Positivity (2017) 21:377–394
DOI 10.1007/s11117-016-0427-z

Positivity

Coercive nonlocal elements in fractional differential equations

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Received: 27 October 2015 / Accepted: 12 May 2016 / Published online: 21 May 2016

Abstract We consider the fractional boundary problem

$$\begin{aligned} -\left[D_{0^+}^{\nu} y\right](t) &= \lambda f\big(t, y(t)\big), 0 < t < 1 \\ y^{(i)}(0) &= 0, 0 \le i \le n-2 \\ \left[D_{0^+}^{\alpha} y\right](1) &= H\big(\varphi(y)\big), \end{aligned}$$

where $n \in \mathbb{N}_4$, $n-1 < \nu \le n$, $\alpha \in [1, n-2]$, and $\lambda > 0$ is a parameter. Here the element φ is a linear functional that represents a nonlocal boundary condition. We show that by introducing a new order cone, we can ensure that this functional is coercive, which is of importance in proving existence results for the above boundary value problem under minimal assumptions on the functions f and H. We also develop a new open set attendant to the cone. By means of examples we investigate both the usefulness of the new set as well as the strength of the coercivity condition and its dependence on the order, ν, of the fractional derivative. Finally, the methods we develop are applicable to a range of fractional-order boundary value problems.

Keywords Riemann–Liouville fractional derivative · coercivity · nonlocal boundary condition · positive solution · Hammerstein integral equation

Mathematics Subject Classification Primary 26A33 · 34A08 · 34B10 · 45G10 · 45M20; Secondary 34B18 · 47H14 · 47H30

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1 Introduction

In this paper we consider the nonlocal boundary value problem

$$\begin{aligned} -\left[D_{0^+}^{\nu}y\right](t) &= \lambda f\big(t, y(t)\big), 0 < t < 1 \\ y^{(i)}(0) &= 0, 0 \le i \le n-2 \\ \left[D_{0^+}^{\alpha}y\right](1) &= H\big(\varphi(y)\big), \end{aligned} \tag{1.1}$$

where $\nu \in (n-1, n]$ for $n \in \mathbb{N}_4 := \{4, 5, 6, \ldots\}$, $\alpha \in [1, n-2]$, and $\lambda > 0$ is a parameter; here we utilize the Riemann–Liouville derivative. In addition, the maps $H : [0, +\infty) \to [0, +\infty)$ and $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ are continuous. The nonlocal element $\varphi : \mathcal{C}([0, 1]) \to \mathbb{R}$ is a linear functional, which is realizable as a Stieltjes integral with signed measure, namely

$$\varphi(y) = \int_0^1 y(t)\, d\alpha(t), \tag{1.2}$$

where $\alpha \in BV([0, 1]; \mathbb{R})$ is not necessarily monotone increasing—i.e., it may occur that $\varphi(y) < 0$ even though y is nonnegative. As the example will demonstrate in Sect. 3, although it may occur that H is nonlinear, it need *not* necessarily be so.

The primary contribution of this paper is to introduce a new order cone and attendant open set with which to study fractional differential equation (1.1). In particular, we utilize the cone

$$\mathcal{K} := \left\{ y \in \mathcal{C}([0, 1]) \ : \ y \ge 0, \varphi(y) \ge \left(\inf_{s \in S_0} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t, s)\, d\alpha(t) \right) \|y\| \right\}, \tag{1.3}$$

where α is as in (1.2) above, G is the Green's function, see (2.1), associated to problem (1.1), and $\mathcal{G}(s) := \max_{t \in [0,1]} G(t, s)$. The set S_0 in (1.3) is a set of full measure on which the infimum is positive. The precise assumption leading to this set is given in Sect. 2. The motivation behind including the coercivity condition in (1.3) is so that we can weaken the growth conditions imposed on both H and f. In particular, we can use asymptotic or even, as we shall see, pointwise conditions on H that, without some sort of coercivity and thus lower control over φ, might not be possible to achieve easily.

As previously intimated, in addition to the use of a new cone in fractional nonlocal boundary value problems, we also make use of the coercivity generated by $\mathcal{K}$ by taking advantage of a new open set in the study of such problems. In particular, we define here the open set $\widehat{V}_{\rho}$ by

$$\widehat{V}_{\rho} := \{y \in \mathcal{K} \ : \ \varphi(y) < \rho\}. \tag{1.4}$$

As shall be seen in both Sects. 2 and 3, the use of this open set is only viable since we have that φ is a coercive functional by means of $\mathcal{K}$. In any case, as shall be shown in Sect. 3, the use of the set $\widehat{V}_{\rho}$ as defined in (1.4) can produce better existence results. In particular, we can treat cases where H is piecewise linear and no particular growth conditions are imposed on f. Alternatively, we can treat cases where H has only pointwise conditions imposed, which is one interesting and somewhat unusual consequence of our approach. And in each case these conditions are accommodated with $\lambda > 0$ essentially unrestricted.

It should be mentioned at this juncture that the methods we introduce here, namely the use of the new cone $\mathcal{K}$ and the set $\widehat{V}_\rho$, are applicable to a range of fractional-order boundary value problems. We prefer the specificity and concreteness of (1.1), as it provides an application of the abstract results. But the techniques developed herein should be able to be extended to a variety of settings such as semipositone fractional BVPs, singular fractional BVPs, and fractional-order differential equations equipped with boundary conditions other than those utilized in (1.1).

