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Strictly positive definite kernels on a product of circles

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Abstract We supply a Fourier characterization for the real, continuous, isotropic and strictly positive definite kernels on a product of circles. In other words, if S^1 is the unit circle in $\mathbb{R}^2$, $\cdot$ is the usual inner product of $\mathbb{R}^2$ and f is a real continuous function on $[-1, 1]^2$, we determine necessary and sufficient conditions in order that $f(x \cdot y, z \cdot w)$ be a strictly positive definite kernel on $S^1 \times S^1$.

Keywords Positive definite · Strictly positive definite · Isotropy · Product of circles · Schoenberg's theorem · Skolem-Mahler-Lech theorem

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1 Introduction

Positive definite functions and kernels have a long history in mathematics, entering as an important tool in harmonic analysis and other areas as well. In the spherical setting, they can be traced back to the remarkable paper of Schoenberg published in 1942 [19], where a characterization for the continuous, isotropic and positive definite kernels on a single sphere was obtained. This characterization is far-reaching, having applications in approximation theory, spatial statistics, geomathematics, discrete geometry, etc. We

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mention [3,4,6,16] and references therein for some applications of positive definite functions and kernels on spheres.

In this paper, we will be concerned with positive definite kernels on a product of circles. As so, we will recast the basic concepts and results from Schoenberg's work that applies to circles, up to the point we can state what the main contribution in the present paper is.

We will write S^1 to denote the unit circle in $\mathbb{R}^2$. Continuity of a kernel K on S^1 will be attached to the usual geodesic distance on S^1 and that will be extended to the product $S^1 \times S^1$ in the usual way. The *isotropy or radiality* of a kernel K on S^1 refers to the existence of a function K_r on $[-1, 1]$ so that

$$K(x, y) = K_r(x \cdot y), \quad x, y \in S^1,$$

in which $\cdot$ is the usual inner product of $\mathbb{R}^2$. For a kernel K on $S^1 \times S^1$, isotropy corresponds to the property

$$K((x, z), (y, w)) = K_r(x \cdot y, z \cdot w), \quad x, y, z, w \in S^1,$$

in which the function K_r has now domain $[-1, 1]^2$. In both cases, we will call K_r the *isotropic part* of the kernel K. Finally, the *positive definiteness* of a real kernel K on an infinite set X refers to the validity of the inequality

$$\sum_{\mu,\nu=1}^{n} c_\mu c_\nu K(x_\mu, x_\nu) \geq 0,$$

whenever n is a positive integer, $x_1, x_2, \ldots, x_n$ are distinct points on X and the c_μ are real scalars. The *strict positive definiteness* of K demands both, its positive definiteness and that the inequalities above be strict whenever at least one of the c_μ is nonzero. We will apply these definitions to the cases in which either $X = S^1$ or $X = S^1 \times S^1$.

According to a result of Schoenberg in [19], a real, continuous and isotropic kernel K on S^1 is positive definite if, and only if, the isotropic part K_r of K has the form

$$K_r(t) = \sum_{k=0}^{\infty} a_k P_k(t), \quad t \in [-1, 1],$$

in which all the a_k are nonnegative, P_k is the Tchebyshev polynomial (of first kind) of degree k (see [23]), and $\sum_{k=0}^{\infty} a_k P_k(1) < \infty$. In the nineties, due to the appearance of the so-called radial basis interpolation on spheres, many attempts were made in order to deduce a similar characterization for the strict case [11–15,18,20,22] but that only appeared in [15] (see also [2]): a kernel having Schoenberg's representation described above is strictly positive definite on S^1 if, and only if, the set $\{k : a_{|k|} > 0\}$ intersects every full arithmetic progression in $\mathbb{Z}$. Both results described above extends to the complex setting, that is, to the case in which S^1 is replaced with the unit circle in $\mathbb{C}$, the positive definite kernel is allowed to assume complex values and the scalars c_μ are now complex numbers. This extension is also discussed in [15]. It is worth to mention [21] where the very same problem was discussed.

Moving to $S^1 \times S^1$, a theorem proved in [8] includes a characterization for the positive definiteness of a real, continuous and isotropic kernel K on $S^1 \times S^1$ as those having an isotropic part in the form

$$K_r(t, s) = \sum_{k,l=0}^{\infty} a_{k,l} P_k(t) P_l(s), \quad t, s \in [-1, 1], \tag{1.1}$$

where all the coefficients $a_{k,l}$ are nonnegative and $\sum_{k,l=0}^{\infty} a_{k,l} P_k(1) P_l(1) < \infty$. This characterization can also be deduced from abstract versions of a classical result of S. Bochner on positive definiteness, a typical example being Theorem 4.11 in [1]. For a function K_r representable as in (1.1), we will write

$$J_K := \left\{(k, l) \in \mathbb{Z}_+^2 : a_{k,l} > 0\right\}.$$

The results in this paper will converge to the characterization for strict positive definiteness on $S^1 \times S^1$ described in Theorem 1.1 below. The result is supplementary to one of the main results proved in [7], where a similar problem was considered and solved for the product of higher dimensional spheres.

