Positivity (2017) 21:825–845
DOI 10.1007/s11117-016-0433-1

Positivity

Positive solutions for fractional differential systems with nonlocal Riemann–Liouville fractional integral boundary conditions

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Received: 9 April 2016 / Accepted: 28 June 2016 / Published online: 5 July 2016

Abstract In this paper, we study the positive solutions of fractional differential system with coupled nonlocal Riemann–Liouville fractional integral boundary conditions. Our analysis relies on Leggett–Williams and Guo–Krasnoselskii's fixed point theorems. Two examples are worked out to illustrate our main results.

Keywords Fractional differential systems · Nonlocal boundary conditions · Riemann–Liouville fractional integral conditions · Positive solutions · Fixed point theorems

Mathematics Subject Classification 26A33 · 34A08 · 34B18

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1 Introduction

The fractional order differential equations has been received much attention due to its various applications in science and engineering such as fluid dynamics, heat conduction, control theory, electroanalytical chemistry, economics, fractal theory, fractional biological neurons, etc. It is proved that the fractional order differential equation is a better tool for the description of hereditary properties of various materials and processes than the corresponding integer order differential equation. For a systematic development of the topic, we refer the books [1–7]. A variety of results on initial and boundary value problems of fractional differential equations and inclusions can easily be found in the literature on the topic. For some recent results, we can refer to [8–18] and references cited therein.

Recently in [19] the authors studied the existence of positive solutions to the boundary value problems of fractional differential equations of the form

$$D^q u(t) + f(t, u(t)) = 0, \quad 1 < q \le 2, \quad 0 < t < 1, \tag{1.1}$$

subject to three point multi-term fractional integral boundary conditions

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m} \alpha_i (I^{p_i} u)(\eta), \quad 0 < \eta < 1, \tag{1.2}$$

where D^q is the standard Riemann–Liouville fractional derivative of order q, I^{p_i} is the Riemann–Liouville fractional integral of order $p_i > 0$, $i = 1, 2, \ldots, m$, $f \in C([0, 1] \times \mathbb{R})$ and $\alpha_i \ge 0$, $i = 1, 2, \ldots, m$, are real constants. The existence and multiplicity of positive solutions were obtained by using fixed point theorems. For some recent results on positive solutions of fractional differential equations we refer to [20–25] and references cited therein.

The main purpose in this paper is to investigate some sufficient conditions for existence of positive solutions to the following fractional system of differential equations subject to the nonlocal Riemann–Liouville fractional integral boundary conditions of the form

$$\begin{cases} D^p x(t) + f(t, x(t), y(t)) = 0, \quad 1 < p \le 2, \ t \in (0, 1), \\ D^q y(t) + g(t, x(t), y(t)) = 0, \quad 1 < q \le 2, \ t \in (0, 1), \\ x(0) = 0, \quad x(1) = \sum\limits_{i=1}^{m} \alpha_i I^{\gamma_i} y(\eta), \\ y(0) = 0, \quad y(1) = \sum\limits_{j=1}^{n} \beta_j I^{\mu_j} x(\xi), \end{cases} \tag{1.3}$$

where D^ϕ are Riemann–Liouville fractional derivatives of orders $\phi \in \{p, q\}$, $f, g \in C([0, 1] \times \mathbb{R}_+^2, \mathbb{R}_+)$, I^Φ are Riemann–Liouville fractional integrals of order $\Phi \in \{\gamma_i, \mu_j\}$, $\alpha_i, \beta_j > 0$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ and the fixed constants $0 < \eta < \xi < 1$.

Many researchers have shown their interest in the study of systems of fractional differential equations. The motivation for those works stems from both the intensive development of the theory of fractional calculus itself and the applications. See for example [26–30] where systems for fractional differential equations were studied by using Banach contraction mapping principle and Schaefer's fixed point theorem.

In this paper, we firstly derive the corresponding Green's function and some of its properties are proved. Consequently problem (1.3) is deduced to a equivalent Fredholm integral equation of the second kind. Finally, by the means of some fixed-point theorems, the existence and multiplicity of positive solutions are obtained. Illustrative examples are also presented.

2 Preliminaries

In this section, we introduce some notations and definitions of Riemann–Liouville fractional calculus (see [4]) and present preliminary results needed in our proofs later.

Definition 2.1 The (left-sided) fractional integral of order $\alpha > 0$ of a function $f\colon (0,\infty) \to \mathbb{R}$ is given by

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t > 0, \tag{2.1}$$

provided the right-hand side is pointwise defined on $(0,\infty)$, where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

Definition 2.2 The Riemann–Liouville fractional derivative of order $\alpha \geq 0$ for a function $f\colon (0,\infty) \to \mathbb{R}$ is given by

$$(D^\alpha f)(t) = \left(\frac{d}{dt}\right)^n (I^{n-\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \quad t > 0, \tag{2.2}$$

$n-1 < \alpha < n$, provided that the right-hand side is pointwise defined on $(0,\infty)$.
We also denote the Riemann–Liouville fractional derivative of f by $D^\alpha f(t)$. If $\alpha = m \in \mathbb{N}$ then $D^m f(t) = f^{(m)}(t)$ for $t > 0$, and if $\alpha = 0$ then $D^0 f(t) = f(t)$ for $t > 0$.

Lemma 2.1 *Let $\alpha > 0$ and $u \in C(0,1) \cap L^1(0,1)$. Then the fractional differential equation $D^\alpha u(t) = 0$ has a unique solution*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad 0 < t < 1, \tag{2.3}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary real constants, and $n-1 < \alpha < n$.

Lemma 2.2 *Let $\alpha > 0$, $n-1 < \alpha \leq n$ and $y \in AC(0,1)$. (By AC we denote the space of absolutely continuous functions). The solution of the fractional differential equation $D^{\alpha}u(t) + y(t) = 0, \quad 0 < t < 1$, is*

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_1 t^{\alpha-1} + \cdots + c_n t^{\alpha-n}, \quad 0 < t < 1, \tag{2.4}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary real constants.

Lemma 2.3 *Assume that $u, v \in AC([0,1], \mathbb{R}^+)$. Then the following system*

$$\begin{cases} D^p x(t) + u(t) = 0, & t \in (0,1), \\ D^q y(t) + v(t) = 0, & t \in (0,1), \\ x(0) = 0, & x(1) = \sum\limits_{i=1}^{n} \alpha_i I^{\gamma_i} y(\eta), \\ y(0) = 0, & y(1) = \sum\limits_{j=1}^{m} \beta_j I^{\mu_j} x(\xi), \end{cases} \tag{2.5}$$

can be written in the equivalent integral equations of the form

$$\begin{aligned} x(t) = & -\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}u(s)ds + t^{p-1}\bigg[\frac{1}{\Omega\Gamma(p)}\int_0^1 (1-s)^{p-1}u(s)ds \\ & -\frac{1}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i}{\Gamma(q+\gamma_i)}\int_0^{\eta}(\eta-s)^{q+\gamma_i-1}v(s)ds + \frac{\Lambda_1}{\Omega\Gamma(q)}\int_0^1 (1-s)^{q-1}v(s)ds \\ & -\frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j}{\Gamma(p+\mu_j)}\int_0^{\xi}(\xi-s)^{p+\mu_j-1}u(s)ds\bigg], \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} y(t) = & -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v(s)ds + t^{p-1}\bigg[\frac{1}{\Omega\Gamma(q)}\int_0^1 (1-s)^{q-1}v(s)ds \\ & -\frac{1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j}{\Gamma(p+\mu_j)}\int_0^{\xi}(\xi-s)^{p+\mu_j-1}u(s)ds + \frac{\Lambda_2}{\Omega\Gamma(p)}\int_0^1 (1-s)^{p-1}u(s)ds \\ & -\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i}{\Gamma(q+\gamma_i)}\int_0^{\eta}(\eta-s)^{q+\gamma_i-1}v(s)ds\bigg], \end{aligned} \tag{2.7}$$

where

$$\Omega := 1 - \Lambda_1\Lambda_2 > 0,$$

with

$$\Lambda_1 := \sum_{i=1}^{m}\frac{\alpha_i \eta^{q+\gamma_i-1}\Gamma(q)}{\Gamma(q+\gamma_i)}, \quad \Lambda_2 := \sum_{j=1}^{n}\frac{\beta_j \xi^{p+\mu_j-1}\Gamma(p)}{\Gamma(p+\mu_j)}.$$

Proof Applying Lemma 2.2, the first two equations of problem (2.5) can be expressed as

$$x(t) = -\frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} u(s) ds + c_1 t^{p-1} + c_2 t^{p-2},$$
$$y(t) = -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s) ds + k_1 t^{q-1} + k_2 t^{q-2},$$

where $c_1, c_2, k_1, k_2 \in \mathbb{R}$.

