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Composition operators on f-algebras

R. Abid[1] · M. A. Ben Amor[2] · K. Boulabiar[3]

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Abstract By a composition operator between two f-algebras we mean a positive algebra homomorphism. This paper intends to give a systematic study of such operators. A particular attention is paid to their connection with separating regular operators as well as to their global behavior in the module of regular operators. The paper ends with some open problems.

Keywords Composition operator · f-algebra · Separating

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✉ K. Boulabiar
karim.boulabiar@ipest.rnu.tn

R. Abid
rim.abid@ipest.rnu.tn

M. A. Ben Amor
mohamedamine.benamor@ipest.rnu.tn

1 Département du Cycle Préparatoire, IPEST, Université de Carthage, B.P. 51, 2070 La Marsa, Tunisia

2 Département du Cycle Agrégatif IPEST, Université de Carthage, B.P. 51, 2070 La Marsa, Tunisia

3 Laboratoire de Recherche LATAO, Département de Mathématiques Faculté des Sciences de Tunis, Université Tunis-El Manar, 2092 El Manar, Tunisia

1 Introduction

Let X, Y be two topological spaces with X *realcompact* (i.e., X is homeomorphic with a closed set in an appropriate product of real lines) and τ be a continuous function from a *clopen* (i.e., closed an open) set Ω into Y. The composition operator $C_{\Omega,\tau}$ defined from $C(X)$ into $C(Y)$ by

$$(Tf)(y) = (f \circ \tau)(y) \text{ if } y \in \Omega \quad \text{and} \quad (Tf)(y) = 0 \text{ if } y \notin \Omega.$$

is a positive algebra homomorphism. Conversely, for any algebra homomorphism C from $C(X)$ into $C(Y)$, there exist a clopen set Ω in X and a continuous function τ from Ω into Y such that T is the composition operator $C_{\Omega,\tau}$ (see Theorem 10.8 in [14]). This remarkable fact lies behind our motivation to introduce the notion of composition operators on the more general setting of f-algebras. A synopsis of the content of this paper seems to be in order.

Let A be an f-algebra with identity e and B be a semiprime f-algebra (both are assumed to be Archimedean). By a *composition operator* from A into B we mean a positive algebra homomorphism from A into B. It should be pointed out by the way that, unlike the $C(X)$-$C(Y)$ case, an algebra homomorphism from A into B need not be positive (an example in this direction can be found in [16, Example 5.2]). It is readily checked that any composition operator C from A into B is *separating*, i.e.,

$$CfCg = 0 \quad \text{for all } f, g \in A \text{ with } fg = 0.$$

The converse does not hold, of course. In spite of that, we shall prove that any separating regular operator T from A into B with Te idempotent is a composition operator. Another characterization of composition operators will be obtained. Indeed, we will show that an operator T from A into B is a composition operator if and only if Te is idempotent and T is a B-discrete element in the B-module $\mathfrak{L}^r(A, B)$ of all regular operators from A into B. This fact can be obtained as a consequence of results by Kutateladze [17] and Zaanen [26] if B is, in addition, Dedekind complete. Our results allow us to extend and reprove in a more direct approach a representation theorem of separating operators in $\mathfrak{L}^r(A, B)$. Indeed, it has been proved in [8] that if $T \in \mathfrak{L}^r(A, B)$ is separating then there exist $w \in B$ and a composition operator C from A into the maximal ring of quotients $Q(B)$ of B such that $T = wC$. We shall prove that we can get the same conclusion if $T \in \mathfrak{L}^r(A, B)$ and Te is *von Neumann regular* (i.e., the equality $(Te)^2 v = Te$ holds for some $v \in B$). In addition to the aforementioned representation theorem, we shall obtain as a corollary of our result that if $T \in \mathfrak{L}^r(A, B)$ is a *biseparating* operator (i.e., T is bijective with T and T^{-1} separating) then A and B are isomorphic as f-algebras. The last application of our main result is global in nature and deals with vector spaces of separating operators in $\mathfrak{L}^r(A, B)$. More precisely, we shall prove that any vector subspace of $\mathfrak{L}^r(A, B)$ the operators in which are separating is contained in a one-dimensional B-submodule of $\mathfrak{L}^r(A, Q(B))$ generated by a composition operator from A into $Q(B)$. This result is based upon the Hausdorff Maximal Principle and a representation theorem by Wickstead [25], who proved that $Q(B)$ can be identified with the f-algebra $\mathrm{Orth}^w(B)$ of all weak orthomorphisms on B. All these results lead to some open problems which are collected in a separate section at the end of the paper.

2 Preliminaries

We take it for granted that the reader is familiar with the notions of vector lattices (also called Riesz spaces) and regular operators such as lattice homomorphisms and orthomorphisms. For terminology, notations and concepts not explained in this paper we refer to the standard monographs [3] by Aliprantis-Burkinshaw and [20] by Luxemburg-Zaanen.