Theorem 1.1 *Let K be a real, continuous, isotropic and positive definite kernel on $S^1 \times S^1$. The following assertions are equivalent.*

(i) *K is strictly positive definite;*
(ii) *The set $\{(k, l) : (|k|, |l|) \in J_K\}$ intersects all the translations of each subgroup of $\mathbb{Z}^2$ having the form $\{(pa, qb) : q, p \in \mathbb{Z}\}$, $a, b > 0$;*
(iii) *The set $\{(k, l) : (|k|, |l|) \in J_K\}$ intersects all the translations of each subgroup of $\mathbb{Z}^2$ having the form $(a, b)\mathbb{Z} + (0, d)\mathbb{Z}$, $a, d > 0$.*

If this is the case, the set $\{(k, l) : (|k|, |l|) \in J_K\}$ intersects all the translations of each lattice of $\mathbb{Z}^2$ infinitely many times.

At this time, we have found no practical problems where the characterization described in Theorem 1.1 could enter in a decisive manner. Strict positive definiteness on a product of manifolds is a quite new subject and we are sure that potential applications will appear. In particular, we expect the result to have applicability in approximation theory, probability theory, stochastic processes and code theory.

The paper proceeds as follows. In Sect. 2, we present a list of technical results that will be needed in the presentation of the proof to Theorem 1.1. The proof itself appears in Sect. 3.

2 Preliminary results

This section contains several technical results to be used in the proof of the main theorem described in Sect. 1. They are presented following the order they will be required in the proof of Theorem 1.1.

In the first proposition, we explore a little bit deeper the concept of strict positive definiteness on $S^1 \times S^1$. The outcome implies an obvious equivalence for that concept.

It is an easy matter to verify that the strict positive definiteness of the kernel K with isotropic part given by (1.1) depends upon J_K only and not on the actual values of the Fourier coefficients $a_{k,l}$. For distinct points $(x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)$ on $S^1 \times S^1$, we will write $A = (A_{\mu\nu})$, in which

$$A_{\mu\nu} = K_r(x_\mu \cdot x_\nu, w_\mu \cdot w_\nu), \quad \mu, \nu = 1, 2, \ldots, n.$$

We will also represent the points above in polar form:

$$x_\mu = (\cos\theta_\mu, \sin\theta_\mu), \quad w_\mu = (\cos\phi_\mu, \sin\phi_\mu), \quad \theta_\mu, \phi_\mu \in [0, 2\pi), \quad \mu = 1, 2, \ldots, n.$$

Proposition 2.1 *Let K be a nonzero, real, continuous, isotropic and positive definite kernel on $S^1 \times S^1$. For distinct points $(x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)$ on $S^1 \times S^1$ and a column vector $c = (c_\mu)$ in $\mathbb{R}^n$, the following statements are equivalent:*

(i) $c^t Ac = 0$*;*
(ii) *The double equality*

$$\sum_{\mu=1}^{n} c_\mu e^{ik\theta_\mu} e^{il\phi_\mu} = \sum_{\mu=1}^{n} c_\mu e^{ik\theta_\mu} e^{-il\phi_\mu} = 0$$

holds for all $(k, l) \in J_K$.

Proof The normalization one decides to adopt for the Tchebyshev polynomials is of no importance in this paper. So, we will write

$$P_k(x_\mu \cdot x_\nu) = \frac{2}{k} \cos k(\theta_\mu - \theta_\nu), \quad k > 0, \quad x_\mu \in S^1, \quad \mu = 1, 2, \ldots, n,$$

while $P_0(x_\mu \cdot x_\nu) = 1$, $\mu = 1, 2, \ldots, n$. The equality $c^t Ac = 0$ is equivalent to

$$\sum_{\mu,\nu=1}^{n} c_\mu c_\nu P_k(x_\mu \cdot x_\nu) P_l(w_\mu \cdot w_\nu) = 0, \quad (k, l) \in J_K.$$

Introducing the polar representation for the points and arranging, the equality appearing in the formula above becomes

$$\left|\sum_{\mu=1}^{n} c_\mu \cos k\theta_\mu \cos l\phi_\mu\right|^2 + \left|\sum_{\mu=1}^{n} c_\mu \cos k\theta_\mu \sin l\phi_\mu\right|^2 + \left|\sum_{\mu=1}^{n} c_\mu \sin k\theta_\mu \cos l\phi_\mu\right|^2 + \left|\sum_{\mu=1}^{n} c_\mu \sin k\theta_\mu \sin l\phi_\mu\right|^2 = 0.$$

