Positivity (2017) 21:255–272
DOI 10.1007/s11117-016-0418-0

Positivity

Sensitivity analysis in constrained set-valued optimization via Studniarski derivatives

Nguyen Le Hoang Anh[1]

Received: 1 October 2015 / Accepted: 16 April 2016 / Published online: 28 April 2016

Abstract In the paper, we develop some calculus of Studniarski derivatives for set-valued maps. Then, we establish relationships between Studniarski derivatives of a given objective map and that of the weak perturbation map. Finally, applications to sensitivity analysis of a constrained set-valued optimization problem are obtained. Several examples are given to illustrate some advantages of our results over recent existing ones in the literature.

Keywords Studniarski detivative · Calculus rules · Directional metric subregularity · Sensitivity analysis · Weak perturbation map · Set-valued optimization

Mathematics Subject Classification 49J52 · 49J53 · 54C60 · 90C31

1 Introduction

Sensitivity analysis means the quantitative analysis. It provides informations about derivatives of the perturbation map in case of an optimization problem being perturbed. For sensitivity results, the readers are referred to [11] (for nonlinear programming) [18,26,30,31] (for nonsmooth optimization) [9,28,29] (for set-valued optimization) and the references therein. Recently, many kinds of higher-order generalized derivatives have been proposed with their applications to optimization. We can divide them into two groups. In the first group, the existence of higher-order derivatives depends on lower-order directions, for example, higher-order contingent (adjacent) derivatives

✉ Nguyen Le Hoang Anh
nlhanh@hcmus.edu.vn

1 Department of Optimization and System Theory, University of Science, Vietnam National University Hochiminh City, 227 Nguyen Van Cu, District 5, Hochiminh City, Vietnam

in [7], higher-order generalized contingent (adjacent) derivatives in [32], higher-order generalized contingent (adjacent) epiderivatives in [19], higher-order variational sets in [6, 17], higher-order radial derivatives in [3, 4], etc. In the second one, a higher-order direction exists without informations of lower-order ones, such as the Studniarski derivative introduced by Studniarski in [27] and higher-order radial derivative in the sense of Studniarski in [2]. In recent years, there has been an increasing interest in the study of the Studniarski derivative. We mention here some papers relative to this concept and its applications to optimization. In [28], the lower Studniarski derivative was introduced and applied to sensitivity analysis in parametrized vector optimization. A new notion of the weak lower Studniarski derivative was proposed in [29] with its applications to optimality conditions for a set-valued optimization problem. The Studniarski derivative was employed to obtain optimality conditons and duality in set-valued optimization in [1] and sensitivity analysis for nonsmooth vector optimization in [9]. Its properties and calculus were discussed in [5]. Almost all results in [5, 9] were obtained by virtue of the semi-Studniarski derivative property [called "proto-contingent-type derivative" and "proto-Studniarki derivative" in [9] and [5], respectively (resp)]. This property is quite heavy. In this case, we expect that it can be replaced by a weaker concept.

Motivated by the preceding observations and [10], in the paper we first improve calculus of the Studniarski derivative for set-valued maps without the semi-Studniarski derivative property. Relationships between a weak perturbation map and the feasible-objective map in terms of Studniarski derivatives are also re-established under relaxed conditions. Then, we apply these results to get sensitivity analysis of a constrained set-valued optimization problem (CSOP) in terms of Studniarski derivatives.

The layout of the paper is as follows. Section 2 is devoted to some main notations and concepts needed for our later use. In Sect. 3, we develop some calculus rules of Studniarski derivatives for set-valued maps by virtue of the directional metric subregularity. Relationships between Studniarski derivatives of a set-valued map and its profile map are discussed in Sect. 4. Then, relationships between Studniarski derivatives of a weak perturbation map and the feasible-objective map in set-valued optimization are implied. Finally, we apply these results to sensitivity analysis of a (CSOP). In detail, we discuss Studniarski derivatives of the feasible-objective map into the decision space. Some possible developments are contained in Sect. 5.