We conclude the introduction by mentioning some of the relevant literature on both fractional-order and nonlocal boundary value problems and its connection to this present work. Problem (1.1) was studied in both the local and nonlocal boundary condition settings by the author [9,10]. Many other authors have studied either extensions of (1.1) or similar problems—see, for example, [7,29,39,50,59]. More generally, there has been substantial research interest in fractional-order differential equations over the past 10 to 15 years, and there are numerous papers within the area. For a representative but certainly not exhaustive sampling, one may consult [1,2,4,8,20,30,41] and the references therein. At the same time, the study of nonlocal boundary value problems has seen intense research recently, and these studies have spanned such topics as nonlinear boundary conditions, linear boundary conditions, semipositone problems, and the construction of various cones to allow for more general linear boundary conditions—see, for example, the contributions by Anderson [3], Cabada, et al. [6], Goodrich [11–15], Graef and Webb [19], Infante [22], Infante and Pietramala [23,25], Infante, Pietramala, and Minhós [24], Infanate, Pietramala, and Tenuta [26], Jankowski [28], Karakostas and Tsamatos [31,32], Karakostas [33], Webb and Infante [47], and Yang [53–56]. More generally, the study of various perturbed Hammerstein integral equations, which, as in our work, are typically utilized in the study of boundary value problems, have also been studied by many authors such as Goodrich [17], Lan and Lin [36], Liu and Wu [38], Webb [46], Xu and Yang [51], Yang [57], and Zhao [60]. The articles by Picone [42] and Whyburn [49], while classical, are of interest for their historical value.

It should also be mentioned in particular that the cone we introduce here is inspired by the not dissimilar cones utilized first by Graef et al. [18] and then subsequently by Webb [45] and Ma and Zhong [40]. In addition, the type of open set we introduce here (i.e., one in which a functional is utilized as part of the defining condition) is similar to constructions found in some other works—see, for example, Infante and Maciejewski [27]. However, we have not seen before the particular cone and open set we introduce here, nor used in the specific ways in which we utilize them here.

All in all, then, in this work we join several of these strands of research by developing a new cone and open set in order to study a nonlocal boundary value problem in the context of fractional differential equations. Furthermore, since ν can be an integer in (1.1) and, in fact, the case $\nu = 4$ is important in beam deflection models, the results here also have some interest in the integer-order setting. And, indeed, there have appeared many works on specifically fourth-order BVPs with nonlocal boundary conditions—see, for example, [5,34,37,48,52,61]. Thus, our results, while couched in the fractional-order setting, also complement those papers specifically treating the fourth-order setting.

2 Preliminary lemmata and notation

We begin by first stating some notation that will be used throughout the reminder of this paper.

Notation 2.1 For use in the sequel, we appeal to the following notational conventions.

- Define the map $\mathcal{G} : [0, 1] \rightarrow [0, +\infty)$ as in Sect. 1—namely, put

$$\mathcal{G}(s) := \sup_{t\in[0,1]} G(t, s),$$

  where the map $(t, s) \mapsto G(t, s)$ is defined in (2.1) below. Note that for each fixed $s \in [0, 1]$ it holds that $G(t, s) \le \mathcal{G}(s)$, for all $t \in [0, 1]$.
- Given a function $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ and a set $X \subseteq [0, +\infty)$ define the number $\widetilde{f}_X^M$ to be

$$\widetilde{f}_X^M := \sup_{(t,y)\in[0,1]\times X} f(t, y).$$

- Given a number $\rho > 0$ define the open set Ω_ρ by $\Omega_\rho := \{y \in \mathcal{K} : \|y\| < \rho\}$, where $\mathcal{K}$ is as in (1.3) above.

A couple of basic definitions regarding fractional derivatives and integrals of Riemann–Liouville type are recalled next. For a more detailed exposition on this and a variety of related topics in the continuous fractional calculus, the reader may consult the monograph by Podlubny [43].

Definition 2.2 Let $\nu > 0$ with $\nu \in \mathbb{R}$. Suppose that $y : [a, +\infty) \rightarrow \mathbb{R}$. Then the νth order Riemann–Liouville fractional integral is defined to be

$$D_{a^+}^{-\nu} y(t) := \frac{1}{\Gamma(\nu)} \int_a^t y(s)(t-s)^{\nu-1}\, ds,$$

whenever the right-hand side is defined.

Definition 2.3 For $y : [a, +\infty) \rightarrow \mathbb{R}$ and with $\nu > 0$ and $\nu \in \mathbb{R}$, we define the νth order Riemann–Liouville fractional derivative to be

$$D_{a^+}^{\nu} y(t) := \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_a^t \frac{y(s)}{(t-s)^{\nu+1-n}}\, ds,$$

where $n \in \mathbb{N}$ is the unique positive integer satisfying $n - 1 \le \nu < n$ and $t > a$, whenever the right-hand side is defined.

We also need to recall some preliminary lemmata regarding the Green's function associated to problem (1.1). In particular, we recall the following results, which may be found in a paper by the author [9].

Lemma 2.4 *Let $g \in \mathcal{C}([0,1])$ be given. Then the unique solution to problem $-D^{\nu}y(t) = g(t)$ together with the boundary conditions*

$$y^{(i)}(0) = 0,\ i \in \{0, 1, 2, \ldots, n-2\}$$
$$\left[D_{0^+}^{\alpha} y\right](1) = 0,$$

where $\alpha \in [1, n-2]$, is

$$y(t) = \int_0^1 G(t,s)g(s)\,ds,$$

where

$$G(t,s) = \begin{cases} \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1}-(t-s)^{\nu-1}}{\Gamma(\nu)}, & 0 \le s \le t \le 1 \\ \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}, & 0 \le t \le s \le 1 \end{cases} \tag{2.1}$$

is the Green's function for this problem.