From the initial conditions of (2.5) that $x(0) = 0$, $y(0) = 0$, we have $c_2 = k_2 = 0$. Therefore, we get the following equations

$$x(t) = -\frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} u(s) ds + c_1 t^{p-1}, \tag{2.8}$$

and

$$y(t) = -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s) ds + k_1 t^{q-1}. \tag{2.9}$$

Taking the Riemann–Liouville fractional integral of orders μ_j and γ_i to (2.8) and (2.9), and also substitution $t = \xi$ and $t = \eta$, respectively, we obtain

$$I^{\mu_j} x(\xi) = -\frac{1}{\Gamma(p+\mu_j)} \int_0^\xi (\xi - s)^{p+\mu_j-1} u(s) ds + c_1 \frac{\xi^{p+\mu_j-1}\Gamma(p)}{\Gamma(p+\mu_j)},$$

and

$$I^{\gamma_i} y(\eta) = -\frac{1}{\Gamma(q+\gamma_i)} \int_0^\eta (\eta - s)^{q+\gamma_i-1} v(s) ds + k_1 \frac{\eta^{q+\gamma_i-1}\Gamma(q)}{\Gamma(q+\gamma_i)}.$$

Using the second nonlocal boundary conditions of (2.5), we deduce the following system

$$\begin{aligned}
c_1 - k_1 \sum_{i=1}^m \frac{\alpha_i \eta^{q+\gamma_i-1}\Gamma(q)}{\Gamma(q+\gamma_i)} &= \frac{1}{\Gamma(p)} \int_0^1 (1-s)^{p-1} u(s) ds \\
&\quad - \sum_{i=1}^m \frac{\alpha_i}{\Gamma(q+\gamma_i)} \int_0^\eta (\eta - s)^{q+\gamma_i-1} v(s) ds, \\
-c_1 \sum_{j=1}^n \frac{\beta_j \xi^{p+\mu_j-1}\Gamma(p)}{\Gamma(p+\mu_j)} + k_1 &= \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s) ds \\
&\quad - \sum_{j=1}^n \frac{\beta_j}{\Gamma(p+\mu_j)} \int_0^\xi (\xi - s)^{p+\mu_j-1} u(s) ds.
\end{aligned}$$

Solving the above system to find constants c_1 and k_1, we obtain

$$c_1 = \frac{1}{\Omega\Gamma(p)}\int_0^1 (1-s)^{p-1}u(s)ds - \frac{1}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i}{\Gamma(q+\gamma_i)}\int_0^{\eta}(\eta-s)^{q+\gamma_i-1}v(s)ds$$
$$+\frac{\Lambda_1}{\Omega\Gamma(q)}\int_0^1 (1-s)^{q-1}v(s)ds - \frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j}{\Gamma(p+\mu_j)}\int_0^{\xi}(\xi-s)^{p+\mu_j-1}u(s)ds,$$

and

$$k_1 = \frac{1}{\Omega\Gamma(q)}\int_0^1 (1-s)^{q-1}v(s)ds - \frac{1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j}{\Gamma(p+\mu_j)}\int_0^{\xi}(\xi-s)^{p+\mu_j-1}u(s)ds$$
$$+\frac{\Lambda_2}{\Omega\Gamma(p)}\int_0^1 (1-s)^{p-1}u(s)ds - \frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i}{\Gamma(q+\gamma_i)}\int_0^{\eta}(\eta-s)^{q+\gamma_i-1}v(s)ds.$$

Substituting the values of c_1 and k_1 in (2.8) and (2.9), we deduce the integral equations (2.6) and (2.7), respectively, as desired. The converse follows by direct computation. This completes the proof. □

Lemma 2.4 (Green's function) *The integral equations* (2.6) *and* (2.7) *in Lemma* 2.3 *can be expressed in the form of Green functions as*

$$x(t) = \int_0^1 G_1(t,s)u(s)ds, \tag{2.10}$$

$$y(t) = \int_0^1 G_2(t,s)v(s)ds, \tag{2.11}$$

where G_1, G_2 are the Green's functions given by

$$G_1(t,s) = g_p(t,s) + \frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j t^{p-1}}{\Gamma(p+\mu_j)}g^p_{\mu_j}(\xi,s)$$
$$+\frac{1}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i t^{p-1}}{\Gamma(q+\gamma_i)}g^q_{\gamma_i}(\eta,s), \tag{2.12}$$
$$G_2(t,s) = g_q(t,s) + \frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i t^{q-1}}{\Gamma(q+\gamma_i)}g^q_{\gamma_i}(\eta,s)$$
$$+\frac{1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j t^{q-1}}{\Gamma(p+\mu_j)}g^p_{\mu_j}(\xi,s), \tag{2.13}$$

where

$$g_\phi(t,s) = \begin{cases} \dfrac{(1-s)^{\phi-1}t^{\phi-1} - (t-s)^{\phi-1}}{\Gamma(\phi)}, & 0 \le s \le t \le 1, \\ \dfrac{(1-s)^{\phi-1}t^{\phi-1}}{\Gamma(\phi)}, & 0 \le t \le s \le 1, \end{cases} \tag{2.14}$$

and

$$g_\psi^\phi(\rho,s) = \begin{cases} \rho^{\phi+\psi-1}(1-s)^{\phi-1} - (\rho-s)^{\phi+\psi-1}, & 0 \le s \le \rho \le 1, \\ \rho^{\phi+\psi-1}(1-s)^{\phi-1}, & 0 \le \rho \le s \le 1, \end{cases} \tag{2.15}$$

with $\phi \in \{p,q\}$, $\psi \in \{\mu_j, \gamma_i\}$, $\rho \in \{\xi, \eta\}$.