2 Preliminaries

Throughout this paper, let X, Y, and Z be normed spaces, $C \subseteq Y$ be a closed convex cone. For $A \subseteq Y$, $\text{int}A$ and $\text{cl}A$ denote the interior and closure of A, resp. $B_X(x, r)$ stands for the open ball in X centered at x with radius $r > 0$. $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ are used for the sets of the natural numbers, real numbers, and nonnegative real numbers, resp. If $\text{int}\, C \neq \emptyset$, $\hat{y} \in A$ is said to be a weak efficient point of A ($\hat{y} \in \text{WMin}_C\, A$) iff $(A - \hat{y}) \cap (-\text{int}\, C) = \emptyset$.

For a set-valued map $F : X \rightrightarrows Y$, the domain and graph of F are defined by

$$\text{dom}\, F := \{x \in X | F(x) \neq \emptyset\}, \quad \text{gr}\, F := \{(x, y) \in X \times Y | y \in F(x)\}, \text{ resp.}$$

$F + C$ is called the profile map of F defined by $(F + C)(x) := F(x) + C$.

Recall that F is said to be metric regular at $(x_0, y_0) \in \text{gr}\, F$ iff there exist $\mu, \lambda > 0$ such that for all $x \in B_X(x_0, \lambda)$, $y \in B_Y(y_0, \lambda)$,

$$d(x, F^{-1}(y)) \leq \mu d(y, F(x)). \tag{1}$$

If we fix $y = y_0$ in (1), F is said to be metric subregular at (x_0, y_0). Let S be a nonempty subset in X, F is metric subregular at (x_0, y_0) wrt S iff there exist $\mu, \lambda > 0$ such that for all $x \in B_X(x_0, \lambda) \cap S$, $y \in B_Y(y_0, \lambda)$,

$$d(x, F^{-1}(y) \cap S) \leq \mu d(y, F(x)).$$

It is well known that the metric regularity property of (the metric subregularity)F is equivalent to the Aubin property (the calmness, resp) of the inverse map $F^{-1} : Y \rightrightarrows X$, see [7]. More properties and applications of metric (sub)regularity can be found in books [7,20,21,25] and papers [8,12,13].

In the paper, we only use a weaker concept of metric subregularity as follows. Let $F : X \times Y \rightrightarrows Z$, $((x_0, y_0), z_0) \in \text{gr}\, F$ and $(u, v) \in X \times Y$. For $m \in \mathbb{N}$, F is said to be directionally metric subregular of order m in Y at $((x_0, y_0), z_0)$ in direction (u, v) with respect to (wrt) a subset S in $X \times Y$ iff there exist $\mu, \lambda > 0$ such that for all $t \in (0, \lambda)$, $u' \in B_X(u, \lambda)$, $v' \in B_Y(v, \lambda)$ with $(x_0 + tu', y_0 + t^m v') \in S$,

$$d((x_0 + tu', y_0 + t^m v'), F^{-1}(z_0) \cap S) \leq \mu d(z_0, F(x_0 + tu', y_0 + t^m v')).$$

It is obvious to see that if F is metric subregular at $((x_0, y_0), z_0)$ wrt a subset S, then F is directionally metric subregular of order m in Y at $((x_0, y_0), z_0)$ in direction (u, v) wrt S, for all $m \in \mathbb{N}$ and $(u, v) \in X \times Y$.

3 Studniarski derivatives of set-valued maps

In this section, we recall the concept of Studniarski derivative for set-valued maps and develop calculus rules of this concept.

Definition 3.1 ([1]) Let $m \in \mathbb{N}$, $F : X \rightrightarrows Y$, and $(x_0, y_0) \in \text{gr}\, F$.