Lemma 2.5 *Let G be as given in the statement of Lemma 2.4. Then we find that:*

(1) *$G(t,s)$ is a continuous function on the unit square $[0,1] \times [0,1]$;*
(2) *$G(t,s) \ge 0$ for each $(t,s) \in [0,1] \times [0,1]$; and*
(3) *$\mathcal{G}(s) = G(1,s)$, for each $s \in [0,1]$.*

We state next the structural and regularity conditions that we impose on problem (1.1). Throughout this work we denote by $\|\cdot\|$ the usual supremum norm on the space $\mathcal{C}([0,1])$. In summary, condition (A1) concerns the basic structure of φ, conditions (A2)–(A3) concern the growth properties of the maps H and f, condition (A4) concerns the existence of the coercivity constant $C_0 := \inf_{s \in S_0} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\,d\alpha(t)$, and, finally, condition (A5) is a technical condition that ensures that the cone $\mathcal{K}$ is neither empty nor trivial. It should be noted that not all of these conditions are used in each existence result—e.g., we are able to weaken or remove conditions (A2) and (A3) by replacing them with other assumptions.

A1: The functional $\varphi(y)$ may be written in the form

$$\varphi(y) := \int_{[0,1]} y(t)\,d\alpha(t),$$

where $\alpha : [0,1] \to \mathbb{R}$ satisfies $\alpha \in BV([0,1])$. Moreover, we denote by $C_1 > 0$ some finite constant such that

$$|\varphi(y)| \le C_1 \|y\|,$$

for each $y \in \mathcal{C}([0,1])$.

A2: The map $H : [0, +\infty) \to [0, +\infty)$ is continuous, and there exists a number $\rho_1 > 0$ such that

$$\frac{H(\rho_1)}{\rho_1} > \frac{1}{\varphi(\beta)},$$

where the map $\beta : [0, 1] \to \mathbb{R}$ is defined by

$$\beta(t) := \frac{\Gamma(\nu - \alpha)}{\Gamma(\nu)} t^{\nu - 1}. \tag{2.2}$$

A3: The function $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and satisfies

$$\lim_{y \to +\infty} \frac{f(t, y)}{y} = 0, \text{ uniformly for } t \in [0, 1].$$

A4: Assume that the map

$$s \mapsto \frac{1}{\mathcal{G}(s)} \int_0^1 G(t, s)\, d\alpha(t)$$

is defined for $s \in S_0$, where $S_0 \subseteq [0, 1]$ has full measure (i.e., $|S_0| = 1$), and the constant defined by

$$C_0 := \inf_{s \in S_0} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t, s)\, d\alpha(t)$$

satisfies $C_0 \in (0, C_1)$.
A5: It holds that

$$\varphi(\beta) \geq C_0 \|\beta\|,$$

where β is defined in (2.2)

Remark 2.6 Seeing as condition (A5) implies that $\beta \in \mathcal{K}$ with $\|\beta\| \neq 0$, it follows that this condition ensures both that $\mathcal{K} \neq \varnothing$ and that $\mathcal{K}$ is nontrivial; here $\mathcal{K}$ is as defined earlier in (1.3). Moreover, observe that condition (A5) implies that $\varphi(\beta) > 0$ since $\|\beta\| > 0$ evidently.

Remark 2.7 As will be shown explicitly by the example in Sect. 3, the range of admissible values for the parameter λ, appearing in (1.1), is explicitly computable. Thus, we do not state our results here for some uncomputable, "sufficiently small" parameter λ. Moreover, the computation of the coercivity constant C_0, appearing in (A4), is also reasonable to compute as will be shown in Sect. 3.

In order to study problem (1.1) we consider instead the operator $T : \mathcal{C}([0, 1]) \to \mathcal{C}([0, 1])$ defined by

$$(Ty)(t) := \beta(t) H(\varphi(y)) + \lambda \int_0^1 G(t, s) f(s, y(s))\, ds \tag{2.3}$$

and then look for solutions of the Hammerstein-type equation $(Ty)(t) = y(t)$. Note that (see [10, Lemma 4.3]) it has been shown that the map β occurring in (2.2) has the property that it is increasing in t, it holds that $\beta^{(i)}(0) = 0$ for each $0 \le i \le n-2$ with $i \in \mathbb{N}$, and that $\left[D_{0+}^{\alpha}\beta\right](1) = 1$. These facts combine to show that a solution of the Hammerstein equation is thus a solution of the boundary value problem (1.1).

Ordinarily it is trivial to show that $T(\mathcal{K}) \subseteq \mathcal{K}$. In this case because of the use of a new cone, we provide part of this proof in detail.

Lemma 2.8 *Let T be the operator defined in* (2.3). *Then it holds that $T(\mathcal{K}) \subseteq \mathcal{K}$.*

Proof It is obvious that whenever $y \in \mathcal{K}$ it holds that $(Ty)(t) \ge 0$, for each $t \in [0, 1]$. Therefore, we focus on the coercivity condition. To this end, let $y \in \mathcal{K}$ be fixed but arbitrary. Observe that

$$\|Ty\| \le H(\varphi(y))\|\beta\| + \lambda \int_0^1 \mathcal{G}(s) f(s, y(s))\, ds.$$

Thus, recalling that S_0 is a set of full measure, we write

$$\begin{aligned}
\varphi(Ty) &= \varphi(\beta)H(\varphi(y)) + \lambda \int_0^1 \int_0^1 G(t,s) f(s, y(s))\, d\alpha(t)\, ds \\
&= \varphi(\beta)H(\varphi(y)) + \lambda \int_{S_0} \left[\int_0^1 G(t,s)\, d\alpha(t)\right] f(s, y(s))\, ds \\
&= \varphi(\beta)H(\varphi(y)) + \lambda \int_{S_0} \left[\frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha(t)\right] \mathcal{G}(s) f(s, y(s))\, ds \\
&\ge \varphi(\beta)H(\varphi(y)) + \lambda \int_{S_0} \left[\inf_{s \in S_0} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha(t)\right] \mathcal{G}(s) f(s, y(s))\, ds \\
&= \varphi(\beta)H(\varphi(y)) + C_0\lambda \int_{S_0} \mathcal{G}(s) f(s, y(s))\, ds \\
&\ge C_0\left[H(\varphi(y))\|\beta\| + \lambda \int_0^1 \mathcal{G}(s) f(s, y(s))\, ds\right] \\
&\ge C_0\|Ty\|.
\end{aligned}$$

As this establishes that $Ty \in \mathcal{K}$, we conclude that $T(\mathcal{K}) \subseteq \mathcal{K}$, as claimed. □

We next discuss the set $\widehat{V}_\rho$, whose definition was given preliminarily in (1.4) in Sect. 1. We first state and prove two elementary lemmata regarding this set. That these hold is essential for the use of $\widehat{V}_\rho$ in the existence arguments.