Beginning with the next paragraph, we shall impose as blanket assumptions that all vector lattices under consideration are real and Archimedean. Moreover, all given operators are supposed to be linear.

The following lines discuss the notion of function algebras as introduced by Birkhoff and Pierce in [5]. A vector lattice A which is simultaneously an associative algebra such that the positive cone

$$A^{+} = \{f \in A : 0 \leq f\}$$

is closed under multiplication, i.e.,

$$fg \in A^{+} \quad \text{for all } f, g \in A^{+},$$

is called a *lattice-ordered algebra* (or a *Riesz algebra*). The Riesz algebra A is called a *function algebra* (briefly, an *f-algebra*) if

$$f \wedge g = 0 \text{ and } 0 \leq h \quad \text{imply} \quad (fh) \wedge g = (hf) \wedge g = 0.$$

We call a subalgebra B of A an *f-subalgebra* of A if B is, in addition, a vector sublattice of the underlaying vector lattice of A. Obviously, f-subalgebras of A are in turn f-algebras. On the other hand, since the underlying vector lattice of the f-algebra A is assumed to be Archimedean, A is commutative and have positive squares. Hence, if A has an identity e then $e \in A^{+}$. By the way, any f-algebra with identity is *semiprime*, that is, 0 is the only nilpotent element in A. In any f-algebra A, the condition $|f| \wedge |g| = 0$ implies $fg = 0$ and the converse holds if A is, in addition, semiprime. Moreover,

$$|fg| = |f|\,|g| \quad \text{for all } f, g \in A.$$

It follows that if the f-algebra is semiprime and $f, g \in A$ then

$$fg = 0 \quad \text{if and only if} \quad |f|\,|g| = 0 \quad \text{if and only if} \quad |f| \wedge |g| = 0.$$

All these elementary properties can be found in [27, Chapter 20] and will be used throughout the paper without further mention. In spite of that, it is worth noting that the class of semiprime f-algebras contains properly the algebra $C(X)$ of all real-valued continuous functions on a topological space X as well as the algebra of all real μ-measurable functions on a set carrying a measure μ. More generally, the set $\mathrm{Orth}(L)$ of all orthomorphisms on a vector lattice L is an Archimedean f-algebra with composition as multiplication.

For brevity's sake, we will assume throughout the paper that A is an Archimedean f-algebra with identity e and B is an Archimedean semiprime f-algebra.

Recall that an operator T from A into B is said to be *regular* if T is the difference of two positive operators in $\mathfrak{L}^r(A, B)$ (equivalenty, if the inequality $T \leq S$ holds for some positive operator S in $\mathfrak{L}^r(A, B)$). The set $\mathfrak{L}^r(A, B)$ of all regular operators from A into B is an ordered vector space with respect to the pointwise addition, scalar multiplication, and ordering. Idempotents elements in f-algebras will play a preponderant role in the context of our problem. Recall that an element p in the f-algebra B is said to be *idempotent* if $p^2 = p$. Obviously, an idempotent element in B is positive. Let $p \in B$ be an idempotent element and put

$$pB = \{pu : u \in B\}.$$

In other words, pB is the principal ring ideal of B generated by p. In particular, pB is a subalgebra of B. Further properties of pB are given next.

Lemma 2.1 *Let p be an idempotent element in B. Then the following assertions hold.*

(i) *pB is a projection band in B.*
(ii) *pB is an f-subalgebra of B with p as identity.*

Proof (i) Define the multiplication operator π_p on B by

$$\pi_p u = pu \quad \text{for all } u \in B.$$

Obviously, π_p is a positive operator. Let $u \in B^+$ and observe that

$$pu(u - pu) = 0.$$

Since B is semiprime, it follows that

$$|u - pu| \wedge |pu| = 0.$$

But then

$$0 \leq pu \leq |u - pu| \vee |pu| = |u - pu + pu| = u.$$

This yields that the inequalities

$$0 \leq \pi_p \leq I_B$$

hold in $\mathfrak{L}^r(A, B)$, where I_B denotes the identity operator of B. Using Theorem 1.44 in [3], we derive that π_p is a projection band and thus the range $\operatorname{Im}\pi_p$ of π_p is a band projection on B. The obvious equality $\operatorname{Im}\pi_p = pB$ gives the conclusion.

(ii) We have observed already that pB is a subalgebra of B. Furthermore, from (i) it follows that pB is a vector sublattice of B. We derive that pB is an f-subalgebra of B. Finally, if $v \in B$ then $v = pu$ for some $u \in B$ and so

$$pv = p(pu) = p^2 u = pu = v.$$

This means that p is an identity in pB and the proof is complete. □

Actually, we may prove quite easily that if p is an idempotent element in B then pB is the principal band of B generated by p.