(i) The mth-order upper Studniarski derivative of F at (x_0, y_0) is a set-valued map $D^m F(x_0, y_0) : X \rightrightarrows Y$ defined by

$$D^m F(x_0, y_0)(u) := \operatorname*{Limsup}_{t \downarrow 0,\, u' \to u} \frac{F(x_0 + tu') - y_0}{t^m}.$$

(ii) The mth-order lower Studniarski derivative of F at (x_0, y_0) is a set-valued map $D_l^m F(x_0, y_0) : X \rightrightarrows Y$ defined by

$$D_l^m F(x_0, y_0)(u) := \operatorname*{Liminf}_{t \downarrow 0,\, u' \to u} \frac{F(x_0 + tu') - y_0}{t^m}.$$

Equivalently, we obtain the following formulae

$$D^m F(x_0, y_0)(u) = \left\{v \in Y | \exists t_n \downarrow 0, \exists (u_n, v_n) \to (u, v), y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\right\},$$
$$D_l^m F(x_0, y_0)(u) = \left\{v \in Y | \forall t_n \downarrow 0, \forall u_n \to u, \exists v_n \to v, y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\right\}.$$

Definition 3.2 ([29]) Let $m \in \mathbb{N}$ and $(x_0, y_0) \in \operatorname{gr} F$. The mth-order weak lower Studniarski derivative of F at (x_0, y_0) is a set-valued map $D_w^m F(x_0, y_0) : X \rightrightarrows Y$ defined by

$$D_w^m F(x_0, y_0)(u) = \left\{v \in Y | \forall t_n \downarrow 0, \exists (u_n, v_n) \to (u, v), y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\right\},$$

It is easy to see that

$$D_l^m F(x_0, y_0)(u) \subseteq D_w^m F(x_0, y_0)(u) \subseteq D^m F(x_0, y_0)(u). \tag{2}$$

The above inclusions were illustrated by Examples 3.4-3.6 in [29]. By virtue of the converse inclusions of (2), we have the following definition.

Definition 3.3 (i) The map F is said to have the mth-order semi-Studniarski derivative at (x_0, y_0) if $D_l^m F(x_0, y_0)(u) = D^m F(x_0, y_0)(u)$ for all $u \in X$.
(ii) The map F is said to have the mth-order proto-Studniarski derivative at (x_0, y_0) if $D_w^m F(x_0, y_0)(u) = D^m F(x_0, y_0)(u)$ for all $u \in X$.

Remark 3.1 (i) If F has the mth-order semi-Studniarski derivative at (x_0, y_0), then F has the mth-order proto-Studniarski derivative at (x_0, y_0).
(ii) Definitions 3.3(i), (ii) are called the mth-order proto-Studniarski derivative and the mth-order strict Studniarski derivative, resp, in [5], while the authors named Definition 3.3(i) the mth-order proto-contingent-type derivative in [9]. In the paper, we use the terminologies "semi-derivative" and "proto-derivative" according to the idea of [23] and [24], resp.

We now consider the following operations.

Definition 3.4 ([5]) (i) Let $F_1, F_2 : X \rightrightarrows Y$, the sum of F_1 and F_2 is the set-valued map $F_1 + F_2 : X \rightrightarrows Y$ defined by $(F_1 + F_2)(x) := \{y_1 + y_2 \in Y | y_1 \in F_1(x), y_2 \in F_2(x)\}$.
(ii) If $Y = \mathbb{R}^k$ (an Euclidean space), then the product of F_1 and F_2 is the set-valued map $\langle F_1, F_2\rangle : X \rightrightarrows \mathbb{R}$ defined by $\langle F_1, F_2\rangle (x) := \{\langle y_1, y_2\rangle \in \mathbb{R} | y_1 \in F_1(x), y_2 \in F_2(x)\}$.
(iii) If $Y = \mathbb{R}$, then the quotient of F_1 and F_2 is the set-valued map $F_1/F_2 : X \rightrightarrows \mathbb{R}$ defined by $(F_1/F_2)(x) := \{y_1/y_2 \in \mathbb{R} | y_1 \in F_1(x), y_2 \in F_2(x), y_2 \neq 0\}$.
(iv) Let $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$, the chain of F and G is the set-valued map $G \circ F : X \rightrightarrows Z$ defined by $(G \circ F)(x) := \{z \in Z | \exists y \in F(x), z \in G(y)\}$.