Lemma 2.9 *For each fixed $\rho > 0$ it holds that*

$$\Omega_{\frac{\rho}{C_1}} \subseteq \widehat{V}_\rho \subseteq \Omega_{\frac{\rho}{C_0}}. \tag{2.4}$$

Proof That (2.4) holds is both a consequence of the linearity of φ and the coercivity condition. To see this, first fix $y \in \widehat{V}_\rho$. Then we note that

$$\rho > \varphi(y) \geq C_0\|y\|.$$

In light of the coercivity condition above, we see that $y \in \widehat{V}_\rho$ implies that

$$\|y\| \leq \frac{\rho}{C_0},$$

and this establishes that

$$\widehat{V}_\rho \subseteq \Omega_{\frac{\rho}{C_0}}.$$

On the other hand, fix $y \in \Omega_{\frac{\rho}{C_1}}$. Then it follows that

$$\varphi(y) \leq C_1\|y\| < \rho,$$

whence $y \in \widehat{V}_\rho$ so that $\widehat{V}_\rho \supseteq \Omega_{\frac{\rho}{C_1}}$.

Finally, let us note that it does, in fact, hold that

$$\Omega_{\frac{\rho}{C_1}} \subsetneq \Omega_{\frac{\rho}{C_0}},$$

for each $\rho > 0$. This follows from the observation that $C_0 < C_1$. Thus, the inclusion is well defined, and this completes the proof. □

Lemma 2.10 *For each fixed $\rho \in (0, +\infty)$ the set $\widehat{V}_\rho$ defined in (1.4) is a (relatively) open set in $\mathcal{K}$ and, furthermore, is bounded.*

Proof Since this is an obvious consequence of the continuity of the linear functional φ, inclusion (2.4), and the strict inequality in the definition of $\widehat{V}_\rho$, we omit the formal proof of this fact. □

Remark 2.11 We note that the set $\widehat{V}_\rho$ is similar in spirit to the set commonly denoted V_ρ in the literature, which is defined, for an appropriate order cone $\mathcal{K}_0$ and fixed $(a, b) \Subset (0, 1)$, by

$$V_\rho := \left\{ y \in \mathcal{K}_0 \ : \ \min_{t\in[a,b]} y(t) < \rho \right\},$$

and which was originally utilized by Lan [35]. In fact, the reason we have denoted our new set in (1.4) by $\widehat{V}_\rho$ is since it is clearly inspired by Lan's construction, V_ρ, above. Moreover, as mentioned in Sect. 1 sets similar to $\widehat{V}_\rho$ have been utilized in other works—e.g., [27].

Remark 2.12 The dual use of the new cone $\mathcal{K}$ together with the new open set $\widehat{V}_\rho$ will make the proof of the existence results very simple. This is one of the advantages of the dual use of these two constructions.

We conclude by stating the fixed point theorem, which we use in the existence arguments in Sect. 3. In particular, our approach here is index theoretic. To this end, we recall a basic result in this direction, and one may consult Infante et al. [26], Guo and Lakshmikantham [21], or Zeidler [58] for further details on these types of results.

Lemma 2.13 *Let D be a bounded open set and, with $\mathcal{K}$ a cone in a Banach space $\mathscr{X}$, suppose both that $D \cap \mathcal{K} \neq \emptyset$ and that $\overline{D} \cap \mathcal{K} \neq \mathcal{K}$. Let D_1 be open in $\mathscr{X}$ with $\overline{D}_1 \subseteq D \cap \mathcal{K}$. Assume that $T : \overline{D} \cap \mathcal{K} \to \mathcal{K}$ is a compact map such that $Tx \neq x$ for $x \in \mathcal{K} \cap \partial D$. If $i_{\mathcal{K}}(T, D \cap \mathcal{K}) = 1$ and $i_{\mathcal{K}}(T, D_1 \cap \mathcal{K}) = 0$, then T has a fixed point in $(D \cap \mathcal{K}) \setminus \left(\overline{D_1 \cap \mathcal{K}}\right)$. Moreover, the same result holds if $i_{\mathcal{K}}(T, D \cap \mathcal{K}) = 0$ and $i_{\mathcal{K}}(T, D_1 \cap \mathcal{K}) = 1$.*

3 Main result and example

We begin by stating and proving the existence results for the Hammerstein-type equation $(Ty)(t) = y(t)$ and thus for problem (1.1).

Theorem 3.1 *Assume that conditions (A1)–(A2) and (A4)–(A5) hold. Fix $\lambda > 0$ and assume that there exists a number $\rho_2 > \frac{C_1}{C_0}\rho_1$ such that*

$$\frac{H\left(\rho_2\right)}{\rho_2}\varphi(\beta) + \frac{\lambda}{\rho_2}\widetilde{f}^M_{\left[0, \frac{\rho_2}{C_0}\right]} \int_0^1 \int_0^1 G(t, s)\, d\alpha(t)\, ds < 1.$$

Then problem (1.1) has at least one positive solution.