Proof From Lemma 2.3, by direct computation, we have

$$\begin{aligned}
x(t) &= -\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}u(s)ds + t^{p-1}\Bigg[\frac{1}{\Omega\Gamma(p)}\int_0^1 (1-s)^{p-1}u(s)ds \\
&\quad - \frac{1}{\Omega}\sum_{i=1}^m \frac{\alpha_i}{\Gamma(q+\gamma_i)}\int_0^\eta (\eta-s)^{q+\gamma_i-1}v(s)ds + \frac{\Lambda_1}{\Omega\Gamma(q)}\int_0^1 (1-s)^{q-1}v(s)ds \\
&\quad - \frac{\Lambda_1}{\Omega}\sum_{j=1}^n \frac{\beta_j}{\Gamma(p+\mu_j)}\int_0^\xi (\xi-s)^{p+\mu_j-1}u(s)ds\Bigg] \\
&= \int_0^1 \frac{(1-s)^{p-1}t^{p-1}}{\Gamma(p)}u(s)ds - \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}u(s)ds \\
&\quad - \frac{(1-\Lambda_1\Lambda_2)}{\Omega\Gamma(p)}\int_0^1 (1-s)^{p-1}t^{p-1}u(s)ds + \frac{1}{\Omega\Gamma(p)}\int_0^1 (1-s)^{p-1}t^{p-1}u(s)ds \\
&\quad - \frac{t^{p-1}}{\Omega}\sum_{i=1}^m \frac{\alpha_i}{\Gamma(q+\gamma_i)}\int_0^\eta (\eta-s)^{q+\gamma_i-1}v(s)ds \\
&\quad + \frac{t^{p-1}\Lambda_1}{\Omega\Gamma(q)}\int_0^1 (1-s)^{q-1}v(s)ds - \frac{t^{p-1}\Lambda_1}{\Omega}\sum_{j=1}^n \frac{\beta_j}{\Gamma(p+\mu_j)}\int_0^\xi (\xi-s)^{p+\mu_j-1}u(s)ds \\
&= \int_0^1 g_p(t,s)u(s)ds + \frac{\Lambda_1}{\Omega}\sum_{j=1}^n \frac{\beta_j t^{p-1}}{\Gamma(p+\mu_j)}\int_0^1 \xi^{p+\mu_j-1}(1-s)^{p-1}u(s)ds \\
&\quad - \frac{\Lambda_1}{\Omega}\sum_{j=1}^n \frac{\beta_j t^{p-1}}{\Gamma(p+\mu_j)}\int_0^\xi (\xi-s)^{p+\mu_j-1}u(s)ds \\
&\quad - \frac{t^{p-1}}{\Omega}\sum_{i=1}^m \frac{\alpha_i}{\Gamma(q+\gamma_i)}\int_0^\eta (\eta-s)^{q+\gamma_i-1}v(s)ds + \frac{t^{p-1}\Lambda_1}{\Omega\Gamma(q)}\int_0^1 (1-s)^{q-1}v(s)ds \\
&= \int_0^1 g_p(t,s)u(s)ds + \frac{\Lambda_1}{\Omega}\sum_{j=1}^n \frac{\beta_j t^{p-1}}{\Gamma(p+\mu_j)}\int_0^1 g_{\mu_j}^p(\xi,s)u(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i t^{p-1}}{\Gamma(q+\gamma_i)}\int_0^1 g_{\gamma_i}^{q}(\eta,s)v(s)ds\\
&=\int_0^1 G_1(t,s)u(s)ds,
\end{aligned}
$$

which implies that (2.10) holds. In a similar way, we obtain

$$
\begin{aligned}
y(t) &= -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v(s)ds + \frac{t^{q-1}}{\Omega\Gamma(q)}\int_0^1 (1-s)^{q-1}v(s)ds\\
&\quad -\frac{t^{q-1}}{\Omega}\sum_{j=1}^{n}\frac{\beta_j}{\Gamma(p+\mu_j)}\int_0^{\xi}(\xi-s)^{p+\mu_j-1}u(s)ds\\
&\quad +\frac{t^{q-1}\Lambda_2}{\Omega\Gamma(p)}\int_0^1 (1-s)^{p-1}u(s)ds\\
&\quad -\frac{t^{q-1}\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i}{\Gamma(q+\gamma_i)}\int_0^{\eta}(\eta-s)^{q+\gamma_i-1}v(s)ds\\
&\quad +\int_0^1 \frac{(1-s)^{q-1}t^{q-1}}{\Gamma(q)}v(s)ds - \int_0^1 \frac{(1-s)^{q-1}t^{q-1}}{\Gamma(q)}v(s)ds\\
&= \int_0^1 \frac{(1-s)^{q-1}t^{q-1}}{\Gamma(q)}v(s)ds - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v(s)ds\\
&\quad +\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i t^{q-1}}{\Gamma(q+\gamma_i)}\int_0^1 (1-s)^{q-1}\eta^{q+\gamma_i-1}v(s)ds\\
&\quad -\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i t^{q-1}}{\Gamma(q+\gamma_i)}\int_0^{\eta}(\eta-s)^{q+\gamma_i-1}v(s)ds\\
&\quad +\frac{1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j t^{q-1}}{\Gamma(p+\mu_j)}\int_0^1 (1-s)^{p-1}\xi^{p+\mu_j-1}u(s)ds\\
&\quad -\frac{1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j t^{q-1}}{\Gamma(p+\mu_j)}\int_0^{\xi}(\xi-s)^{p+\mu_j-1}u(s)ds\\
&= \int_0^1 g_q(t,s)v(s)ds + \frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i t^{q-1}}{\Gamma(q+\gamma_i)}\int_0^1 g_{\gamma_i}^{q}(\eta,s)v(s)ds\\
&\quad +\frac{1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j t^{q-1}}{\Gamma(p+\mu_j)}\int_0^1 g_{\mu_j}^{p}(\xi,s)u(s)ds\\
&= \int_0^1 G_2(t,s)v(s)ds,
\end{aligned}
$$

which proves that (2.11) is true. This completes the proof. □

Before establishing some properties of the Green's functions, we set the following constants

$$\Lambda_3 = \frac{\Gamma(p)}{\Gamma(2p)} + \frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\left(\frac{\mu_j + p(1-\xi)}{p\Gamma(p+\mu_j+1)}\right)\beta_j \xi^{p+\mu_j-1}$$
$$+\frac{1}{\Omega}\sum_{i=1}^{m}\left(\frac{\gamma_i + q(1-\eta)}{q\Gamma(q+\gamma_i+1)}\right)\alpha_i \eta^{q+\gamma_i-1},$$
$$\Lambda_4 = \frac{\Gamma(q)}{\Gamma(2q)} + \frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\left(\frac{\gamma_i + q(1-\eta)}{q\Gamma(q+\gamma_i+1)}\right)\alpha_i \eta^{q+\gamma_i-1}$$
$$+\frac{1}{\Omega}\sum_{j=1}^{n}\left(\frac{\mu_j + p(1-\xi)}{p\Gamma(p+\mu_j+1)}\right)\beta_j \xi^{p+\mu_j-1},$$
$$\Lambda_5 = \frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j \xi^{2p+\mu_j-2}(1-\xi)^p}{p\Gamma(p+\mu_j)} + \frac{1}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i \eta^{p+q+\gamma_i-2}(1-\eta)^q}{q\Gamma(q+\gamma_i)},$$
$$\Lambda_6 = \frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i \eta^{2q+\gamma_i-2}(1-\eta)^q}{q\Gamma(q+\gamma_i)} + \frac{1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j \xi^{p+q+\mu_j-2}(1-\xi)^p}{p\Gamma(p+\mu_j)}.$$

Lemma 2.5 *The Green's functions $G_1(t,s)$ and $G_2(t,s)$ in (2.12)–(2.13) satisfy the following properties:*

(P_1) $G_1(t,s)$, $G_2(t,s)$ *are continuous on* $[0,1]\times[0,1]$*;*
(P_2) $G_1(t,s)$, $G_2(t,s) \geq 0$ *for all* $0 \leq s, t \leq 1$*;*
(P_3) $G_1(t,s) \leq \sup_{0\leq t\leq 1} G(t,s) \leq g_p(s,s) + \frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j g^p_{\mu_j}(\xi,s)}{\Gamma(p+\mu_i)} + \frac{1}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i g^q_{\gamma_i}(\eta,s)}{\Gamma(q+\gamma_i)}$, $G_2(t,s) \leq \sup_{0\leq t\leq 1} G(t,s) \leq g_q(s,s) + \frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i g^q_{\gamma_i}(\eta,s)}{\Gamma(q+\gamma_i)} + \frac{1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j g^p_{\mu_j}(\xi,s)}{\Gamma(p+\mu_j)}$*;*
(P_4) $\int_0^1 \sup_{0\leq t\leq 1} G_1(t,s)ds \leq \Lambda_3$ *and* $\int_0^1 \sup_{0\leq t\leq 1} G_2(t,s)ds \leq \Lambda_4$*;*
(P_5) $\inf_{\xi\leq t\leq 1} G_1(t,s) \geq \frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j \xi^{p-1}}{\Gamma(p+\mu_j)} g^p_{\mu_j}(\xi,s) + \frac{1}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i \eta^{p-1}}{\Gamma(q+\gamma_i)} g^q_{\gamma_i}(\eta,s)$, $\inf_{\xi\leq t\leq 1} G_2(t,s) \geq \frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\alpha_i \eta^{q-1}}{\Gamma(q+\gamma_i)} g^q_{\gamma_i}(\eta,s) + \frac{1}{\Omega}\sum_{j=1}^{n}\frac{\beta_j \xi^{q-1}}{\Gamma(p+\mu_j)} g^p_{\mu_j}(\xi,s)$*;*
(P_6) $\int_\eta^1 \inf_{\xi\leq t\leq 1} G_1(t,s)ds \geq \Lambda_5$ *and* $\int_\eta^1 \inf_{\xi\leq t\leq 1} G_2(t,s)ds \geq \Lambda_6$.