Calculus rules for the above-mentioned operations in terms of Studniarski derivatives were discussed in [5]. The semi-Studniarski derivative property plays an essential role to get inclusions concerning calculus for the operators of set-valued maps, especially Propositions 3.1-3.4 in [5]. However, it is a quite strong condition. Thus, we prefer to lighten this assumption by using a weaker hypothesis of the proto-Studniarski derivative property.

Proposition 3.1 *Let* $F_1, F_2 : X \rightrightarrows Y$, $x_0 \in \operatorname{dom} F_1 \cap \operatorname{dom} F_2$, *and* $y_i \in F_1(x_0)$, $i = 1, 2$. *Suppose that either* F_1 *or* F_2 *has the* m*th-order proto-Studniarski derivative at* (x_0, y_1) *or* (x_0, y_2)*, resp, and the map* $g : (X \times Y)^2 \to \mathbb{R}_+$ *defined by* $g(\alpha, \beta, \gamma, \delta) := ||\alpha - \gamma||^m$ *is directionally metric subregular of order* m *in* $Y \times Y$ *at* $((x_0, y_1, x_0, y_2), 0)$ *in the direction* $(u, \overline{v}, u, \hat{v})$ *wrt* $\operatorname{gr} F_1 \times \operatorname{gr} F_2$*, for all* $(u, \overline{v}) \in \operatorname{gr} D^m F_1(x_0, y_1)$ *and* $(u, \hat{v}) \in \operatorname{gr} D^m F_2(x_0, y_2)$*. Then,*

(i) $D^m F_1(x_0, y_1)(u) + D^m F_2(x_0, y_2)(u) \subseteq D^m(F_1 + F_2)(x_0, y_1 + y_2)(u)$.

(ii) *If* $Y = \mathbb{R}^k$*, then*

$$\left\langle y_2, D^m F_1(x_0, y_1)(u)\right\rangle + \left\langle y_1, D^m F_2(x_0, y_2)(u)\right\rangle \subseteq D^m(\langle F_1, F_2\rangle)(x_0, \langle y_1, y_2\rangle)(u).$$

(iii) *If* $Y = \mathbb{R}$ *and* $y_2 \neq 0$*, then*

$$\frac{1}{y_2^2}(y_2 D^m F_1(x_0, y_1)(u) - y_1 D^m F_2(x_0, y_2)(u)) \subseteq D^m(F_1/F_2)(x_0, y_1/y_2)(u).$$

Proof Let $\overline{v} \in D^m F_1(x, y_1)(u)$ and $\hat{v} \in D^m F_2(x, y_2)(u)$, then there exist $t_n \downarrow 0$, $(\overline{u}_n, \overline{v}_n) \to (u, \overline{v})$ such that

$$y_1 + t_n^m \overline{v}_n \in F_1(x_0 + t_n \overline{u}_n).$$

Suppose that F_2 has the mth-order proto-Studniarski derivative at (x_0, y_2), with t_n above, there are $(\hat{u}_n, \hat{v}_n) \to (u, \hat{v})$ such that

$$y_2 + t_n^m \hat{v}_n \in F_2(x_0 + t_n \hat{u}_n).$$

It follows from the directionally metric subregularity assumption that there exist $\mu > 0$ and $\lambda > 0$ such that for every $t \in (0, \lambda)$ and $(u_1, v_1, u_2, v_2) \in B_{X\times Y}((u, \overline{v}), \lambda) \times B_{X\times Y}((u, \hat{v}), \lambda)$ with $(x_0 + tu_1, y_2 + t^m v_1, x_0 + tu_2, y_2 + t^m v_2) \in \operatorname{gr} F_1 \times \operatorname{gr} F_2$,

$$d\left((x_0 + tu_1, y_2 + t^m v_1, x_0 + tu_2, y_2 + t^m v_2), g^{-1}(0) \cap (\operatorname{gr} F_1 \times \operatorname{gr} F_2)\right) \leq \mu d\left(0, g\left(x_0 + tu_1, y_2 + t^m v_1, x_0 + tu_2, y_2 + t^m v_2\right)\right). \quad (3)$$