Proof Having showed that $T(\mathcal{K}) \subseteq \mathcal{K}$ in Lemma 2.4 and in observation of the fact that T is completely continuous, we focus on the actual index calculations. To this end, we begin by noting that by condition (A2) there exists $\rho_1 > 0$ such that

$$\frac{H\left(\rho_1\right)}{\rho_1} > \frac{1}{\varphi(\beta)}. \tag{3.1}$$

We show now for each $y \in \partial \widehat{V}_{\rho_1}$ and each $\mu \geq 0$ that $y \neq Ty + \mu e$, where we put $e(t) := \beta(t)$. Recall here that $\beta \in \mathcal{K}$ by condition (A5), and, moreover, that $\|\beta\| \neq 0$; thus, β represents a valid selection for the map $t \mapsto e(t)$. So, suppose for contradiction the existence of fixed $y \in \partial \widehat{V}_{\rho_1}$ and $\mu \geq 0$ such that $y = Ty + \mu e$. Then applying φ to both sides of the equality $y = Ty + \mu e$ and using the fact that $\varphi(\mu e) \geq 0$, we obtain that

$$\begin{aligned}
\rho_1 &= \varphi(y) \\
&\geq \varphi(\beta) H(\varphi(y)) + \lambda \int_0^1 \int_0^1 G(t, s) f(s, y(s))\, d\alpha(t)\, ds \\
&\geq \varphi(\beta) H(\varphi(y)) = \varphi(\beta) H\left(\rho_1\right),
\end{aligned}$$

whence

$$\frac{H\left(\rho_1\right)}{\rho_1} \leq \frac{1}{\varphi(\beta)}.$$

Since this is a contradiction to inequality (3.1), we deduce that

$$i_{\mathcal{K}}\left(T, \widehat{V}_{\rho_1}\right) = 0. \tag{3.2}$$

On the other hand, we now show that for $\rho_2 > \frac{C_1}{C_0}\rho_1$, where ρ_2 is the number given in the statement of the theorem, we have

$$i_{\mathcal{K}}\left(T, \widehat{V}_{\rho_2}\right) = 1. \tag{3.3}$$

In order to prove (3.3) we show that for each $\mu \geq 1$ and each $y \in \widehat{V}_{\rho_2}$ it holds that $Ty \neq \mu y$. Therefore, suppose not. Then we have $\varphi(Ty) = \mu\varphi(y) \geq \varphi(y)$ for some $y \in \partial\widehat{V}_{\rho_2}$. Consequently, we may write

$$\varphi(y) \leq \varphi(\beta)H(\varphi(y)) + \lambda\int_0^1\int_0^1 G(t,s)f(s,y(s))\,d\alpha(t)\,ds, \tag{3.4}$$

which, since $\varphi(y) = \rho_2$, implies that

$$\begin{aligned}1 &\leq \frac{H(\rho_2)}{\rho_2}\varphi(\beta) + \frac{\lambda}{\rho_2}\int_0^1\int_0^1 G(t,s)f(s,y(s))\,d\alpha(t)\,ds\\ &\leq \frac{H(\rho_2)}{\rho_2}\varphi(\beta) + \frac{\lambda}{\rho_2}\int_0^1\int_0^1 G(t,s)\widetilde{f}^M_{\left[0,\frac{\rho_2}{C_0}\right]}\,d\alpha(t)\,ds\\ &< 1,\end{aligned} \tag{3.5}$$

which is a contradiction, and so, establishes (3.3). Observe that to establish (3.5) we are tacitly using the fact that since $s \mapsto \mathcal{G}(s)$ is a nonnegative map, it follows from condition (A4) that

$$\int_0^1 G(t,s)\,d\alpha(t) > 0,$$

for a.e. $s \in [0,1]$, seeing as $\left|[0,1]\backslash S_0\right| = 0$. We are also appealing to Lemma 2.9 so that since $y \in \overline{\widehat{V}}_{\rho_2}$, it thus follows that $y \in \Omega_{\frac{\rho_2}{C_0}}$, whence

$$0 \leq f(s,y(s)) \leq \widetilde{f}^M_{\left[0,\frac{\rho_2}{C_0}\right]},$$

for each $s \in [0,1]$.

All in all, then, combining (3.2)–(3.3) we conclude the existence of a map

$$y_0 \in \widehat{V}_{\rho_2}\backslash\overline{\widehat{V}}_{\rho_1}$$

such that $Ty_0 = y_0$. Note that

$$\widehat{V}_{\rho_2}\backslash\overline{\widehat{V}}_{\rho_1} \neq \varnothing \tag{3.6}$$

since from Lemma 2.9 we see that $\widehat{V}_{\rho_1} \subseteq \Omega_{\frac{\rho_1}{C_0}}$ and that $\widehat{V}_{\rho_2} \supseteq \Omega_{\frac{\rho_2}{C_1}}$. Thus, we see that if $\rho_2 > \frac{C_1}{C_0}\rho_1$, then it follows that

$$\widehat{V}_{\rho_2} \supseteq \Omega_{\frac{\rho_2}{C_1}} \supsetneq \overline{\Omega}_{\frac{\rho_1}{C_0}} \supset \overline{\widehat{V}}_{\rho_1} \supsetneq \widehat{V}_{\rho_1},$$

which establishes (3.6). Since this map y_0 solves (1.1) and satisfies $\|y_0\| \neq 0$, the proof is thus complete. □

Remark 3.2 As the proof of Theorem 3.1 demonstrates, the solution to (1.1) guaranteed by Theorem 3.1 satisfies the norm localization

$$0 < \frac{\rho_1}{C_1} < \|y_0\| < \frac{\rho_2}{C_0},$$

where here we appeal to Lemma 2.9, noting especially that

$$\mathcal{K}\backslash\overline{\widehat{V}}_{\rho_1} \subseteq \mathcal{K}\backslash\overline{\Omega}_{\frac{\rho_1}{C_1}} \tag{3.7}$$

and

$$\widehat{V}_{\rho_2} \subseteq \Omega_{\frac{\rho_2}{C_0}}, \tag{3.8}$$

whereupon combining inclusions (3.7)–(3.8) we obtain

$$\widehat{V}_{\rho_2}\backslash\overline{\widehat{V}}_{\rho_1} \subseteq \Omega_{\frac{\rho_2}{C_0}}\backslash\overline{\Omega}_{\frac{\rho_1}{C_1}},$$

which provides the desired localization.