Proof It is easy to prove that the condition (P_1) holds. For $0 \leq s, t \leq 1$, using the results in [19], we have $g_\phi(t,s) \geq 0$, $g^\phi_\psi(\rho,s) \geq 0$, where $\phi \in \{p,q\}$, $\rho \in \{\eta,\xi\}$ and

$\psi \in \{\mu_j, \gamma_i\}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, which leads to $G_1(t, s), G_2(t, s) \geq 0$. Therefore, the property (P_2) is true.

From [19], we have $g_\phi(t, s) \leq g_\phi(s, s)$ for $\phi \in \{p, q\}, (t, s) \in [0, 1]$, which yields

$$G_1(t, s) \leq \sup_{0 \leq t \leq 1} G_1(t, s) \leq g_p(s, s) + \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\beta_j g_{\mu_j}^p(\xi, s)}{\Gamma(p + \mu_i)} + \frac{1}{\Omega} \sum_{i=1}^{m} \frac{\alpha_i g_{\gamma_i}^q(\eta, s)}{\Gamma(q + \gamma_i)}, s \in [0, 1], \tag{2.16}$$

and also

$$G_2(t, s) \leq \sup_{0 \leq t \leq 1} G_2(t, s) \leq g_q(s, s) + \frac{\Lambda_2}{\Omega} \sum_{i=1}^{m} \frac{\alpha_i g_{\gamma_i}^q(\eta, s)}{\Gamma(q + \gamma_i)} + \frac{1}{\Omega} \sum_{j=1}^{n} \frac{\beta_j g_{\mu_j}^p(\xi, s)}{\Gamma(p + \mu_j)}, s \in [0, 1]. \tag{2.17}$$

Thus the condition (P_3) is proved. Consequently, by direct integration, we get

$$\int_0^1 \sup_{0 \leq t \leq 1} G_1(t, s) ds \leq \int_0^1 g_p(s, s) ds + \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\beta_j}{\Gamma(p + \mu_i)} \int_0^1 g_{\mu_j}^p(\xi, s) ds + \frac{1}{\Omega} \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(q + \gamma_i)} \int_0^1 g_{\gamma_i}^q(\eta, s) ds = \Lambda_3$$

and

$$\int_0^1 \sup_{0 \leq t \leq 1} G_2(t, s) ds \leq \int_0^1 g_q(s, s) ds + \frac{\Lambda_2}{\Omega} \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(q + \gamma_i)} \int_0^1 g_{\gamma_i}^q(\eta, s) ds + \frac{1}{\Omega} \sum_{j=1}^{n} \frac{\beta_j}{\Gamma(p + \mu_j)} \int_0^1 g_{\mu_j}^p(\xi, s) ds = \Lambda_4.$$

Therefore, the condition (P_4) holds.

From the positivity of the Green functions in (P_2), we have

$$\inf_{\xi \leq t \leq 1} G_1(t, s) = \inf_{\xi \leq t \leq 1} \left(g_p(t, s) + \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\beta_j t^{p-1}}{\Gamma(p + \mu_j)} g_{\mu_j}^p(\xi, s) + \frac{1}{\Omega} \sum_{i=1}^{m} \frac{\alpha_i t^{p-1}}{\Gamma(q + \gamma_i)} g_{\gamma_i}^q(\eta, s) \right)$$

$$
\begin{aligned}
&\geq \inf_{\xi \leq t \leq 1} g_p(t,s) + \inf_{\xi \leq t \leq 1} \left(\frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\beta_j t^{p-1}}{\Gamma(p+\mu_j)} g^p_{\mu_j}(\xi,s) \right) \\
&\quad + \inf_{\xi \leq t \leq 1} \left(\frac{1}{\Omega} \sum_{i=1}^{m} \frac{\alpha_i t^{p-1}}{\Gamma(q+\gamma_i)} g^q_{\gamma_i}(\eta,s) \right) \\
&\geq \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\beta_j \xi^{p-1}}{\Gamma(p+\mu_j)} g^p_{\mu_j}(\xi,s) + \inf_{\eta \leq t \leq 1} \left(\frac{1}{\Omega} \sum_{i=1}^{m} \frac{\alpha_i t^{p-1}}{\Gamma(q+\gamma_i)} g^q_{\gamma_i}(\eta,s) \right) \\
&\geq \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\beta_j \xi^{p-1}}{\Gamma(p+\mu_j)} g^p_{\mu_j}(\xi,s) + \frac{1}{\Omega} \sum_{i=1}^{m} \frac{\alpha_i \eta^{p-1}}{\Gamma(q+\gamma_i)} g^q_{\gamma_i}(\eta,s).
\end{aligned}
$$

In the same method of the above inequalities, we obtain

$$
\inf_{\xi \leq t \leq 1} G_2(t,s) \geq \frac{\Lambda_2}{\Omega} \sum_{i=1}^{m} \frac{\alpha_i \eta^{q-1}}{\Gamma(q+\gamma_i)} g^q_{\gamma_i}(\eta,s) + \frac{1}{\Omega} \sum_{j=1}^{n} \frac{\beta_j \xi^{q-1}}{\Gamma(p+\mu_j)} g^p_{\mu_j}(\xi,s).
$$

Therefore, the inequalities in (P_5) are satisfied.

To prove (P_6), by directly integration, we have

$$
\begin{aligned}
\int_\eta^1 \inf_{\xi \leq t \leq 1} G_1(t,s)ds &\geq \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\beta_j \xi^{p-1}}{\Gamma(p+\mu_j)} \int_\eta^1 g^p_{\mu_j}(\xi,s)ds \\
&\quad + \frac{1}{\Omega} \sum_{i=1}^{m} \frac{\alpha_i \eta^{p-1}}{\Gamma(q+\gamma_i)} \int_\eta^1 g^q_{\gamma_i}(\eta,s)ds \\
&\geq \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\beta_j \xi^{p-1}}{\Gamma(p+\mu_j)} \int_\xi^1 g^p_{\mu_j}(\xi,s)ds \\
&\quad + \frac{1}{\Omega} \sum_{i=1}^{m} \frac{\alpha_i \eta^{p-1}}{\Gamma(q+\gamma_i)} \int_\eta^1 g^q_{\gamma_i}(\eta,s)ds = \Lambda_5
\end{aligned}
$$

and

$$
\begin{aligned}
\int_\eta^1 \inf_{\xi \leq t \leq 1} G_2(t,s)ds &\geq \frac{\Lambda_2}{\Omega} \sum_{i=1}^{m} \frac{\alpha_i \eta^{q-1}}{\Gamma(q+\gamma_i)} \int_\eta^1 g^q_{\gamma_i}(\eta,s)ds \\
&\quad + \frac{1}{\Omega} \sum_{j=1}^{n} \frac{\beta_j \xi^{q-1}}{\Gamma(p+\mu_j)} \int_\xi^1 g^p_{\mu_j}(\xi,s)ds = \Lambda_6.
\end{aligned}
$$

Therefore, we get the required inequality in (P_6).