For n large enough, we have $t_n \in (0, \lambda)$ and $(\overline{u}_n, \overline{v}_n, \hat{u}_n, \hat{v}_n) \in B_{X\times Y}((u, \overline{v}), \lambda) \times B_{X\times Y}((u, \hat{v}), \lambda)$. Thus, from (3), there exist $(\overline{x}_n, \overline{y}_n, \hat{x}_n, \hat{y}_n) \in \operatorname{gr} F_1 \times \operatorname{gr} F_2$ with $\overline{x}_n = \hat{x}_n$ for all n such that

$$\left\|\left(x_0 + t_n\overline{u}_n, y_1 + t_n^m\overline{v}_n, x_0 + t_n\hat{u}_n, y_2 + t_n^m\hat{v}_n\right) - \left(\overline{x}_n, \overline{y}_n, \hat{x}_n, \hat{y}_n\right)\right\| \leq \mu {t_n}^m \left|\left|\overline{u}_n - \hat{u}_n\right|\right|^m,$$

which implies

$$\begin{aligned}\|x_0 + t_n\overline{u}_n - \overline{x}_n\| &\leq \left\|\left(x_0 + t_n\overline{u}_n, y_1 + t_n^m\overline{v}_n, x_0 + t_n\hat{u}_n, y_2 + t_n^m\hat{v}_n\right)\right. \\ &\quad \left. - \left(\overline{x}_n, \overline{y}_n, \hat{x}_n, \hat{y}_n\right)\right\| \\ &\leq \mu t_n^m \left|\left|\overline{u}_n - \hat{u}_n\right|\right|^m.\end{aligned}$$

Similarly, we have

$$\left\|y_1 + t_n^m\overline{v}_n - \overline{y}_n\right\| \leq \mu t_n^m||\overline{u}_n - \hat{u}_n||^m, \quad \left\|y_2 + t_n^m\hat{v}_n - \hat{y}_n\right\| \leq \mu t_n^m \left|\left|\overline{u}_n - \hat{u}_n\right|\right|^m.$$

Consequently,

$$\left\|\frac{\overline{x}_n - x_0}{t_n} - \overline{u}_n\right\| \leq \mu t_n^{m-1}||\overline{u}_n - \hat{u}_n||^m, \quad \left\|\frac{\overline{y}_n - y_1}{t_n^m} - \overline{v}_n\right\| \leq \mu \left|\left|\overline{u}_n - \hat{u}_n\right|\right|^m,$$

$$\left\|\frac{\hat{y}_n - y_2}{t_n^m} - \hat{v}_n\right\| \leq \mu \left|\left|\overline{u}_n - \hat{u}_n\right|\right|^m. \tag{4}$$

By setting $v_n^1 := \dfrac{\overline{y}_n - y_1}{t_n^m}$, $v_n^2 := \dfrac{\hat{y}_n - y_2}{t_n^m}$ and $u_n := \dfrac{\overline{x}_n - x_0}{t_n}$, then $v_n^1 \to \overline{v}$, $v_n^2 \to \hat{v}$, $u_n \to u$ (take $n \to +\infty$ in (4)) and

$$y_1 + t_n^m v_n^1 = \overline{y}_n \in F_1(\overline{x}_n) = F_1(x_0 + t_n u_n),$$

$$y_2 + t_n^m v_n^2 = \hat{y}_n \in F_2(\hat{x}_n) = F_2(\overline{x}_n) = F_2(x_0 + t_n u_n).$$

Thus,

$$(y_1 + y_2) + t_n^m(v_n^1 + v_n^2) \in (F_1 + F_2)(x_0 + t_n u_n),$$

i.e., $\overline{v} + \hat{v} \in D^m(F_1 + F_2)(x_0, y_1 + y_2)(u)$.

For the proofs of parts (ii) and (iii), one refers to Propositions 3.3 and 3.4 in [5].

□

Proposition 3.2 *Let* $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$, $(x_0, y_0) \in \mathrm{gr}F$, *and* $(y_0, z_0) \in \mathrm{gr}G$.

