic to an OBA A with the property that all positive elements in A have the upper Browder spectrum property relative to arbitrary Banach algebra homomorphisms $T : A \to B$—see [11, Corollary 5.13]. In addition, we have the following result:

Theorem 4.7.4 ([11], Corollaries 5.4 and 5.3) *Let A be a semisimple OBA with closed and normal algebra cone C, B a Banach algebra and $T : A \to B$ a homomorphism with closed range satisfying the Riesz property. If $a \in C$, then under each of the following conditions a has the upper Browder spectrum property:*

1. *A is commutative.*
2. *C is inverse-closed.*

The research on Fredholm theory in OBAs continues.

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