3 Main results

Let $E = C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ be the Banach space with the norm $\|(x, y)\| := \|x\| + \|y\|$, where $\|x\| = \sup_{t\in[0,1]} |x(t)|$, $\|y\| = \sup_{t\in[0,1]} |y(t)|$. Then we define the positive cone $\mathcal{P} \subset E$ by

$$\mathcal{P} = \{(x, y) \in E : x(t) \geq 0 \quad \text{and} \quad y(t) \geq 0, \quad 0 \leq t \leq 1\}.$$

Define an operator $\mathcal{Q}$ on E by

$$\mathcal{Q}(x, y)(t) = (A(x, y)(t), B(x, y)(t)), \quad \text{for all } t \in [0, 1], \tag{3.1}$$

where the operators $A\colon \mathcal{P} \to E$ and $B\colon \mathcal{P} \to E$ are defined by

$$\begin{cases} A(x, y)(t) := \displaystyle\int_0^1 G_1(t, s) f(s, x(s), y(s))ds, \\ B(x, y)(t) := \displaystyle\int_0^1 G_2(t, s) g(s, x(s), y(s))ds. \end{cases} \tag{3.2}$$

Lemma 3.1 *The operator $\mathcal{Q}\colon \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof Since $G_1(t, s) \geq 0, G_2(t, s) \geq 0$ for $s, t \in [0, 1]$, we have $A(x, y) \geq 0$, $B(x, y) \geq 0$ for all $x, y \in \mathcal{P}$. Hence, $A, B\colon \mathcal{P} \to \mathcal{P}$.

For a constant $R > 0$, we define $U = \{(x, y) \in \mathcal{P} : \|(x, y)\| < R\}$. Let

$$L = \max_{0\leq t\leq 1,\ 0\leq x\leq R,\ 0\leq y\leq R} |f(t, x, y)|.$$

Then for $(x, y) \in U$, from Lemma 2.5, one has

$$\begin{aligned} |A(x, y)(t)| &= \left|\int_0^1 G_1(t, s) f(s, x(s), y(s))ds\right| \\ &\leq L \int_0^1 G_1(t, s)ds \\ &\leq L \int_0^1 \left(g_p(s, s) + \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\beta_j g^p_{\mu_j}(\xi, s)}{\Gamma(p + \mu_i)} + \frac{1}{\Omega} \sum_{i=1}^{m} \frac{\alpha_i g^q_{\gamma_i}(\eta, s)}{\Gamma(q + \gamma_i)} \right) ds \\ &= L\Bigg[\frac{\Gamma(p)}{\Gamma(2p)} + \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \left(\frac{\mu_j + p(1 - \xi)}{p\Gamma(p + \mu_j + 1)} \right) \beta_j \xi^{p+\mu_j-1} \\ &\quad + \frac{1}{\Omega} \sum_{i=1}^{m} \left(\frac{\gamma_i + q(1 - \eta)}{q\Gamma(q + \gamma_i + 1)} \right) \alpha_i \eta^{q+\gamma_i-1} \Bigg] := M_1. \end{aligned}$$

Therefore, $\|A(x, y)\| \leq M_1$. Similarly we can prove that $\|B(x, y)\| \leq M_2$, and so $\mathcal{Q}(U)$ is uniformly bounded. In addition, it follows from the continuity of f, g, the uniform continuity of $G_1(t,s), G_2(t,s)$ on $[0,1]\times[0,1]$ that $\mathcal{Q}\colon E\to E$ is continuous.

Also as in Lemma 2.5 of [19] we can prove that $\mathcal{Q}(U)$ is equi-continuous. Applying the Arzelá-Ascoli Theorem, we have that $\overline{\mathcal{Q}(U)}$ is compact, i.e., $\mathcal{Q}\colon\mathcal{P}\to\mathcal{P}$ is a completely continuous operator. This completes the proof. □

3.1 Existence result via Leggett–Williams fixed point theorem

In this subsection, the existence of at least three positive solutions will be proved using the Leggett–Williams fixed point theorem.

Definition 3.1 A continuous mapping $\omega\colon\mathcal{P}\to[0,\infty)$ is said to be a nonnegative continuous concave functional on the cone $\mathcal{P}$ of a real Banach space E provided that

$$\omega(\lambda x+(1-\lambda)y)\geq\lambda\omega(x)+(1-\lambda)\omega(y)$$

for all $x,y\in\mathcal{P}$ and $\lambda\in[0,1]$.

Let $a,b,d>0$ be given constants and define $\mathcal{P}_d=\{(x,y)\in\mathcal{P}:\|(x,y)\|<d\}$, $\overline{\mathcal{P}}_d=\{(x,y)\in\mathcal{P}:\|(x,y)\|\leq d\}$ and $\mathcal{P}(\omega,a,b)=\{(x,y)\in\mathcal{P}:\omega((x,y))\geq a,\ \|(x,y)\|\leq b\}$.

Theorem 3.1 (Leggett–Williams fixed point theorem) *Let $\mathcal{P}$ be a cone in the real Banach space E and $c>0$ be a constant. Assume that there exists a concave nonnegative continuous functional ω on $\mathcal{P}$ with $\omega(x)\leq\|x\|$ for all $x\in\overline{\mathcal{P}}_c$. Let $Q\colon\overline{\mathcal{P}}_c\to\overline{\mathcal{P}}_c$ be a completely continuous operator. Suppose that there exist constants $0<a<b<d\leq c$ such that the following conditions hold:*

(i) *$\{x\in\mathcal{P}(\omega,b,d):\omega(x)>b\}\neq\emptyset$ and $\omega(Qx)>b$ for $x\in\mathcal{P}(\omega,b,d)$;*
(ii) *$\|Qx\|<a$ for $x\leq a$;*
(iii) *$\omega(Qx)>b$ for $x\in\mathcal{P}(\omega,b,c)$ with $\|Qx\|>d$.*

Then Q has at least three fixed points x_1,x_2 and x_3 in $\overline{\mathcal{P}}_c$. In addition, $\|x_1\|<a$, $\omega(x_2)>b$, $\|x_3\|>a$ with $\omega(x_3)<b$.

Theorem 3.2 *Let functions $f,g\colon[0,1]\times\mathbb{R}_+^2\to\mathbb{R}_+$ be continuous functions. Suppose that there exist constants $0<a<b<c$ such that the following assumptions hold:*

(H_1) *$f(t,x,y)<\dfrac{a}{2\Lambda_3}$ and $g(t,x,y)<\dfrac{a}{2\Lambda_4}$, for $(t,x,y)\in[0,1]\times[0,a]\times[0,a]$;*
(H_2) *$f(t,x,y)>\dfrac{b}{2\Lambda_5}$ and $g(t,x,y)>\dfrac{b}{2\Lambda_6}$, for $(t,x,y)\in[\eta,1]\times[b,c]\times[b,c]$;*
(H_3) *$f(t,x,y)\leq\dfrac{c}{2\Lambda_3}$ and $g(t,x,y)\leq\dfrac{c}{2\Lambda_4}$, for $(t,x,y)\in[0,1]\times[0,c]\times[0,c]$.*

Then, the problem (1.3) *has at least three positive solutions* (x_1, y_1), (x_2, y_2) *and* (x_3, y_3) *such that* $\|(x_1, y_1)\| < a$, $\inf_{\xi \le t \le 1}(x_2, y_2)(t) > b$ *and* $\|(x_3, y_3)\| > a$ *with* $\inf_{\xi \le t \le 1}(x_3, y_3)(t) < b$.