Thus, $c^t Ac = 0$ is equivalent to

$$\sum_{\mu=1}^{n} c_\mu \cos k\theta_\mu \cos l\phi_\mu = \sum_{\mu=1}^{n} c_\mu \cos k\theta_\mu \sin l\phi_\mu = 0$$

and

$$\sum_{\mu=1}^{n} c_\mu \sin k\theta_\mu \cos l\phi_\mu = \sum_{\mu=1}^{n} c_\mu \sin k\theta_\mu \sin l\phi_\mu = 0$$

for $(k, j) \in J_K$. An obvious manipulation of these equations taking into account the fact that c_μ are real numbers leads to the double equality in (ii). □

From now on, we will deal with subgroups of $\mathbb{Z}^2$ and their translations. We will need a classification for the nontrivial subgroups of $\mathbb{Z}^2$. Let S be such a subgroup and let$\{(1, 0), (0, 1)\}$ be the canonical basis of $\mathbb{Z}^2$. Write p_1 to denote the canonical projection of $\mathbb{Z}^2$ onto its first component. If $p_1(S) = 0$, then S is a subgroup of $(0, 1)\mathbb{Z}$. Otherwise, $p_1(S)$ is a nontrivial subgroup of $(1, 0)\mathbb{Z}$, say, $(a, 0)\mathbb{Z}$, with $a > 0$, and we can pick $y \in \mathbb{Z}^2$ so that $p_1(y) = a$. Now, if $x \in S$, then $p_1(x) \in a\mathbb{Z}$, that is, $x - \alpha y \in (0, 1)\mathbb{Z}$, for some $\alpha \in \mathbb{Z}$. In other words, $S = y\mathbb{Z} \oplus (S \cap (0, 1)\mathbb{Z})$. If $S \cap (0, 1)\mathbb{Z} = 0$, then $S = y\mathbb{Z}$. Otherwise, $S = y\mathbb{Z} \oplus b\mathbb{Z}$, in which $b\mathbb{Z}$ is a subgroup of $S \cap (0, 1)\mathbb{Z}$. The outcome of this brief discussion is this one.

Lemma 2.2 *A nontrivial subgroup of $\mathbb{Z}^2$ belongs to one of the following categories:*

(i) $(0, b)\mathbb{Z} := \{(0, pb) : p \in \mathbb{Z}\}$, $b > 0$;
(ii) $(a, b)\mathbb{Z} := \{(pa, pb) : p \in \mathbb{Z}\}$, $a > 0$;
(iii) $(a, b)\mathbb{Z} + (0, d)\mathbb{Z} := \{(pa, pb + qd) : p, q \in \mathbb{Z}\}$, $a, d > 0$.

We advise the reader that there are different ways to describe the subgroups of $\mathbb{Z}^2$ (for instance, the one presented in [17] is slightly different and quite elegant). The subgroups that fit into Lemma 2.2-(iii) will be called *lattices*. The set of lattices of $\mathbb{Z}^2$ encompasses all the subgroups of rank 2. If $ad = 1$, then a lattice becomes the whole $\mathbb{Z}^2$, otherwise it is a proper subgroup of $\mathbb{Z}^2$. The lattices having the form

$$(a\mathbb{Z}, b\mathbb{Z}) := \{(pa, qb) : q, p \in \mathbb{Z}\}, \quad a, b > 0,$$

will be called *rectangular lattices* of $\mathbb{Z}^2$. By *translates of subgroups* of $\mathbb{Z}^2$, we will mean sets of the form $(j, j') + S$, in which (j, j') is a fixed element of $\mathbb{Z}^2$ and S is a subgroup of $\mathbb{Z}^2$.

Lemma 2.3 below provides a decomposition of a lattice through translations of rectangular lattices.

Lemma 2.3 *The lattice $L = (a, b)\mathbb{Z} + (0, d)\mathbb{Z}$, $a, d > 0$, can be decomposed in the form*

$$L = \bigcup_{(j,j')\in A} \left[(j, j') + (ad\mathbb{Z}, ad\mathbb{Z})\right],$$

in which $A = L \cap \{(\alpha, \beta) \in \mathbb{Z}^2 : 0 \leq \alpha, \beta < ad\}$.