Next we state two selected corollaries of Theorem 3.1. Since the proofs are mostly obvious modifications of the proof of Theorem 3.1, we omit parts of them. In particular, the first of these, Corollary 3.3, demonstrates that we may replace the pointwise condition in (A2) with a limit condition; the upshot of this is that we no longer have to require that $\rho_2 > \frac{C_1}{C_0}\rho_1$. On the other hand, the second of these, Corollary 3.4, demonstrates that if we assume condition (A3) in addition to (A2) and a limit condition on the behavior of $\frac{H(z)}{z}$ as $z \to +\infty$, then we can obtain an existence result that is applicable no matter the value of $\lambda > 0$.

Corollary 3.3 *Suppose that conditions* (*A1*) *and* (*A4*)–(*A5*) *hold. Fix* $\lambda > 0$ *and assume that there exists a number* $\rho_2 > 0$ *such that*

$$\frac{H(\rho_2)}{\rho_2}\varphi(\beta) + \frac{\lambda}{\rho_2}\widetilde{f}^{M}_{\left[0,\frac{\rho_2}{C_0}\right]}\int_0^1\int_0^1 G(t,s)\,d\alpha(t)\,ds < 1.$$

If, in addition, it holds that

$$\liminf_{z\to 0^+}\frac{H(z)}{z} > \frac{1}{\varphi(\beta)},$$

then problem (1.1) *has at least one positive solution.*

Proof Omitted. □

Corollary 3.4 *Suppose that conditions (A1)–(A5) hold. Let* $\lambda \in (0, +\infty)$ *be fixed but arbitrary. If, in addition, it holds that*

$$\limsup_{z \to +\infty} \frac{H(z)}{z} < \frac{1}{\varphi(\beta)},$$

then problem (1.1) *has at least one positive solution.*

Proof The first part of the proof is identical to the first part of the proof of Theorem 3.1. For the second part, by the assumption in the statement of the corollary we may fix a number $\varepsilon > 0$ sufficiently small such that for each $z \geq z_0 := z_0(\varepsilon)$ it holds that

$$\frac{H\,(z)}{z} < \frac{1}{\varphi(\beta)} - \varepsilon.$$

Note that without loss of generality we may assume that ε satisfies the inequality

$$\varepsilon < \frac{1}{\varphi(\beta)},$$

where here we use the assumption from (A5) that $\varphi(\beta) > 0$. It then follows that there exists a number $\rho_2 > 0$ sufficiently large, which may be assumed without loss to satisfy $\rho_2 > \max\left\{z_0, \frac{C_1}{C_0}\rho_1\right\}$, with ρ_1 as before, such that each of

$$\frac{\widetilde{f}^M_{[0,\rho_2]}}{\rho_2} < \varepsilon\varphi(\beta)\left(\lambda \int_0^1 \int_0^1 G(t,s)\,d\alpha(t)\,ds\right)^{-1} \tag{3.9}$$

and

$$\frac{H\,(\rho_2)}{\rho_2} < \frac{1}{\varphi(\beta)} - \varepsilon \tag{3.10}$$

holds; note that (3.9) is well defined due to condition (A4). Here to obtain inequality (3.9) we are using [16, Lemma 3.2]—see also [44, Lemma 2.8]; this, in particular, allows us to assert that if condition (A3) is in force, then it follows that $\lim_{\rho \to +\infty} \frac{\widetilde{f}^M_{[0,\rho]}}{\rho} = 0$. Then putting estimates (3.9)–(3.10) into inequality (3.4), for any $y \in \partial\widehat{V}_{\rho_2}$ we obtain

$$\begin{aligned}
\rho_2 &< \varphi(\beta)\left(\frac{1}{\varphi(\beta)} - \varepsilon\right)\rho_2 + \lambda\rho_2 \int_0^1 \int_0^1 \frac{\widetilde{f}^M_{\left[0, \frac{\rho_2}{C_0}\right]}}{\rho_2} G(t,s)\,d\alpha(t)\,ds \\
&< \varphi(\beta)\left(\frac{1}{\varphi(\beta)} - \varepsilon\right)\rho_2 + \lambda\rho_2\left(\lambda \int_0^1 \int_0^1 G(t,s)\,d\alpha(t)\,ds\right)^{-1} \\
&\quad \times \varepsilon\varphi(\beta) \int_0^1 \int_0^1 G(t,s)\,d\alpha(t)\,ds,
\end{aligned}$$

from which it follows that

$$1 < 1 - \varepsilon\varphi(\beta) + \varepsilon\varphi(\beta) = 1,$$

which is a contradiction, and so, (3.3) holds. Hence, as in the proof of Theorem 3.1 we conclude that problem (1.1) has at least one positive solution. □

Remark 3.5 Note that Corollary 3.4 applies for *any* value of $\lambda > 0$—c.f., [6, Theorem 3.1] and [16, Theorem 3.3]. In some sense, both Theorem 3.1 and Corollary 3.3 also allow for an unrestricted λ, although, in general, the larger λ is, the smaller the quantity $\widetilde{f}^M_{\left[0,\frac{\rho_2}{C_0}\right]}$ will have to be.