Proof Firstly, we will show that $\mathcal{Q}\colon \overline{\mathcal{P}}_c \to \overline{\mathcal{P}}_c$. For $(x, y) \in \overline{\mathcal{P}}_c$, it follows that $\|(x, y)\| \le c$. From the condition (H_3) and Lemma 2.5, we have

$$\begin{aligned}
\|\mathcal{Q}(x, y)\| &= \sup_{t\in[0,1]} |A(x, y)(t)| + \sup_{t\in[0,1]} |B(x, y)(t)| \\
&\le \int_0^1 \sup_{t\in[0,1]} G_1(t, s) f(s, x(s), y(s))ds \\
&\quad + \int_0^1 \sup_{t\in[0,1]} G_2(t, s) g(s, x(s), y(s))ds \\
&\le \frac{c}{2\Lambda_3} \int_0^1 \sup_{t\in[0,1]} G_1(t, s)ds + \frac{c}{2\Lambda_4} \int_0^1 \sup_{t\in[0,1]} G_2(t, s)ds = c.
\end{aligned}$$

This implies that $Q\colon \overline{\mathcal{P}}_c \to \overline{\mathcal{P}}_c$.

Let $(x, y) \in \overline{\mathcal{P}}_a$. The condition (H_1) implies that

$$\begin{aligned}
\|\mathcal{Q}(x, y)\| &\le \int_0^1 \sup_{t\in[0,1]} G_1(t, s) f(s, x(s), y(s))ds \\
&\quad + \int_0^1 \sup_{t\in[0,1]} G_2(t, s) g(s, x(s), y(s))ds \\
&< \frac{a}{2\Lambda_3} \int_0^1 \sup_{t\in[0,1]} G_1(t, s)ds + \frac{a}{2\Lambda_4} \int_0^1 \sup_{t\in[0,1]} G_2(t, s)ds = a.
\end{aligned}$$

Hence, the condition (ii) of Theorem 3.1 is fulfilled.

Now, we let a concave nonnegative continuous functional ω on $\mathcal{P}$ by $\omega(x, y) = \inf_{t\in[\xi,1]} |x(t)| + \inf_{t\in[\xi,1]} |y(t)|$. Choosing $(x, y)(t) = ((b + c)/2, (b + c)/2)$ for all $t \in [0, 1]$, we have that $(x, y)(t) \in \overline{\mathcal{P}}(\omega, b, c)$ and $\omega((x, y)) = \omega((b + c)/2, (b + c)/2)) > b$. Then we obtain $\{(x, y) \in \mathcal{P}(\omega, b, c) : \omega((x, y)) > b\} \ne \emptyset$. Thus, if $(x, y) \in \overline{P}(\omega, b, c)$, then $b \le x(t) \le c$ and $b \le y(t) \le c$ for $t \in [\xi, 1]$. Using the condition (H_2) and Lemma 2.5, we have

$$\begin{aligned}
\omega(Q(x, y)(t)) &= \inf_{\xi \le t \le 1} |A(x, y)(t)| + \inf_{\xi \le t \le 1} |B(x, y)(t)| \\
&\ge \int_\eta^1 \inf_{\xi \le t \le 1} G_1(t, s) f(s, x(s), y(s))ds \\
&\quad + \int_\eta^1 \inf_{\xi \le t \le 1} G_2(t, s) g(s, x(s), y(s))ds \\
&> \frac{b}{2\Lambda_5} \int_\eta^1 \inf_{\xi \le t \le 1} G_1(t, s)ds + \frac{b}{2\Lambda_6} \int_\eta^1 \inf_{\xi \le t \le 1} G_2(t, s)ds = b.
\end{aligned}$$

Hence $\omega(\mathcal{Q}(x, y)) > b$ for all $(x, y) \in \mathcal{P}(\omega, b, c)$. This implies that the condition (i) of Theorem 3.1 is fulfilled.

Finally, we suppose that $(x, y) \in \mathcal{P}(\omega, b, c)$ with $\|\mathcal{Q}(x, y)\| > d$, where $b < d \leq c$. This implies that $b \leq x(t) \leq c$ and $b \leq y(t) \leq c$ for all $t \in [\xi, 1]$. By (H_2) and Lemma 2.5, we obtain

$$\begin{aligned}\omega(Q(x, y)(t)) &= \inf_{\xi \leq t \leq 1} |A(x, y)(t)| + \inf_{\xi \leq t \leq 1} |B(x, y)(t)| \\ &> \frac{b}{2\Lambda_5} \int_\eta^1 \inf_{\xi \leq t \leq 1} G_1(t, s)ds + \frac{b}{2\Lambda_6} \int_\eta^1 \inf_{\xi \leq t \leq 1} G_2(t, s)ds = b,\end{aligned}$$

which leads to satisfy condition (iii) of Theorem 3.1. Therefore, by applying Theorem 3.1, we deduce that the problem (1.3) has at least three positive solutions (x_1, y_1), (x_2, y_2) and (x_3, y_3) such that

$$\begin{gathered}\|(x_1, y_1)\| < a,\ \inf_{\xi \leq t \leq 1}(x_2, y_2)(t) > b \quad \text{and} \\ \|(x_3, y_3)\| > a \text{ with } \inf_{\xi \leq t \leq 1}(x_3, y_3)(t) < b.\end{gathered}$$

This completes the proof. □

3.2 Existence result via Guo–Krasnoselskii fixed point theorem

In this subsection, the existence theorems of at least one positive solution will be established using the Guo–Krasnoselskii fixed point theorem.

Theorem 3.3 (Guo–Krasnoselskii fixed point theorem) *Let E be a Banach space, and let $\mathcal{P} \subset E$ be a cone. Assume that Φ_1, Φ_2 are bounded open subsets of E with $0 \in \Phi_1$, $\overline{\Phi}_1 \subset \Phi_2$, and let $Q: \mathcal{P} \cap (\overline{\Phi}_2 \backslash \Phi_1) \to \mathcal{P}$ be a completely continuous operator such that:*

(i) $\|Qx\| \geq \|x\|$, $x \in \mathcal{P} \cap \partial\Phi_1$, *and* $\|Qx\| \leq \|x\|$, $x \in \mathcal{P} \cap \partial\Phi_2$; *or*
(ii) $\|Qx\| \leq \|x\|$, $x \in \mathcal{P} \cap \partial\Phi_1$, *and* $\|Qx\| \geq \|x\|$, $x \in \mathcal{P} \cap \partial\Phi_2$.

Then Q has a fixed point in $\mathcal{P} \cap (\overline{\Phi}_2 \backslash \Phi_1)$.

Theorem 3.4 *Let $f, g: [0, 1] \times \mathbb{R}_+^2 \to \mathbb{R}_+$ be continuous functions. Suppose that there exist constants $\lambda_2 > \lambda_1 > 0$, $\kappa_1 \in (\Lambda_5^{-1}, \infty)$, $\kappa_2 \in (\Lambda_6^{-1}, \infty)$, $\kappa_3 \in (0, \Lambda_3^{-1})$ and $\kappa_4 \in (0, \Lambda_4^{-1})$. In addition, assume the the following condition hold:*

(H_4) $f(t, x, y) \geq \dfrac{\kappa_1\lambda_1}{2}$ *for* $(t, x, y) \in [0, 1] \times [0, \lambda_1] \times [0, \lambda_1]$ *and* $g(t, x, y) \geq \dfrac{\kappa_2\lambda_1}{2}$ *for* $(t, x, y) \in [0, 1] \times [0, \lambda_1] \times [0, \lambda_1]$;
(H_5) $f(t, x, y) \leq \dfrac{\kappa_3\lambda_2}{2}$ *for* $(t, x, y) \in [0, 1] \times [0, \lambda_2] \times [0, \lambda_2]$ *and* $g(t, x, y) \leq \dfrac{\kappa_4\lambda_2}{2}$ *for* $(t, x, y) \in [0, 1] \times [0, \lambda_2] \times [0, \lambda_2]$.