Proof For $p, q \in \mathbb{Z}$, we can certainly write

$$(pa, pb + qd) = (j + \alpha ad, j' + \beta ad),$$

in which $j, j' \in \{0, 1, \ldots, ad - 1\}$. Since

$$(j, j') = (p - \alpha d)(a, b) + (\alpha b + q - \beta a)(0, d),$$

it is clear that $(j, j') \in L$. These arguments show that L is a subset of the union quoted in the statement of the lemma. As for the reverse inclusion, first observe that if $\alpha, \beta \in \mathbb{Z}$, we have that

$$(\alpha ad, \beta ad) = d\alpha(a, b) + (a\beta - b\alpha)(0, d) \in L.$$

Since L is a subgroup of $\mathbb{Z}^2$, $(j, j') + (\alpha ad, \beta ad) \in L$ whenever $(j, j') \in A$. □

Next, we recall an elementary bi-dimensional version of the Skolem-Mahler-Lech Theorem due to Laurent [9,10]. The original Skolem-Mahler-Lech Theorem is discussed in details in [5]. This very same bi-dimensional version was used in [17] in order to characterize certain strictly positive definite kernels on complex Hilbert spaces.

Theorem 2.4 *Let* $\{(x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)\}$ *be a subset of* $(\mathbb{C}\backslash\{0\})^2$. *For n complex numbers* $c_1, c_2, \ldots, c_n$, *define a double sequence* $\{b_{k,l} : k, l \in \mathbb{Z}\}$ *through the formula*

$$b_{k,l} := \sum_{\mu=1}^{n} c_\mu x_\mu^k w_\mu^l, \quad k, l \in \mathbb{Z}.$$

Then, the set $\{(k, l) : b_{k,l} = 0\}$ *is the union of a finite number of translates of subgroups of* $\mathbb{Z}^2$.

The technical lemma below adds to Theorem 2.4 when the points are distinct and belong to $\Omega_2 \times \Omega_2$, in which Ω_2 is the unit circle in $\mathbb{C}$.

Lemma 2.5 *Let* $(x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)$ *be distinct points in* $\Omega_2 \times \Omega_2$. *For complex numbers* $c_1, c_2, \ldots, c_n$, *define*

$$b_{k,l} = \sum_{\mu=1}^{n} c_\mu x_\mu^k w_\mu^l, \quad k, l \in \mathbb{Z}.$$

If $\{(k, l) : b_{k,l} = 0\} = \mathbb{Z}^2$, *then all the* c_μ *are zero.*

Proof We will write the components of the points in polar form $x_\mu = e^{i\theta_\mu}$, $w_\mu = e^{i\phi_\mu}$, $\mu = 1, 2, \ldots, n$, and will assume, as we can, that the n points (θ_1, ϕ_1), (θ_2, ϕ_2), $\ldots$, (θ_n, ϕ_n) are distinct in $[0, 2\pi)^2$. Choose $\alpha, \beta \in \mathbb{Z}$ in such a way that all the elements in the set

$$\left\{\alpha\frac{\theta_\mu - \theta_\nu}{2\pi} + \beta\frac{\phi_\mu - \phi_\nu}{2\pi} : \mu, \nu = 1, 2, \ldots, n; \mu \neq \nu\right\}$$

are nonzero. Next, pick $\gamma \in \mathbb{Z}_+$ arbitrarily large so that

$$\left\{\frac{\alpha}{\gamma}\frac{\theta_\mu - \theta_\nu}{2\pi} + \frac{\beta}{\gamma}\frac{\phi_\mu - \phi_\nu}{2\pi} : \mu, \nu = 1, 2, \ldots, n; \mu \neq \nu\right\} \subset (-1, 1)\backslash\{0\}.$$

For each pair (μ, ν), $\mu \neq \nu$, for which

$$\frac{\alpha}{\gamma}\frac{\theta_\mu - \theta_\nu}{2\pi} + \frac{\beta}{\gamma}\frac{\phi_\mu - \phi_\nu}{2\pi} \in \mathbb{Q},$$

let $p_{\mu\nu}$ be a positive integer $> \gamma$ satisfying

$$p_{\mu\nu}\left(\frac{\alpha}{\gamma}\frac{\theta_\mu - \theta_\nu}{2\pi} + \frac{\beta}{\gamma}\frac{\phi_\mu - \phi_\nu}{2\pi}\right) \in \mathbb{Z}.$$

Finally, select an integer q so that q is greater then all the $p_{\mu\nu}$ and each set $\{q, p_{\mu\nu}\}$ is coprime. If $\{(k, l) : b_{k,l} = 0\} = \mathbb{Z}^2$, then we may infer that

$$\sum_{\mu=1}^{n} c_\mu e^{i\theta_\mu k} e^{i\phi_\mu l} = 0,\ (k, l) = (0, 0), (\alpha q, \beta q), \ldots, ((n-1)\alpha q, (n-1)\beta q).$$