The next paragraph deal with separating regular operators on f-algebras. An operator $T \in \mathfrak{L}^r(A, B)$ is said to be *separating* (or *disjointness preserving*) if

$$fg = 0 \text{ in } A \quad \text{implies} \quad TfTg = 0 \text{ in } B.$$

Since both A and B are semiprime, we derive that $T \in \mathfrak{L}^r(A, B)$ is separating if and only if

$$|Tf| \wedge |Tg| = 0 \quad \text{for all } f, g \in A \text{ with } |f| \wedge |g| = 0.$$

Thus, a positive operator $T \in \mathfrak{L}^r(A, B)$ is separating if and only if T is a lattice homomorphism. Recall that an operator $T \in \mathfrak{L}^r(A, B)$ is called a *lattice* (or *Riesz*) homomorphism if $Tf \wedge Tg = 0$ in B whenever $f \wedge g = 0$ in A. It is well-known that the $\mathfrak{L}^r(A, B)$ need not be a vector lattice (unless, for instance, B is Dedekind complete). However, it turns out that any separating operator $T \in \mathfrak{L}^r(A, B)$ has a *modulus* (i.e., an absolute value) $|T|$ in $\mathfrak{L}^r(A, B)$. Moreover, if $T \in \mathfrak{L}^r(A, B)$ then $|T|$ is a lattice homomorphism and the equalities

$$|Tf| = ||T|\, f| = |T|\, |f| \tag{1}$$

hold in B for every $f \in A$. Therefore, the positive part T^+ and the negative part T^- of the separating operator $T \in \mathfrak{L}^r(A, B)$ exist in $\mathfrak{L}^r(A, B)$ and we have

$$T^+ f = (Tf)^+ \text{ and } T^- f = (Tf)^- \quad \text{for all } f \in A^+.$$

Notice that T^+ and T^- are again lattice homomorphisms (see Theorem 3.1.4 in [22]). More about disjointness preserving operators on vector lattices can be found in the survey paper [9]. We end this preliminaries section with a lemma which will often come in handy.

Lemma 2.2 *Let $T \in \mathfrak{L}^r(A, B)$ be separating. Then $T = 0$ if and only if $Te = 0$.*

Proof Only sufficiency needs a proof. Hence, we assume that $Te = 0$. Let $f \in A$ and n be a natural number. Since A is commutative and squares in A are positive, we may check quite easily that

$$0 \leq 2n\,|f| \leq f^2 + n^2 e.$$

Combining these inequalities with (1), we get

$$\begin{aligned} 0 \leq 2n\,|Tf| &= 2n\,|T|\,|f| \\ &\leq |T|\left(f^2\right) + n^2\,|T|\,e \\ &= |T|\left(f^2\right) + n^2\,|Te| = |T|\left(f^2\right). \end{aligned}$$

But then $Tf = 0$ because B is Archimedean, which completes the proof of the lemma. □

3 Characterizations of composition operators

Recall from the introduction that a positive operator $C \in \mathfrak{L}^r(A, B)$ is a composition operator if

$$C(fg) = CfCg \quad \text{for all } f, g \in A.$$

The set of all composition operators in $\mathfrak{L}^r(A, B)$ is denoted by $\mathfrak{C}(A, B)$. It is easily seen that if $C \in \mathfrak{C}(A, B)$ then C is separating and Ce is an idempotent element of B. It turns out that the converse holds as we can see next.

Proposition 3.1 *Let $T \in \mathfrak{L}^r(A, B)$. Then T is a composition operator if and only if T is separating and Te is an idempotent element of B.*

Proof Necessity being obvious, we prove sufficiency. From the equality $(Te)^2 = Te$ it follows that Te is positive. Therefore, we may write

$$T^- e = (Te)^- = 0.$$

Moreover, T^- is a lattice homomorphism and so a separating operator in $\mathfrak{L}^r(A, B)$. Using Lemma 2.2, we derive that $T^- = 0$. Accordingly, $T = T^+$ and thus T is a lattice homomorphism with Te idempotent. Corollary 5.5 in [16] ends the proof. □