Then the problem (1.3) *has at least one positive solution* (x, y) *such that*

$$\lambda_1 < \|(x, y)\| < \lambda_2.$$

Proof From Lemma 3.1, the operator $\mathcal{Q}\colon \mathcal{P} \to \mathcal{P}$ is completely continuous. Let $\Phi_1 = \{(x, y) \in E : \|(x, y)\| < \lambda_1\}$. Hence, for any $(x, y) \in \mathcal{P} \cap \partial\Phi_1$, we get $0 \le x(t) \le \lambda_1$ and $0 \le y(t) \le \lambda_1$ for all $t \in [0, 1]$. Using the condition (H_4) and Lemma 2.5, one has

$$\begin{aligned}
\|\mathcal{Q}(x, y)\| &= \sup_{t\in[0,1]} \int_0^1 G_1(t, s) f(s, x(s), y(s))ds \\
&\quad + \sup_{t\in[0,1]} \int_0^1 G_2(t, s) g(s, x(s), y(s))ds \\
&\ge \int_\eta^1 \inf_{\xi\le t\le 1} G_1(t, s) f(s, x(s), y(s))ds \\
&\quad + \int_\eta^1 \inf_{\eta\le t\le 1} G_2(t, s) g(s, x(s), y(s))ds \\
&\ge \frac{\kappa_1\lambda_1}{2} \int_\eta^1 \inf_{\xi\le t\le 1} G_1(t, s)ds + \frac{\kappa_2\lambda_1}{2} \int_\eta^1 \inf_{\eta\le t\le 1} G_2(t, s)ds \ge \lambda_1,
\end{aligned}$$

which means that $\|\mathcal{Q}(x, y)\| \ge \|(x, y)\|$ for $(x, y) \in \mathcal{P} \cap \partial\Phi_1$.

Define $\Phi_2 = \{(x, y) \in E : \|(x, y)\| < \lambda_2\}$. Therefore, for any $(x, y) \in \mathcal{P} \cap \partial\Phi_2$, we get $0 \le x(t) \le \lambda_2$ and $0 \le y(t) \le \lambda_2$ for all $t \in [0, 1]$. From assumption (H_5), we obtain

$$\begin{aligned}
\|\mathcal{Q}(x, y)\| &\le \int_0^1 \sup_{t\in[0,1]} G_1(t, s) f(s, x(s), y(s))ds \\
&\quad + \int_0^1 \sup_{t\in[0,1]} G_2(t, s) g(s, x(s), y(s))ds \\
&\le \frac{\kappa_3\lambda_2}{2} \int_0^1 \sup_{t\in[0,1]} G_1(t, s)ds + \frac{\kappa_4\lambda_2}{2} \int_0^1 \sup_{t\in[0,1]} G_2(t, s)ds \le \lambda_2,
\end{aligned}$$

which yields $\|\mathcal{Q}(x, y)\| \le \|(x, y)\|$ for $(x, y) \in \mathcal{P} \cap \partial\Phi_2$.

Therefore, the first part of Theorem 3.3 implies that the operator $\mathcal{Q}$ has a fixed point in $\mathcal{P} \cap (\overline{\Phi}_2 \backslash \Phi_1)$ which is a positive solution of problem (1.3). Hence, the problem (1.3) has at least on positive solution (x, y) such that

$$\lambda_1 < \|(x, y)\| < \lambda_2.$$

The proof is complete. □

Similarly to the previous theorem, we can prove the following result.

Theorem 3.5 *Let $f, g\colon [0,1]\times\mathbb{R}_+^2 \to \mathbb{R}_+$ be continuous functions. Assume that there exist constants $\lambda_2 > \lambda_1 > 0$, $\kappa_1 \in (\Lambda_5^{-1}, \infty)$, $\kappa_2 \in (\Lambda_6^{-1}, \infty)$, $\kappa_3 \in (0, \Lambda_3^{-1})$ and $\kappa_4 \in (0, \Lambda_4^{-1})$. In addition, assume the the following condition hold.*

(H_6) $f(t,x,y) \le \dfrac{\kappa_3\lambda_1}{2}$ *for* $(t,x,y) \in [0,1]\times[0,\lambda_1]\times[0,\lambda_1]$ *and*
$g(t,x,y) \le \dfrac{\kappa_4\lambda_1}{2}$ *for* $(t,x,y) \in [0,1]\times[0,\lambda_1]\times[0,\lambda_1]$*;*
(H_7) $f(t,x,y) \ge \dfrac{\kappa_1\lambda_2}{2}$ *for* $(t,x,y) \in [0,1]\times[0,\lambda_2]\times[0,\lambda_2]$ *and*
$g(t,x,y) \ge \dfrac{\kappa_2\lambda_2}{2}$ *for* $(t,x,y) \in [0,1]\times[0,\lambda_2]\times[0,\lambda_2]$.

Then the problem (1.3) has at least one positive solution (x, y) such that

$$\lambda_1 < \|(x,y)\| < \lambda_2.$$

4 Examples

In this section, we present two examples to illustrate our results.

Example 4.1 Consider following fractional system of differential equations subject to the nonlocal fractional integral boundary conditions of the form

$$\begin{cases} D^{\frac{3}{2}}x(t) + f(t,x(t),y(t)) = 0, \quad t\in(0,1),\\ D^{\frac{5}{3}}y(t) + g(t,x(t),y(t)) = 0, \quad t\in(0,1),\\ x(0)=0,\ x(1) = \dfrac{2}{3}I^{\frac{1}{2}}y\left(\dfrac{1}{4}\right) + \dfrac{\sqrt{3}}{5}I^{\frac{3}{2}}y\left(\dfrac{1}{4}\right) + \dfrac{\pi}{12}I^{\frac{5}{2}}y\left(\dfrac{1}{4}\right),\\ y(0)=0,\ y(1) = \dfrac{\sqrt{2}}{5}I^{\frac{1}{3}}x\left(\dfrac{4}{7}\right) + \dfrac{1}{\sqrt{7}}I^{\frac{2}{3}}x\left(\dfrac{4}{7}\right) + \dfrac{2}{\sqrt{e}}I^{\frac{4}{3}}x\left(\dfrac{4}{7}\right) + \dfrac{8}{13}I^{\frac{5}{3}}x\left(\dfrac{4}{7}\right), \end{cases} \tag{4.1}$$

where

$$f(t,x,y) = \begin{cases} x\left(\dfrac{2}{3}-x\right) + \dfrac{1}{2}y\left(\dfrac{2}{3}-y\right) + \dfrac{1}{5}(t+1); & 0\le t\le 1; 0\le x,y\le 2/3,\\ \dfrac{1}{5}(t+1)|\cos(x\pi)| + \dfrac{1}{5}(t+1)|\cos(y\pi)| & \\ +45\left(x-\dfrac{2}{3}\right)\left(y-\dfrac{2}{3}\right); & 0\le t\le 1; 2/3\le x,y\le 4/3,\\ \dfrac{1}{5}(t+101) + \sin^2\left(\left(x-\dfrac{4}{3}\right)\left(y-\dfrac{4}{3}\right)\right); & 0\le t\le 1; 4/3\le x,y<\infty, \end{cases}$$

and

$$g(t,x,y)=\begin{cases} xy\left(\frac{2}{3}-x\right)\left(\frac{2}{3}-y\right)+\frac{1}{8}(t+2); & 0\le t\le 1;\, 0\le x,y\le 2/3,\\ \frac{1}{8\sqrt{3}}(t+2)|\sin(x\pi)|+\frac{1}{8\sqrt{3}}(t+2)|\sin(y\pi)| & \\ +27\left(x-\frac{2}{3}\right)\left(y-\frac{2}{3}\right); & 0\le t\le 1;\, 2/3\le x,y\le 4/3,\\ \frac{1}{8}(t+98)+\sin^4\left(\left(x-\frac{4}{3}\right)\left(y-\frac{4}{3}\right)\right); & 0\le t\le 1;\, 4/3\le x,y<\infty. \end{cases}$$

Here $p=3/2, q=5/3, m=3, \eta=1/4, \alpha_1=2/3, \gamma_1=1/2, \alpha_2=\sqrt{3}/5, \gamma_2=3/2$, $\alpha_3=\pi/12$, $\gamma_3=5/2$, $n=4$, $\xi=4/7$, $\beta_1=\sqrt{2}/5$, $\mu_1=1/3$, $\beta_2=1/\sqrt{7}$, $\mu_2=2/3$, $\beta_3=2/\sqrt{e}$, $\mu_3=4/3$, $\beta_4=8/13$, $\mu_4=5/3$. We find that $\Lambda_1=0.1173432604$ and $\Lambda_2=0.6208838470$ which leads to $\Omega=0.9271434651>0$. In addition, we can compute that $\Lambda_3=0.5487565277$, $\Lambda_4=0.6859288172$, $\Lambda_5=0.03857691941$ and $\Lambda_6=0.1101600049$.