The matrix of the system above has $\mu\nu$-entries given by

$$\left[e^{i(\alpha\theta_\mu + \beta\phi_\mu)q}\right]^\nu, \quad \mu, \nu = 1, 2, \ldots, n,$$

and, consequently, it is a Vandermonde matrix. So, the proof of the lemma will be complete as long as we show that the n points $e^{i(\alpha\theta_\mu+\beta\phi_\mu)q}$, $\mu = 1, 2, \ldots, n$, are distinct. But, for $\mu \neq \nu$,

$$e^{i(\alpha\theta_\mu+\beta\phi_\mu)q} = e^{i(\alpha\theta_\nu+\beta\phi_\nu)q}$$

if, and only if,

$$q\left(\alpha\frac{\theta_\mu - \theta_\nu}{2\pi} + \beta\frac{\phi_\mu - \phi_\nu}{2\pi}\right) \in \mathbb{Z}.$$

If all the numbers

$$\left(\alpha\frac{\theta_\mu-\theta_\nu}{2\pi}+\beta\frac{\phi_\mu-\phi_\nu}{2\pi}\right),\quad \mu\neq\nu,$$

are irrational, we are done. Otherwise, there would be integers j and j' such that

$$\frac{j}{\gamma q}=\frac{j'}{p_{\mu\nu}}\in(-1,1)\setminus\{0\},$$

for some pair (μ,ν), $\mu\neq\nu$. Since $\{q,p_{\mu\nu}\}$ is coprime, then $p_{\mu\nu}$ would divide γ, contradicting our choice of $p_{\mu\nu}$. □

The next result reveals that if a proper subset A of $\mathbb{Z}^2$ is a finite union of translates of subgroups of $\mathbb{Z}^2$, then there exists a rectangular lattice H of $\mathbb{Z}^2$ and $(j,j')\in\mathbb{Z}^2$ so that $[(j,j')+H]\cap A=\emptyset$.

Lemma 2.6 *Let A be a proper subset of $\mathbb{Z}^2$. If A is a finite union of translates of subgroups of $\mathbb{Z}^2$ and $(j,j')\in\mathbb{Z}^2\setminus A$, then there exists a rectangular lattice H of $\mathbb{Z}^2$ such that $(j,j')+H\subset\mathbb{Z}^2\setminus A$.*

Proof If A is a finite union of translates of subgroups of $\mathbb{Z}^2$, we can write

$$A=F\cup[(j_1,j_1')+G_1]\cup[(j_2,j_2')+G_2]\cup\ldots\cup[(j_r,j_r')+G_r]$$

in which F is a finite (possibly empty) subset of $\mathbb{Z}^2$, $(j_1,j_1'),(j_2,j_2'),\ldots,(j_r,j_r')\in\mathbb{Z}^2$ and $G_1,G_2,\ldots,G_r$ are nontrivial subgroups of $\mathbb{Z}^2$. It suffices to prove the lemma in the case in which $F=\emptyset$. Indeed, if a solution $(j,j')+H$ is available for that case, we can pick a convenient subgroup H_1 of H so that $(j,j')+H_1$ avoids all the elements of F. So, assume that $F=\emptyset$ and fix $(j,j')\in\mathbb{Z}^2\setminus A$. We can assume all the G_i have rank 2. Indeed, if G_i has rank 1 for some i, we can pick $(\alpha,\beta)\in\mathbb{Z}^2$ such that

$$(\alpha,\beta)\mathbb{Z}\cap[(j_i-j,j_i'-j')+G_i]=\emptyset.$$

Hence,

$$[(j,j')+(\alpha,\beta)\mathbb{Z}]\cap[(j_i,j_i')+G_i]=\emptyset,$$

and, therefore,

$$(j,j')\notin(j_i,j_i')+(\alpha,\beta)\mathbb{Z}+G_i.$$

In particular, $(\alpha,\beta)\mathbb{Z}+G_i$ is a subgroup of rank 2 and we can replace $(j_i,j_i)+G_i$ with $(j_i,j_i')+(\alpha,\beta)\mathbb{Z}+G_i$ in the union decomposition for A keeping (j,j') in $\mathbb{Z}^2\setminus A$. If all the G_i have rank 2, the proof of the lemma proceeds as follows. Let m_i be the index of G_i in $\mathbb{Z}^2$, $i=1,2,\ldots,r$, and pick a common multiple m of all the m_i.