(i) Suppose that G *has the* m*th-order proto-Studniarski derivative at* (y_0, z_0) *and the map* $g_1 : X \times Y \times Y \times Z \to \mathbb{R}_+$ *defined by* $g_1(\alpha, \beta, \gamma, \delta) := ||\beta - \gamma||^m$ *is directionally metric subregular of order* m *in* Z *at* $((x_0, y_0, y_0, z_0), 0)$ *in the direction* (u, v, v, w) *wrt* $\mathrm{gr}\, F \times \mathrm{gr}\, G$, *for all* $(u, v) \in \mathrm{gr}\, D^1F(x_0, y_0)$ *and* $(v, w) \in \mathrm{gr}\, D^m F_2(y_0, z_0)$. *Then,*

$$D^m G(y_0, z_0)(D^1 F(x_0, y_0)(u)) \subseteq D^m(G \circ F)(x_0, z_0)(u).$$

(ii) Suppose that G has the first order proto-Studniarski derivative at (y_0, z_0) *and the map* $g_2 : X \times Y \times Y \times Z \to \mathbb{R}_+$ *defined by* $g_2(\alpha, \beta, \gamma, \delta) := ||\beta - \gamma||$ *is directionally metric subregular of order m in* $Y \times Y \times Z$ *at* $((x_0, y_0, y_0, z_0), 0)$ *in the direction* (u, v, v, w) *wrt* $\text{gr}\, F \times \text{gr}\, G$*, for all* $(u, v) \in \text{gr}\, D^m F(x_0, y_0)$ *and* $(v, w) \in \text{gr}\, D^1 G(y_0, z_0)$*. Then,*

$$D^1 G(y_0, z_0)(D^m F(x_0, y_0)(u)) \subseteq D^m (G \circ F)(x_0, z_0)(u).$$

Proof By the similarity, we only prove (ii). Let $w \in D^1 G(y_0, z_0)(D^m F(x_0, y_0)(u))$, then there exists $v \in D^m F(x_0, y_0)(u)$ such that $w \in D^1 G(y_0, z_0)(v)$. For v, there are $t_n \downarrow 0$, $(\overline{u}_n, \overline{v}_n) \to (u, v)$ with

$$y_0 + t_n^m \overline{v}_n \in F(x_0 + t_n \overline{u}_n).$$

Since G has the first order proto-Studniarski derivative at (y_0, z_0), with t_n above, there are $(\hat{v}_n, \hat{w}_n) \to (v, w)$ such that

$$z_0 + t_n^m \hat{v}_n \in G(y_0 + t_n^m \hat{w}_n).$$

By the directionally metric subregularity assumption, there exist $\mu > 0$ and $\lambda > 0$ such that for every $t \in (0, \lambda)$ and $(u', v_1, v_2, w') \in B_{X \times Y}((u, v), \lambda) \times B_{Y \times Z}((v, w), \lambda)$ with $(x_0 + tu', y_0 + t^m v_1, y_0 + t^m v_2, z_0 + t^m w') \in \text{gr}\, F \times \text{gr}\, G$,

$$d\left((x_0 + tu', y_0 + t^m v_1, y_0 + t^m v_2, z_0 + t^m w'), g_2^{-1}(0) \cap (\text{gr}\, F \times \text{gr}\, G\right) \leq \mu d\left(0, g_2(x_0 + tu', y_0 + t^m v_1, y_0 + t^m v_2, z_0 + t^m w')\right). \quad (5)$$