Choosing $a=2/3$, $b=4/3$, $c=24$, we get

$$f(t,x,y)\le 0.5666666667 \quad \text{and} \quad g(t,x,y)\le 0.4120370370,$$

which yields for $0\le t\le 1$ and $0\le x,y\le 2/3$,

$$f(t,x,y)<0.6074339286=\frac{a}{2\Lambda_3} \quad \text{and} \quad g(t,x,y)<0.4859590749=\frac{a}{2\Lambda_4}.$$

In addition, we obtain

$$f(t,x,y)\ge 20.25000000 \quad \text{and} \quad g(t,x,y)\ge 12.28125000,$$

which leads to

$$f(t,x,y)>17.28149051=\frac{b}{2\Lambda_5} \quad \text{and} \quad g(t,x,y)>6.051803167=\frac{b}{2\Lambda_6},$$

for $1/4\le t\le 1$ and $4/3\le x,y\le 24$. Also we have

$$f(t,x,y)\le 21.40000000 \quad \text{and} \quad g(t,x,y)\le 13.37500000,$$

which gives

$$f(t,x,y)<21.86762143=\frac{c}{2\Lambda_3} \quad \text{and} \quad g(t,x,y)<17.49452670=\frac{c}{2\Lambda_4},$$

for $0\le t\le 1$ and $0\le x,y\le 24$.

Therefore, the conditions $(H_1$–$H_3)$ of Theorem 3.2 hold. Applying Theorem 3.2, we deduce that the problem (4.1) has at least three positive solutions (x_1,y_1), (x_2,y_2) and

(x_3, y_3) such that $\|(x_1, y_1)\| < 2/3$, $\inf_{\frac{4}{7}\le t\le 1}(x_2, y_2)(t) > 4/3$ and $\|(x_3, y_3)\| > 2/3$ with $\inf_{\frac{4}{7}\le t\le 1}(x_3, y_3)(t) < 4/3$.

Example 4.2 Consider following fractional system of differential equations subject to the nonlocal fractional integral boundary conditions of the form

$$\begin{cases} D^{\frac{7}{4}}x(t) + f(t, x(t), y(t)) = 0, \quad t \in (0, 1), \\ D^{\frac{9}{5}}y(t) + g(t, x(t), y(t)) = 0, \quad t \in (0, 1), \\ x(0) = 0,\ x(1) = \dfrac{\sqrt{\pi}}{13} I^{\frac{1}{4}} y\left(\dfrac{3}{4}\right) + \dfrac{7}{12} I^{\frac{1}{2}} y\left(\dfrac{3}{4}\right) + \dfrac{\sqrt{2}}{15} I^{\frac{3}{4}} y\left(\dfrac{3}{4}\right) \\ \qquad\qquad\qquad + \dfrac{4}{\sqrt{11}} I^{\frac{5}{4}} y\left(\dfrac{3}{4}\right), \\ y(0) = 0,\ y(1) = \dfrac{3}{16} I^{\frac{1}{5}} x\left(\dfrac{16}{19}\right) + \dfrac{2}{\sqrt{5}} I^{\frac{2}{5}} x\left(\dfrac{16}{19}\right) + \dfrac{1}{3e^2} I^{\frac{3}{5}} x\left(\dfrac{16}{19}\right) \\ \qquad\qquad\qquad + \dfrac{3}{8\pi} I^{\frac{4}{5}} x\left(\dfrac{16}{19}\right) + \dfrac{4}{13\sqrt{7}} I^{\frac{6}{5}} x\left(\dfrac{16}{19}\right), \end{cases} \tag{4.2}$$

where

$$f(t, x, y) = \begin{cases} x\,(2-x) + y\,(2-y) + 8(1+t); & 0 \le t \le 1;\ 0 \le x, y \le 2, \\ 8(1+t)e^{2-x} + \sin^2(\pi y); & 0 \le t \le 1;\ 2 \le x, y < \infty, \end{cases}$$

and

$$g(t, x, y) = \begin{cases} xy(2-x)^2 e^{-y} + 7(\sqrt{t}+1) + |\sin(\pi y)|; & 0 \le t \le 1;\ 0 \le x, y \le 2, \\ 7(\sqrt{t}+1)\cos^2(2-x) + 4\sin^2(\pi x)\cos^4(\pi y); & 0 \le t \le 1;\ 2 \le x, y < \infty. \end{cases}$$

Here $p = 7/4$, $q = 9/5$, $m = 4$, $\eta = 3/4$, $\alpha_1 = \sqrt{\pi}/13$, $\gamma_1 = 1/4$, $\alpha_2 = 7/12$, $\gamma_2 = 1/2$, $\alpha_3 = \sqrt{2}/15$, $\gamma_3 = 3/4$, $\alpha_4 = 4/\sqrt{11}$, $\gamma_4 = 5/4$, $n = 5$, $\xi = 16/19$, $\beta_1 = 3/16$, $\mu_1 = 1/5$, $\beta_2 = 2/\sqrt{5}$, $\mu_2 = 2/5$, $\beta_3 = 1/3e^2$, $\mu_3 = 3/5$, $\beta_4 = 3/8\pi$, $\mu_4 = 4/5$, $\beta_5 = 4/13\sqrt{7}$, $\mu_5 = 6/5$. We find that $\Lambda_1 = 0.7502528482$ and $\Lambda_2 = 0.9064443536$ which yields $\Omega = 0.3199375420 > 0$. Further, we can compute that $\Lambda_3 = 0.8012222566$, $\Lambda_4 = 0.9066096747$, $\Lambda_5 = 0.1389153084$ and $\Lambda_6 = 0.1437856901$.

Choosing $\lambda_1 = 2$, $\lambda_2 = 40$, $\kappa_1 = 8 \in (\Lambda_5^{-1}, \infty) = (7.198630673, \infty)$, $\kappa_2 = 7 \in (\Lambda_6^{-1}, \infty) = (6.954795010, \infty)$, $\kappa_3 = 1 \in (0, \Lambda_3^{-1}) = (0, 1.248093137)$ and $\kappa_4 = 1 \in (0, \Lambda_4^{-1}) = (0, 1.103010510)$, we have

$$f(t, x, y) \ge 8 \ge \frac{\kappa_1\lambda_1}{2} \quad \text{and} \quad g(t, x, y) \ge 7 \ge \frac{\kappa_2\lambda_1}{2},$$

for $0 \le t \le 1$, $0 \le x, y \le 2$. Also we have

$$f(t, x, y) \le 17 \le \frac{\kappa_3\lambda_2}{2} \quad \text{and} \quad g(t, x, y) \le 18 \le \frac{\kappa_4\lambda_2}{2},$$

for $0 \le t \le 1$, $2 \le x, y < \infty$.

Thus the conditions (H_4–H_5) hold. By Theorem 3.4, we conclude that the problem (4.2) has at least one positive solution (x, y) such that $2 < \|(x, y)\| < 40$.

Acknowledgements The authors thank the reviewer for his/her constructive comments that led to the improvement of the original manuscript. This research was funded by King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-GOV-60-22.

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