We conclude this section and the paper with an example to illustrate, in particular, the computation and application of the coercivity constant C_0 as well as the application of the existence theorems.

Example 3.6 In this example we consider the nonlocal element

$$\varphi(y) := \frac{1}{2}y\left(\frac{1}{4}\right) - \frac{1}{20}y\left(\frac{1}{10}\right).$$

We complete our calculations in the first place with arbitrary $\alpha \in [1, n-2]$ and $\nu \in (3, +\infty)$ with $n-1 < \nu \leq n$. In this general case we find that

$$\frac{1}{\mathcal{G}(s)}\int_0^1 G(t,s)\,d\alpha(t) = \begin{cases} \dfrac{\left[\frac{1}{2}\left(\frac{1}{4}\right)^{\nu-1} - \frac{1}{20}\left(\frac{1}{10}\right)^{\nu-1}\right](1-s)^{\nu-\alpha-1} - \frac{1}{2}\left(\frac{1}{4}-s\right)^{\nu-1} + \frac{1}{20}\left(\frac{1}{10}-s\right)^{\nu-1}}{(1-s)^{\nu-\alpha-1}-(1-s)^{\nu-1}}, & 0 < s < \frac{1}{10} \\ \dfrac{\frac{1}{2}\left[\left(\frac{1}{4}\right)^{\nu-1}(1-s)^{\nu-\alpha-1} - \left(\frac{1}{4}-s\right)^{\nu-1}\right] - \frac{1}{20}\left(\frac{1}{10}\right)^{\nu-1}(1-s)^{\nu-\alpha-1}}{(1-s)^{\nu-\alpha-1}-(1-s)^{\nu-1}}, & \frac{1}{10} \leq s < \frac{1}{4} \\ \dfrac{\left[\frac{1}{2}\left(\frac{1}{4}\right)^{\nu-1} - \frac{1}{20}\left(\frac{1}{10}\right)^{\nu-1}\right](1-s)^{\nu-\alpha-1}}{(1-s)^{\nu-\alpha-1}-(1-s)^{\nu-1}}, & \frac{1}{4} \leq s < 1 \end{cases}. \tag{3.11}$$

Observe from (3.11) that $S_0 := (0, 1)$ here, and so, S_0 is indeed a set of full measure.

To compute the value of

$$C_0 := \inf_{s \in S_0} \frac{1}{\mathcal{G}(s)}\int_0^1 G(t,s)\,d\alpha(t)$$

some elementary calculus is required. These calculations, in part, rely on the observation that there exists a number $N_0 \in \mathbb{N}$ such that $\nu - \alpha - 1 - N_0 \leq 0$ but $\nu - 1 - N_0 > 0$. (This invokes the fact that $\alpha \geq 1$.) Thus, N_0 applications of L'Hôpital's rule yield

$$\lim_{s\to 1^-} \frac{\left[\frac{1}{2}\left(\frac{1}{4}\right)^{\nu-1} - \frac{1}{20}\left(\frac{1}{10}\right)^{\nu-1}\right](1-s)^{\nu-\alpha-1}}{(1-s)^{\nu-\alpha-1} - (1-s)^{\nu-1}}$$

$$\overset{L'H}{=} \lim_{s\to 1^-} \frac{\left[\frac{1}{2}\left(\frac{1}{4}\right)^{\nu-1} - \frac{1}{20}\left(\frac{1}{10}\right)^{\nu-1}\right](-1)^{N_0}(1-s)^{\nu-\alpha-N_0-1}\prod_{j=1}^{N_0}(\nu-\alpha-j)}{\left[(-1)^{N_0}(1-s)^{\nu-\alpha-1-N_0}\prod_{j=1}^{N_0}(\nu-\alpha-j)\right] - \left[(-1)^{N_0}(1-s)^{\nu-1-N_0}\prod_{j=1}^{N_0}(\nu-j)\right]}$$

$$= \lim_{s\to 1^-} \frac{\frac{1}{2}\left(\frac{1}{4}\right)^{\nu-1} - \frac{1}{20}\left(\frac{1}{10}\right)^{\nu-1}}{1 - \frac{(1-s)^{\nu-1-N_0}}{(1-s)^{\nu-\alpha-N_0-1}} \cdot \frac{\prod_{j=1}^{N_0}(\nu-j)}{\prod_{j=1}^{N_0}(\nu-\alpha-j)}}$$

$$= \frac{1}{2}\left(\frac{1}{4}\right)^{\nu-1} - \frac{1}{20}\left(\frac{1}{10}\right)^{\nu-1}.$$

We also calculate by means of a single application of L'Hôpital's rule that

$$\lim_{s\to 0^+} \frac{\left[\frac{1}{2}\left(\frac{1}{4}\right)^{\nu-1} - \frac{1}{20}\left(\frac{1}{10}\right)^{\nu-1}\right](1-s)^{\nu-\alpha-1} - \frac{1}{2}\left(\frac{1}{4}-s\right)^{\nu-1} + \frac{1}{20}\left(\frac{1}{10}-s\right)^{\nu-1}}{(1-s)^{\nu-\alpha-1} - (1-s)^{\nu-1}}$$

$$\overset{L'H}{=} \frac{1}{\alpha}\left\{\frac{1}{2}\left[\left(\frac{1}{4}\right)^{\nu-2}(\nu-1) - \left(\frac{1}{4}\right)^{\nu-1}(\nu-\alpha-1)\right] + \frac{1}{20}\left[\left(\frac{1}{10}\right)^{\nu-1}(\nu-\alpha-1) - \left(\frac{1}{10}\right)^{\nu-2}(\nu-1)\right]\right\}.$$

Then essentially routine but tedious computations, whose details we omit, demonstrate that for each ν and each admissible α we have that

$$C_0 := C_0(\nu) = \frac{1}{2}\left(\frac{1}{4}\right)^{\nu-1} - \frac{1}{20}\left(\frac{1}{10}\right)^{\nu-1} > 0.$$

The table provided below can then be generated from the map $\nu \mapsto C_0(\nu)$, and it summarizes how the coercivity constant changes as we alter the order, ν, of the fractional differential equation; here we have approximated C_0 to three decimal places of accuracy.