At this point, recall after Kusraev [18] that if M is an ordered module over B (see [24]) then $a \in M^+$ is called a B-*discrete element* if the equality $b = wa$ holds for some $w \in B$ whenever $b \in M$ and $0 \leq b \leq a$. This extends in a natural way the usual concept of discrete elements in ordered vector spaces (see, e.g., Definition 1.42 in [4]). Moreover, if $v \in B$ and $T \in \mathfrak{L}^r(A, B)$ then we may define $vT \in \mathfrak{L}^r(A, B)$ by putting

$$(vT)\,f = vTf \quad \text{for all } f \in A.$$

Clearly, this turns $\mathfrak{L}^r(A, B)$ into an ordered module over B. Next, we intend to characterize composition operators in terms of B-discrete elements in $\mathfrak{L}^r(A, B)$. Let's discuss the question in a particular case. Assume that B is Dedekind complete (and so $\mathfrak{L}^r(A, B)$ is a Dedekind complete vector lattice) and let $T \in \mathfrak{L}^r(A, B)$ with Te idempotent. By Zaanen's Theorem [3, Theorem 2.60], we have $B = \mathrm{Orth}(B)$. Moreover, using Kutateladze's Theorem [3, Theorem 2.50], it turns out that T is a lattice homomorphism if and only if T is an $\mathrm{Orth}(B)$-discrete element in $\mathfrak{L}^r(A, B)$ (notice here that $\mathfrak{L}^r(A, B)$ has a natural structure of a module over $\mathrm{Orth}(B)$ [18]). As Te is idempotent, it follows directly that T is a composition operator if and only if T is a B-discrete element in $\mathfrak{L}^r(A, B)$. Surprisingly enough, the same conclusion can be obtained without imposing any extra conditions on B. The details are given in the following.

Theorem 3.2 *Let $T \in \mathfrak{L}^r(A, B)$. Then T is a composition operator if and only if T is a B-discrete element in $\mathfrak{L}^r(A, B)$ with Te idempotent.*

Proof Put $p = Te$ and assume that T is a composition operator. Hence, p is an idempotent element in B. We claim that T is B-discrete in $\mathfrak{L}^r(A, B)$. To this end, choose $S \in \mathfrak{L}^r(A, B)$ with $0 \leq S \leq T$. Hence,

$$0 \leq Se \leq Te = p.$$

Taking into consideration Lemma 2.1, we derive that

$$Se \in pB \quad \text{and} \quad pSe = Se.$$

Consequently,

$$(S - SeT)\, e = Se - pSe = 0. \tag{2}$$

On the other hand, T is a lattice homomorphism and $0 \leq S \leq T$. Hence, an easy calculation shows that $S - SeT$ is separating. This together with (2) and Lemma 2.2 yields that $S = SeT$, which means that T is a B-discrete element in $\mathfrak{L}^r(A, B)$, as required.

Conversely, we suppose that p is idempotent and T is B-discrete in $\mathfrak{L}^r(A, B)$. First, we claim that T sends A to pB. To this end, let $f \in A^+$ and $n \in \{1, 2, \ldots\}$. By Theorem 2.57 in [3], we have

$$0 \leq n(f - f \wedge ne) \leq f^2.$$

As T is positive, we get

$$0 \leq nTf - nT(f \wedge ne) \leq T\left(f^2\right). \tag{3}$$

Whence,

$$0 \leq npTf - npT(f \wedge ne) \leq pT\left(f^2\right). \tag{4}$$

Moreover,

$$0 \leq T(f \wedge ne) \leq nTe = np,$$

which shows by Lemma 2.1 (i) that $T(f \wedge ne) \in pB$. Using Lemma 2.1 (ii), we see that

$$pT(f \wedge ne) = T(f \wedge ne)$$

and (4) becomes

$$0 \leq npTf - nT(f \wedge ne) \leq pT\left(f^2\right).$$

Combining these inequalities with (3), we may write

$$0 \leq n\,|Tf - pTf| \leq T\left(f^2\right) + pT\left(f^2\right).$$

But then

$$Tf = pTf \in pB$$

because n is arbitrary in $\{1, 2, \ldots\}$ and B is Archimedean. Consequently, T maps A into pB, as desired. Accordingly, T can be seen as a positive operator in $\mathfrak{L}^r(A, pB)$. Moreover, p is the identity of the f-algebra pB (see again Lemma 2.1). That is,

$$T \in \mathfrak{M} = \left\{S \in \mathfrak{L}^r(A, pB) : S \text{ is positive and } Se = p\right\}.$$

Observe that $\mathfrak{M}$ the convex set of the so-called Markov operators from A into pB (for Markov operators see, e.g., [16]). Let $R, S \in \mathfrak{M}$ such that

$$2T = (R + S).$$

In particular,

$$0 \leq R \leq 2T.$$

Considering R as an operators in $\mathfrak{L}^r(A, B)$ and using the fact that T is B-discrete in $\mathfrak{L}^r(A, B)$, we derive that the equality

$$R = uT \tag{5}$$

holds for some $u \in B$. In particular,

$$pu = uTe = Re = p. \tag{6}$$

Now, recall that both R and T take their values in pB. Hence, multiplying (5) by p (which is the identity of pB) and using (6), we obtain

$$R = pR = puT = pT = T.$$

But then T is an extreme point in the convex set $\mathfrak{M}$. Taking into account Theorem 7.5 in [16], we derive that T is separating and Proposition 3.1 completes the proof. □

4 Weighted composition operators

Recall once again that A is an Archimedean f-algebra with identity e and B is an Archimedean semiprime f-algebra. The symbol $\mathfrak{C}(A, B)$ is used to indicate the set of all composition operators in $\mathfrak{L}^r(A, B)$. An operator $T \in \mathfrak{L}^r(A, B)$ is called a *weighted composition operator* if there exist $w \in B$ and $C \in \mathfrak{C}(A, B)$ such that $T = wC$. Obviously, any weighted composition operator in $\mathfrak{L}^r(A, B)$ is separating. The next simple example shows that a separating operator in $\mathfrak{L}^r(A, B)$ need not be a weighted composition operator.