The subgroup $(m\mathbb{Z}, m\mathbb{Z})$ is a rectangular lattice and, by the definition of index of a subgroup, it follows that

$$(m\mathbb{Z}, m\mathbb{Z}) \subset G_i \quad i = 1, 2, \ldots, r.$$

In particular,

$$[(j, j') + (m\mathbb{Z}, m\mathbb{Z})] \cap G_i = \emptyset, \quad i = 1, 2, \ldots, r,$$

and, consequently, $(j, j') + (m\mathbb{Z}, m\mathbb{Z}) \subset \mathbb{Z}^2 \backslash A$. □

We conclude the section with a technical result on sets that intersect all the translations of each lattice in $\mathbb{Z}^2$.

Lemma 2.7 *If A is a subset of $\mathbb{Z}^2$ that intersects all the translations of each lattice in $\mathbb{Z}^2$, then each intersection is an infinite set.*

Proof Let A be a subset of $\mathbb{Z}^2$ that intersects all the translations of each lattice in $\mathbb{Z}^2$. Let $L = (j, j') + (a, b)\mathbb{Z} + (0, d)\mathbb{Z}$, $a, d > 0$, and assume that $A \cap L$ is finite. Write

$$(j+p_1a,\, j'+p_1b+q_1d), (j+p_2a,\, j'+p_2b+q_2d), \ldots, (j+p_na,\, j'+p_nb+q_nd),$$

to denote the elements in the intersection and define

$$p := \max\{|p_1|, |p_2|, \ldots, |p_n|\} \quad \text{and} \quad q := \max\{|q_1|, |q_2|, \ldots, |q_n|\}.$$

We will reach a contradiction, analyzing four different cases.

Case 1. $p = q = 0$: The intersection contains just one element, (j, j'). We now look at the translation

$$L' := (j + 2a, j' + 2b) + (3a, 3b)\mathbb{Z} + (0, d)\mathbb{Z} \subset L$$

of the sublattice $(3a, 3b)\mathbb{Z} + (0, d)\mathbb{Z}$ of $(a, b)\mathbb{Z} + (0, d)\mathbb{Z}$. If $(j, j') \in L'$, then

$$\begin{cases} j + 2a + 3ar = j \\ j' + 2b + 3br + ds = j' \end{cases}$$

for some $r, s \in \mathbb{Z}$. But, since $a(3r + 2) \neq 0$, $r \in \mathbb{Z}$, this is impossible. In particular, $A \cap L' = \emptyset$, a contradiction to our basic assumption.

Case 2. $p = 0$ *and* $q > 0$: Here we consider the sublattice $(a, b)\mathbb{Z} + (0, 2(2q + 1)d)\mathbb{Z}$ of $(a, b)\mathbb{Z} + (0, d)\mathbb{Z}$ and look at its translation

$$L'' := (j, j' + 2qd) + (a, b)\mathbb{Z} + (0, 2(2q + 1)d)\mathbb{Z} \subset L.$$

If $(j+ra, j'+2qd+rb+2s(2q+1)d) = (j, j'+q_\mu d)$ for some $\mu \in \{1, 2, \ldots, n\}$ and $r, s \in \mathbb{Z}$, then

$$\begin{cases} ra = 0 \\ 2qd + rb + 2s(2q+1)d = q_\mu d \end{cases}$$

and, consequently, $2q + 2s(2q+1) = q_\mu$. However, due to the definition of q, no integer s can satisfy the previous equality. Thus, $L'' \cap A = \emptyset$, another contradiction.

Case 3. $p > 0$ *and* $q = 0$: Its is similar to the previous case.

Case 4. $p, q > 0$: Here we consider the sublattice $(2(2p+1)a, 2(2p+1)b)\mathbb{Z} + (0, qd)\mathbb{Z}$ of $(a, b)\mathbb{Z} + (0, d)\mathbb{Z}$ and its translation

$$L''' := (j + 2pa, j' + 2pb) + (2(2p+1)a, 2(2p+1)b)\mathbb{Z} + (0, qd)\mathbb{Z} \subset L.$$

If

$$\begin{aligned} &(j + 2pa + 2r(2p+1)a, j' + 2pb + 2r(2p+1)b + sqd) \\ &\quad = (j + p_\mu a, j' + p_\mu b + q_\mu d) \end{aligned}$$

for some $\mu \in \{1, 2, \ldots, n\}$ and $r, s \in \mathbb{Z}$, we will have that $2p + 2r(2p+1) = p_\mu$. As in Case 2, we can deduce that $L''' \cap A = \emptyset$, a contradiction to our initial assumption on A. □

3 The proof of Theorem 1.1

This section contains a proof for the main theorem announced in the introduction.