ν	3.01	3.05	3.5	3.8	4	5.5
C_0	0.030	0.029	0.015	0.010	0.008	0.001

Note, in particular, that $C_0(\nu) > 0$, for each $\nu > 3$, and, moreover, that

$$\lim_{\nu\to+\infty}\left(\frac{1}{2}\left(\frac{1}{4}\right)^{\nu-1} - \frac{1}{20}\left(\frac{1}{10}\right)^{\nu-1}\right) = \lim_{\nu\to+\infty}\left(2^{1-2\nu} - 2^{-\nu-1}5^{-\nu}\right) = 0.$$

In addition, note that

$$\varphi(\beta) = \frac{\Gamma(\nu-\alpha)}{\Gamma(\nu)}\left[\frac{1}{2}\left(\frac{1}{4}\right)^{\nu-1} - \frac{1}{20}\left(\frac{1}{10}\right)^{\nu-1}\right] = \frac{\Gamma(\nu-\alpha)}{\Gamma(\nu)}C_0(\nu).$$

In observance of the fact that $\|\beta\| = \dfrac{\Gamma(\nu-\alpha)}{\Gamma(\nu)}$, it follows that

$$\varphi(\beta) \geq C_0\|\beta\| = \frac{\Gamma(\nu-\alpha)}{\Gamma(\nu)}C_0(\nu).$$

Thus, condition (A5) will be satisfied for any admissible choice of α and ν for the functional φ given in this example. Since the quantity $\dfrac{1}{\varphi(\beta)}$ occurs in the application of the existence results, we present in the following table values of $\dfrac{1}{\varphi(\beta)}$ for selected choices of α and ν.

ν	3.01	3.01	3.01	3.8	3.8	4
α	1	1.5	1.95	1	1.5	1
$\frac{1}{\varphi(\beta)}$	5.449	75.064	68.699	273.721	393.319	386.473

Finally, to see how to apply the existence theorems, let us suppose that $\nu = 3.01$ and $\alpha = 1$. Let us also define the map H by

$$H(z) := \begin{cases} 6z, & z < 2 \\ 5(z-2)+12, & z \geq 2 \end{cases}.$$

Note that H is a piecewise linear map. Then we conclude that the boundary value problem

$$\begin{aligned} -\left[D_{0^+}^{3.01}y\right](t) &= \lambda f(t, y(t)), 0 < t < 1 \\ y(0) &= y'(0) = y''(0) = 0 \\ y'(1) &= H(\varphi(y)) \end{aligned} \tag{3.12}$$

has at least one positive solution:

- for any $\lambda \in (0, +\infty)$ if condition (A3) holds; or
- for each given $\lambda \in (0, +\infty)$ such that there exists a number $\rho_2 > 0$ satisfying

$$\frac{H(\rho_2)}{\rho_2}\varphi(\beta) + \frac{\lambda}{\rho_2}\widetilde{f}^M_{\left[0,\rho_2\left(\frac{1}{2}\left(\frac{1}{4}\right)^{2.01}-\frac{1}{20}\left(\frac{1}{10}\right)^{2.01}\right)^{-1}\right]}\int_0^1\int_0^1 G(t,s)\,d\alpha(t)\,ds < 1,$$

where these conclusions hold, respectively, by Corollaries 3.4 and 3.3 since we note that

$$5 = \lim_{z\to+\infty}\frac{H(z)}{z} < \frac{1}{\varphi(\beta)} < \lim_{z\to 0^+}\frac{H(z)}{z} = 6.$$

By altering the form of H and utilizing the above tables, we can apply the existence results for other choices of α and ν. Finally, we note that the boundary condition at $t = 1$ can be written in the form

$$y'(1) = \begin{cases} 3y\left(\frac{1}{4}\right) - \frac{3}{10}y\left(\frac{1}{10}\right), & 0 \le \frac{1}{2}y\left(\frac{1}{4}\right) - \frac{1}{20}y\left(\frac{1}{10}\right) < 2 \\ \frac{5}{2}y\left(\frac{1}{4}\right) - \frac{1}{4}y\left(\frac{1}{10}\right) + 2, & 2 \le \frac{1}{2}y\left(\frac{1}{4}\right) - \frac{1}{20}y\left(\frac{1}{10}\right) \end{cases}.$$

Remark 3.7 The localization from Remark 3.2 assures us that the solution, say y_0, of (3.12) must satisfy

$$0 < \frac{20}{11}\rho_1 < \|y_0\| < \rho_2\left[\frac{1}{2}\left(\frac{1}{4}\right)^{2.01} - \frac{1}{20}\left(\frac{1}{10}\right)^{2.01}\right]^{-1} \approx 32.969\rho_2,$$

where the numbers ρ_1 and ρ_2 would depend upon which existence result was used as well as the choice of f and H.

Remark 3.8 Note that in Example 3.6 we see that the strength of the coercivity condition is maximized as $\nu \to 3^+$ and weakened as ν increases away from 3. This can affect the applicability of the existence results since, in general, a larger value of C_0 will impose less of a restriction on f in Theorem 3.1 and Corollary 3.3, for example.

Acknowledgements The author would like to thank the anonymous referee for his or her helpful suggestions and, in particular, pointing out references [18,40,45] and their relation to this paper.

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