Example 4.1 Let A be the vector lattice $C(\mathbb{R})$ equipped with its usual structure of f-algebra. On the other hand, assume that B is the same vector lattice furnished with the multiplication $*$ defined by

$$(f * g)(x) = xf(x)g(x) \quad \text{for all } f, g \in C(\mathbb{R}) \text{ and } x \in \mathbb{R}.$$

Clearly, B is an Archimedean semiprime f-algebra. Obviously, the identity operator I of $C(\mathbb{R})$ is a separating operator in $\mathfrak{L}^r(A, B)$. However, I is not a weighted composition operator for the simple reason that 0 is the unique idempotent element of B and so $\mathfrak{L}^r(A, B)$ contains no non-trivial composition operators.

In spite of the counter-example provided above, we shall obtain a quite satisfactory condition on Te for a separating operator $T \in \mathfrak{L}^r(A, B)$ to be a weighted composition operator. The extra condition we are talking about is largely motivated by the recent paper [12]. In this prospect, we call a *von Neumann regular element* in B any element $w \in B$ for which the equality $w^2u = w$ holds for some $u \in B$. We have gathered thus all the ingredients we need for the proof of the central result of this section (compare with Theorem 3.3 in [12]).

Theorem 4.2 *Let $T \in \mathfrak{L}^r(A, B)$ with Te von Neumann regular. Then T is separating if and only if there exists $C \in \mathfrak{C}(A, B)$ such that $T = TeC$.*

Proof Choose $u \in B$ such that

$$(Te)^2 u = Te$$

and define a map $C \in \mathfrak{L}^r(A, B)$ by

$$C = uT.$$

Clearly, C is a separating. Moreover,

$$(Ce)^2 = (uTe)^2 = u^2(Te)^2 = u(Te)^2u = uTe = Ce.$$

This means that Ce is an idempotent element of B, which together with Proposition 3.1 yields that C is a composition operator. Now, we claim that

$$T = TeC.$$

To this end, let $f, g \in A$ with $fg = 0$. Since $TfTg = 0$, we get

$$\begin{aligned}(T - TeC)\, f\, (T - TeC)\, g &= (T - uTeT)\, f\, (T - uTeT)\, g \\ &= (Tf - uTeTf)\,(Tg - uTeTg) \\ &= TfTg - 2uTeTfTg + (uTe)^2\, TfTg = 0.\end{aligned}$$

Thus, $T - TeC$ is separating. Furthermore,

$$\begin{aligned}(T - TeC)\, e &= Te - TeCe = Te - TeuTe \\ &= Te - (Te)^2\, u = Te - Te = 0.\end{aligned}$$

From Lemma 2.2 it follows that $T - TeC = 0$, completing the proof of the theorem. □

We proceed to the first application of Theorem 4.2, which is a characterization of separating operators in $\mathfrak{L}^r(A, B)$. We first have to recall some of the relevant notions.

From now on, let $Q(B)$ denote the maximal ring of quotients of B (see [19]). Since B is an Archimedean semiprime f-algebra, so is $Q(B)$. Actually, B is an f-subalgebra of $Q(B)$ (see, e.g., [21] by Martinez and [25] by Wickstead). The 'good news' is that $Q(B)$ is *von Neumann regular*, that is, all elements in $Q(B)$ are von Neumann regular (see again [19]). This leads, via Theorem 4.2, to the following.

Corollary 4.3 *An operator $T \in \mathfrak{L}^r(A, B)$ is separating if and only if there exists $C \in \mathfrak{C}(A, Q(B))$ such that $T = TeC$.*

Proof It suffices to consider T as an operator in $\mathfrak{L}^r(A, Q(B))$ then apply the result of the previous theorem. □

We should point out that Corollary 4.3 has been obtained in [8] using a completely different (and much more difficult) method.