Proof $(i) \Rightarrow (ii)$ Assume K is strictly positive definite and write $S = (a\mathbb{Z}, b\mathbb{Z})$ with $a, b > 0$. We will show that

$$\{(k, l) : (|k|, |l|) \in J_K\}$$

intersects $(j, j') + S$, whenever $j \in \{0, 1, \ldots, a-1\}$ and $j' \in \{0, 1, \ldots, b-1\}$. There is nothing to prove if $a = b = 1$. In the other cases, we will assume that

$$\{(k, l) : (|k|, |l|) \in J_K\} \cap (j + a\mathbb{Z}, j' + b\mathbb{Z}) = \emptyset,$$

and will reach a contradiction. In the case in which $a = 1$ and $b \geq 2$, the assumption on $\{(k, l) : (|k|, |l|) \in J_K\}$ implies that $l - j', -l - j' \notin b\mathbb{Z}$, whenever $(k, l) \in J_K$. In particular,

$$\sum_{\mu=1}^{b} \left(e^{i2\pi\mu/b}\right)^{l-j'} = \sum_{\mu=1}^{b} \left(e^{i2\pi\mu/b}\right)^{-l-j'} = 0,$$

and, consequently,

$$\sum_{\mu=1}^{b}\left[\text{ Re }\left(e^{i2\pi\mu j'/b}\right)\right]\left(e^{i2\pi\mu/b}\right)^{l}=0,\quad (k,l)\in J_K.$$

The real scalars $c_\mu :=$ Re $(e^{i2\pi\mu j'/b})$, $\mu = 1, 2, \ldots, b$, are not all zero and the points

$$(x_\mu, w_\mu) = (1, e^{i2\pi\mu/b}),\quad \mu = 1, 2, \ldots, b,$$

are distinct in $S^1 \times S^1$. Thus, under the light of Proposition 2.1, we have a contradiction with the strict positive definiteness of K. The case in which $a \geq 2$ and $b = 1$ is similar. To conclude the proof, we now assume $a, b \geq 2$ and adapt the procedure employed in the first case. If $(k,l) \notin (j + a\mathbb{Z}, j' + b\mathbb{Z})$, then either $k - j \notin a\mathbb{Z}$ or $l - j' \notin b\mathbb{Z}$. Hence, we may conclude that

$$\sum_{\mu=1}^{a}\left(e^{i2\pi\mu/a}\right)^{k-j}\sum_{\nu=1}^{b}\left(e^{i2\pi\nu/b}\right)^{l-j'}=0,$$

that is,

$$\sum_{\mu=1}^{a}\sum_{\nu=1}^{b}e^{-i2\pi\mu j/a}e^{-i2\pi\nu j'/b}\left(e^{i2\pi\mu/a}\right)^{k}\left(e^{i2\pi\nu/b}\right)^{l}=0.$$

Repeating the argument with the assumption $(-k,-l) \notin (j + a\mathbb{Z}, j' + b\mathbb{Z})$, we conclude that

$$\sum_{\mu=1}^{a}\sum_{\nu=1}^{b}e^{i2\pi\mu j/a}e^{i2\pi\nu j'/b}\left(e^{i2\pi\mu/a}\right)^{k}\left(e^{i2\pi\nu/b}\right)^{l}=0.$$

Thus, since (k,l) is arbitrary,

$$\sum_{\mu=1}^{a}\sum_{\nu=1}^{b}\left[\text{ Re }\left(e^{i2\pi\mu j/a}e^{i2\pi\nu j'/b}\right)\right]\left(e^{i2\pi\mu/a}\right)^{k}\left(e^{i2\pi\nu/b}\right)^{l}=0,\quad (k,l)\in J_K.$$

By an analogous procedure, now taking into account that

$$(-k,l), (k,-l) \notin (j + a\mathbb{Z}, j' + b\mathbb{Z}),$$

the conclusion is

$$\sum_{\mu=1}^{a}\sum_{\nu=1}^{b}\left[\text{ Re }\left(e^{i2\pi\mu j/a}e^{i2\pi\nu j'/b}\right)\right]\left(e^{i2\pi\mu/a}\right)^{k}\left(e^{-i2\pi\nu/b}\right)^{l}=0,\quad (k,l)\in J_K.$$