Now, a bijective operator $T \in \mathfrak{L}^r(A, B)$ is said to be *biseparating* if both T and T^{-1} are separating (see, e.g., [15]). Hence, $T \in \mathfrak{L}^r(A, B)$ is biseparating if and only if T is bijective and

$$fg = 0 \text{ in } A \quad \text{if and only if} \quad TfTg = 0 \text{ in } B.$$

It plausible to think that if there exists a biseparating operator in $\mathfrak{L}^r(A, B)$ then A and B are isomorphic as f-algebras (i.e., there exists an algebra and lattice isomorphism). Unfortunately, this fails to be true as the next example shows.

Example 4.4 Keep the same f-algebras as previously defined in Example 4.1. Obviously, the identity operator $I \in \mathfrak{L}^r(A, B)$ biseparating. However, A and B are not isomorphic as f-algebras since B has no identity.

It could be expected therefore that in the presence of such an identity the situation improves and it does. That's what we prove in the last result of this section.

Corollary 4.5 *Assume that B has identity and that $\mathfrak{L}^r(A, B)$ contains a biseparating operator. Then A and B are isomorphic as f-algebras.*

Proof Let u denote the identity of B and pick a biseparating operator $T \in \mathfrak{L}^r(A, B)$. Fix $f \in A$ and define $S \in \mathfrak{L}^r(A, B)$ by

$$Sg = TeT(fg) - TfTg \quad \text{for all } g \in A.$$

Let $g, h \in A$ with $gh = 0$. Hence,

$$(fg)(fh) = f^2 gh = 0.$$

Since T is separating, we get

$$T(fg)T(fh) = T(fg)Th = T(fh)Tg = TgTh = 0.$$

It follows that

$$SgSh = (TeT(fg) - TfTg)(TeT(fh) - TfTh) = 0.$$

Hence, S is separating. Moreover,

$$Se = TeTf - TfTe = 0$$

and thus $S = 0$, where we use Lemma 2.2. This means that

$$TeT(fg) = TfTg \quad \text{for all } f, g \in A.$$

Putting

$$f = g = T^{-1}u$$

in the above equality, we obtain

$$TeT\left(\left(T^{-1}u\right)^2\right) = u.$$

This implies that Te has an inverse in B. In particular, Te is von Neumann regular in B. By Theorem 4.2, there exist $C \in \mathfrak{C}(A, B)$ such that

$$T = TeC.$$

Finally, it easily verified that C is an algebra and lattice isomorphism, which gives the conclusion. □

As a matter a fact, Corollary 4.5 can also be obtained in an alternative way applying Theorem 3.3 in [10].

5 Vector spaces of separating regular operators

This section contains another application of Theorem 4.2 (or rather Corollary 4.3). We call a *vector space of separating regular operators* from A into B any vector subspace $\mathfrak{W}$ of $\mathfrak{L}^r(A, B)$ the operators in which are separating. Our main purpose is to give a complete description of such vector spaces. In order to achieve this aim, we need further background information.

Let C be a composition operator in $\mathfrak{L}^r(A, Q(B))$, where $Q(B)$ is the maximal ring of quotients of B. Put

$$V_C = \{w \in B : wC \text{ maps } A \text{ into } B\}.$$

Obviously, V_C is a vector subspace of B. Moreover, if W is a vector subspace of V_C then the set

$$\mathfrak{W} = \{wC : w \in W\}$$

is a vector space of separating regular operators from A into B. As we shall see next, it turns out that all vector spaces of separating regular operators from A into B are of this form. The proof of this fact relies in part on a result by Wickstead on weak orthomorphisms. Indeed, in his remarkable paper [25], Wickstead introduced weak orthomorphism on a vector lattice L as a variant of the so-called extended orthomorphisms (see, e.g., [23]). The set of such orthomorphisms is denoted by $\mathrm{Orth}^w(L)$. What we are concerned with here is that $\mathrm{Orth}^w(B)$ is a laterally complete semiprime f-algebra. We may recall in passing that a vector lattice L is said to be *laterally complete* if the supremum of every disjoint set in L^+ exists in L (see Chapter 7 in [2]). Moreover, $\mathrm{Orth}^w(B)$ is isomorphic as an f-algebra with $Q(B)$ and so B can be considered as an f-subalgebra of $\mathrm{Orth}^w(B)$. In summary, we have to keep in mind in the following proof that $Q(B)$ is a laterally complete and von Neumann regular f-algebra that contains B as an f-subalgebra. We also need to recall that any Archimedean laterally complete vector lattice has a weak order unit.

As we have already mentioned, the ordered vector space $\mathfrak{L}^r(A, B)$ need not be a vector lattice. Hence, we cannot speak about vector sublattices of $\mathfrak{L}^r(A, B)$. To workaround this terminological problem, Abramovich and Wickstead in [1] call a *generalized vector sublattice* of $\mathfrak{L}^b(A, B)$ any ordered vector subspace $\mathfrak{V}$ of $\mathfrak{L}^r(A, B)$ which is a vector lattice such that the modulus of $T \in \mathfrak{V}$ in $\mathfrak{L}^r(A, B)$ exists and coincides with its modulus in $\mathfrak{V}$. On the other hand, a nonvoid subset $\mathfrak{S}$ of $\mathfrak{L}^r(A, B)$ is called a *separating set* if $fg = 0$ in A implies $SfTg = 0$ for all $S, T \in \mathfrak{S}$. This concept has been introduced in some form or others in [7]. A separating set $\mathfrak{M}$ in $\mathfrak{L}^r(A, B)$ is said to be *maximal* if there is no strictly large separating set in $\mathfrak{L}^r(A, B)$. It is shown in [9] that any maximal separating set $\mathfrak{M}$ in $\mathfrak{L}^r(A, B)$ is a generalized vector sublattice of $\mathfrak{L}^r(A, B)$ the lattice operations of which are given pointwise, namely, if $S, T \in \mathfrak{M}$ and $f \in A^+$ then

$$(S \vee T) f = Sf \vee Tf \quad \text{and} \quad (S \wedge T) f = Sf \wedge Tf.$$

Moreover, the Hausdorff Maximal Principal (i.e., Zorn's Lemma) tells us that any separating set in $\mathfrak{L}^r(A, B)$ is contained in a maximal separating set in $\mathfrak{L}^r(A, B)$.

All these results will be used in the proof of the last result of this paper.

Theorem 5.1 *A subset $\mathfrak{W}$ of $\mathfrak{L}^r(A, B)$ is a vector space of separating regular operators from A into B if and only if there exist $C \in \mathfrak{C}(A, Q(B))$ and a vector subspace W of V_C such that*

$$\mathfrak{W} = \{wC : w \in W\}.$$

Proof The 'if' part is quite clear but the 'only if' is much less evident. Let $\mathfrak{W}$ be a vector subspace of $\mathfrak{L}^r(A, B)$ such that all operators in $\mathfrak{W}$ are separating. We claim that $\mathfrak{W}$ is a separating set in $\mathfrak{L}^r(A, B)$. To this end, choose $S, T \in \mathfrak{W}$ and $f, g \in A$ such that $fg = 0$. Since S, T are separating, we have

$$SfSg = TfTg = 0.$$

Moreover, $S + T$ is again separating and thus

$$(Sf + Tf)(Sg + Tg) = (S + T) f (S + T) g = 0.$$

It follows that

$$SfTg + TfSg = 0.$$

Therefore,

$$0 \leq (SfTg)^2 = Sf(SfTg)Tg = -SfTfSgTg = 0.$$

But then $SfTg = 0$ since B is semiprime. This shows that $\mathfrak{W}$ is a separating set in $\mathfrak{L}^r(A, B)$, as required.

Now, we shall consider $\mathfrak{W}$ as a separating set in $\mathfrak{L}^r(A, Q(B))$. Hence, $\mathfrak{W}$ is contained in a maximal separating set $\mathfrak{M}$ in $\mathfrak{L}^r(A, Q(B))$. Using Theorem 5.3 in [9], we derive that $\mathfrak{M}$ is a generalized vector sublattice of $\mathfrak{L}^r(A, Q(B))$ the lattice operations of which are given pointwise. Since $Q(B)$ is a laterally complete vector lattice, so is $\mathfrak{M}$ (see Theorem 4 in [6]). But then $\mathfrak{M}$ has a positive weak order unit E, where we use [2, Theorem 7.2]. Moreover, E is a separating operator in $\mathfrak{L}^r(A, Q(B))$. From Corollary 4.3, there exists $C \in \mathfrak{C}(A, Q(B))$ such that

$$E = EeC.$$

We claim that if $T \in \mathfrak{M}$ then

$$T = TeC.$$

Indeed, observe that

$$(Te - TeCe)(EeCe) = TeEeCe - TeEe(Ce)^2 = 0.$$

Hence,

$$|Te - TeCe| \wedge EeCe = 0.$$

Since $T - TeC$ is separating, we may write

$$|Te - TeCe| = |T - TeC|\, e.$$

Furthermore, as observed before, the supremum in $\mathfrak{M}$ is given pointwise. This yields that

$$\begin{aligned}(|T - TeC| \wedge E)\, e &= (|T - TeC| \wedge EeC)\, e \\ &= |Te - TeCe| \wedge EeCe = 0.\end{aligned}$$

Using Lemma 2.2, we derive that the equality

$$|T - TeC| \wedge E = 0$$

holds in $\mathfrak{M}$. But then $T = TeC$ because E is a weak order unit in $\mathfrak{M}$. Putting

$$W = \{Te : T \in \mathfrak{W}\},$$

we conclude that W is a vector subspace of B and that

$$\mathfrak{W} = \{wC : w \in W\}.$$

This completes the proof. □

In particular, if $\mathfrak{W}$ is a vector space of separating regular operators from A into B, then $\mathfrak{W}$ is contained in a one-dimensional B-submodule of the B-module $\mathfrak{L}^r(A, Q(B))$ generated by some composition operator C from A into $Q(B)$.