Therefore, since the numbers $\operatorname{Re}\left[e^{i2\pi\mu j/a}e^{i2\pi\nu j'/b}\right]$ are not all zero and the ab points

$$(x_\mu, w_\nu) = (e^{i2\pi\mu/a}, e^{i2\pi\nu/b}), \quad \mu = 1, 2, \ldots, a, \quad \nu = 1, 2, \ldots, b,$$

are distinct in $S^1 \times S^1$, we have reached a contradiction once again.
$(ii) \Leftrightarrow (iii)$ One implication is a consequence of Lemma 2.3. The other one is obvious.
$(ii) \Rightarrow (i)$ Let us assume that $\{(k,l) : (|k|,|l|) \in J_K\}$ intersects all the translations of each rectangular lattice of $\mathbb{Z}^2$. For a fixed $n \geq 2$, n distinct points $(\theta_1,\phi_1), (\theta_2,\phi_2), \ldots, (\theta_n,\phi_n)$ in $[0,2\pi)^2$ and real numbers $c_1, c_2, \ldots, c_n$, not all zero, we intend to show that either $\sum_{\mu=1}^n c_\mu e^{i\theta_\mu k}e^{-i\phi_\mu l} \neq 0$ or $\sum_{\mu=1}^n c_\mu e^{i\theta_\mu k}e^{i\phi_\mu l} \neq 0$, for some $(k,l) \in J_K$. A help of Proposition 2.1 will lead to the strict positive definiteness of K. In order to achieve the conclusion mentioned above, define

$$b_{k,l} = \sum_{\mu=1}^{n} c_\mu e^{i\theta_\mu k}e^{i\phi_\mu l}, \quad k, l \in \mathbb{Z}.$$

On one hand, Lemma 2.5 and the fact that at least one c_μ is nonzero imply that $\{(k,l) : b_{k,l} = 0\} \neq \mathbb{Z}^2$. Theorem 2.4 asserts that $\{(k,l) : b_{k,l} = 0\}$ is the union of a finite number of translations of subgroups of $\mathbb{Z}^2$ while Lemma 2.6 guarantees the existence of a rectangular lattice of $\mathbb{Z}^2$, a translation of which belongs to $\mathbb{Z}^2 \backslash \{(k,l) : b_{k,l} = 0\}$. Thus, due to our assumption on $\{(k,l) : (|k|,|l|) \in J_K\}$, we immediately have that

$$\{(k,l) : (|k|,|l|) \in J_K\} \not\subset \{(k,l) : b_{k,l} = 0\}.$$

Therefore, there must exist at least one pair (k,l) in $\{(k,l) : (|k|,|l|) \in J_K\}$ for which

$$\sum_{\mu=1}^{n} c_\mu e^{i\theta_\mu k}e^{i\phi_\mu l} \neq 0.$$

Since the c_μ are real, the result follows.

The final statement in the proof of Theorem 1.1 is a consequence of Lemma 2.7. □

The results demonstrated in this paper can be adapted to hold for positive definiteness on the complex circle Ω_2. In that case, we replace S^1 with Ω_2, we allow the kernels to assume complex values and the scalars c_μ in the definition of positive definiteness can be complex numbers (the quadratic form in the definition of positive definiteness is Hermitian). We will sketch what these results are and refer the interested reader to [8,15] where the necessary adaptations for the proofs can be prospected from.

Let $K : \Omega_2 \times \Omega_2 \to \mathbb{C}$ be a continuous kernel and assume that

$$K((x,z),(y,w)) = K_r(x \cdot y, z \cdot w), \quad x, y, z, w \in \Omega_2,$$

for some function $K_r : \Omega_2 \times \Omega_2 \to \mathbb{C}$, in which $\cdot$ is now the usual inner product of $\mathbb{C}$. It is positive definite if, and only if, the function K_r is of the form

$$K_r(z, w) = \sum_{k,l \in \mathbb{Z}} a_{k,l} z^k w^l, \quad z, w \in \Omega_2,$$

in which $a_{k,l} \geq 0$, $k, l \in \mathbb{Z}$ and $\sum_{k,l \in \mathbb{Z}} a_{k,l} < \infty$. Taking the above representation for granted, we can define $I_K := \{(k, l) : a_{k,l} > 0\}$. For distinct points $(x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)$ on $\Omega_2 \times \Omega_2$ and a column vector c in $\mathbb{C}^n$, the quadratic form $\overline{c}^t Ac = 0$ corresponds to

$$\sum_{\mu=1}^{n} c_\mu e^{ik\theta_\mu} e^{il\phi_\mu} = 0, \quad (k, l) \in I_K,$$

in which θ_μ and ϕ_μ are the arguments in the polar representation of x_μ and w_μ respectively. In particular, this reveals that the proofs we have developed in Sects. 2 and 3 simplify in the present complex setting. In particular, a continuous and positive definite kernel K on $\Omega_2 \times \Omega_2$ as described above is strictly positive definite if, and only if, I_K intersects all the translations of each rectangular lattice of $\mathbb{Z}^2$.

It is worth to mention that, after some period of research, we have not found yet a characterization for the real, continuous, isotropic and strictly positive definite kernels on $S^1 \times S^m$. An elegant characterization seems to demand a mix of arguments, some presented here and others developed in [7].

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