Let us end where we began. Consider two topological spaces X, Y with X realcompact. From Theorem 3.2 in [9] it follows that $T \in \mathfrak{L}^r(C(X), C(Y))$ is separating if and only if there exists $w \in C(Y)$ and a function τ from Y into X such that τ is continuous on

$$coz(w) = \{y \in Y : w(y) \neq 0\}$$

and

$$(Tf)(y) = w(y)(f \circ \tau)(y) \quad \text{for all } f \in C(X) \text{ and } y \in Y.$$

It is not hard to see that for each $f \in C(X)$ the function $f \circ \tau$ is an element of the maximal ring of quotients $Q(C(Y))$ of the f-algebra $C(Y)$ (see [13]). A composition operator C_τ from $C(X)$ into $Q(C(Y))$ can thus be defined by putting

$$C_\tau f = f \circ \tau \quad \text{for all } f \in C(X).$$

It follows from Theorem 5.1 that if $\mathfrak{W}$ is a vector space of separating regular operators from $C(X)$ into $C(Y)$ then there exist a vector subspace W of $C(Y)$ and a function τ from Y into X such that

$$\mathfrak{W} = \{wC_\tau : w \in W\}.$$

Moreover, τ has to be continuous on $\bigcup_{w\in W} coz(w)$.

6 Open problems

In this section, we discuss three open problems.

6.1 First problem

In Theorem 4.2, we have shown that if $T \in \mathfrak{L}^r(A, B)$ with Te von Neumann regular, then T is separating if and only if there exists $C \in \mathfrak{C}(A, B)$ such that $T = TeC$. Closely examining the proof 4.2, we realize that the equality $TeB = CeB$ holds. Accordingly Theorem 4.2 can be stated alternatively as follows.

Theorem 6.1 *Let B be a semiprime f-algebra and $p \in B^+$. Consider the following assertions.*

(i) *p is von Neumann regular.*
(ii) *For every f-algebra A with identity e and every separating operator $T \in \mathfrak{L}^r(A, B)$ with $p = Te$, there exists $C \in \mathfrak{C}(A, B)$ such that $T = pC$ and $pB = CeB$.*

Then (i) *implies* (ii).

It is plausible now that the converse holds, viz., (ii) implies (i). Indeed, adapting the proof of Theorem 4.2 in [12], we derive that this is true for the $C(X)$-$C(Y)$ case. But the situation remains unclear in general. Notice by the way that nothing about p can be expected without the condition $pB = CeB$.

6.2 Second problem

Throughout the paper, the domain f-algebra A is supposed to have an identity. What we think to be a very legitimate issue is to look at the situation when only semiprimeness is assumed. For instance, does the result of Corollary 4.3 hold if we suppose that A is semiprime with no identity? This question seems natural since this is true for the $C_0(X)$-$C_0(Y)$ case (see Theorem 3.3 in [9]). Unfortunately, the result fails in general (see Example 6.6 in [11]). In spite of that, a partial answer can be derived from Theorem 3.1 in [11]. Indeed, it is proved there that if A is semiprime and nth-root closed (i.e., for every $n \in \{1, 2, \ldots\}$ and $f \in A^+$, there exists $g \in A^+$ such that $g^n = f$) and $T \in \mathfrak{L}^r(A, B)$ is separating, then there exists $C \in \mathfrak{C}(\mathrm{Orth}(A), \mathrm{Orth}^w(B^{ru}))$ such that $T = TeC$, where B^{ru} is the relatively uniform completion of B (see [20] for the relatively uniform topology on vector lattices). So, the question can be stated as follows. What condition is missing on A for the conclusion to hold?

6.3 Third problem

In Theorem 5.1, we describe vector subspaces of $\mathfrak{L}^r(A, B)$ the operators in which are separating. Assuming that $A = B$, we can speak about subalgebras of $\mathfrak{L}^r(A, B) = \mathfrak{L}^r(A)$. For instance, Orth (A) is a subalgebra of $\mathfrak{L}^r(A)$ such that all operators in Orth (A) are separating. We can thus ask about the general case, that is, how can we characterize subalgebras of separating operators in $\mathfrak{L}^r(A)$? In this prospect, we conjecture the following. Let A be an f-algebra with identity and $\mathfrak{A}$ be a subalgebra of $\mathfrak{L}^r(A)$ such that all operators in $\mathfrak{A}$ are separating. Then there exists a vector lattice L such that $\mathfrak{A}$ has an algebra and lattice isomorphic copy in Orth (L). Nevertheless, we have not been able, so far, to prove or disprove this conjecture.

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