For n large enough, we have $(\overline{u}_n, \overline{v}_n, \hat{v}_n, \hat{w}_n) \in B_{X \times Y}((u, v), \lambda) \times B_{Y \times Z}((v, w), \lambda)$ and $t_n \in (0, \lambda)$. Thus, it follows from (5) that there exist $(\overline{x}_n, \overline{y}_n, \hat{y}_n, \hat{z}_n) \in \text{gr}\, F \times \text{gr}\, G$ with $\overline{y}_n = \hat{y}_n$ for all n such that

$$\left\| \left(x_0 + t_n \overline{u}_n, y_0 + t_n^m \overline{v}_n, y_0 + t_n^m \hat{v}_n, z_0 + t_n^m \hat{w}_n\right) - \left(\overline{x}_n, \overline{y}_n, \hat{y}_n, \hat{z}_n\right) \right\| \leq \mu t_n^m \left|\left|\overline{v}_n - \hat{v}_n\right|\right|,$$

which implies

$$\begin{aligned} \left\| y_0 + t_n^m \overline{v}_n - \overline{y}_n \right\| &\leq \left\| \left(x_0 + t_n \overline{u}_n, y_0 + t_n^m \overline{v}_n, y_0 + t_n^m \hat{v}_n, z_0 + t_n^m \hat{w}_n\right) - \left(\overline{x}_n, \overline{y}_n, \hat{y}_n, \hat{z}_n\right) \right\| \\ &\leq \mu t_n^m \left|\left|\overline{v}_n - \hat{v}_n\right|\right|, \end{aligned}$$

and

$$\|x_0 + t_n \overline{u}_n - \overline{x}_n\| \leq \mu t_n^m \left|\left|\overline{u}_n - \hat{u}_n\right|\right|, \quad \left\|z_0 + t_n^m \hat{w}_n - \hat{z}_n\right\| \leq \mu t_n^m \left|\left|\overline{u}_n - \hat{u}_n\right|\right|.$$

Thus,

$$\left\| \frac{\overline{y}_n - y_0}{t_n^m} - \overline{v}_n \right\| \le \mu ||\overline{v}_n - \hat{v}_n||, \quad \left\| \frac{\overline{x}_n - x_0}{t_n} - \overline{u}_n \right\| \le \mu t_n^{m-1} ||\overline{v}_n - \hat{v}_n||,$$
$$\left\| \frac{\hat{z}_n - z_0}{t_n^m} - \hat{w}_n \right\| \le \mu ||\overline{v}_n - \hat{v}_n||. \quad (6)$$

Let $v_n := \dfrac{\overline{y}_n - y_0}{t_n^m}$, $z_n := \dfrac{\hat{z}_n - z_0}{t_n^m}$ and $u_n := \dfrac{\overline{x}_n - x_0}{t_n}$. Take $n \to +\infty$ in (6), then $v_n \to v$, $z_n \to w$, $u_n \to u$ and

$$y_0 + t_n^m v_n = \overline{y}_n \in F(\overline{x}_n) = F(x_0 + t_n u_n),$$
$$z_0 + t_n^m z_n = \hat{z}_n \in G(\hat{y}_n) = G(\overline{y}_n) = G(y_0 + t_n^m v_n).$$

Hence, $z_0 + t_n^m z_n \in (G \circ F)(x_0 + t_n u_n)$, i.e., $w \in D^m(G \circ F)(x_0, z_0)(u)$. □

For the inverse inclusions of Propositions 3.1, 3.2, the readers are referred to Section 3 in [5].

The following simple example gives a case where our results can be employed, while some earlier existing ones cannot.

Example 3.1 Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $G : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x, y) = (x, y)$ and

$$G(x, y) := \begin{cases} \emptyset, & if\ x, y \in \left\{ \frac{1}{n^2} | n \in \mathbb{N} \right\}, \\ x + y, & otherwise. \end{cases}$$

Then, we have

$$(G \circ F)(x, y) := \begin{cases} \emptyset, & if\ x, y \in \left\{ \frac{1}{n^2} | n \in \mathbb{N} \right\}, \\ x + y, & otherwise. \end{cases}$$

We can check that $DF((0, 0), (0, 0))(u, v) = (u, v)$ and

$$DG((0, 0), 0)(u, v) = D_w G((0, 0), 0)(u, v) = u + v, \quad D_l G((0, 0), 0)(0) = \emptyset,$$

i.e., G has the first order proto-Studniarski derivative at $(0, 0)$, but does not have the first order semi-Studniarski derivative at $(0, 0)$. Thus, Proposition 3.2 in [5] does not work (see Remark 3.1). However, the directionally metric subregularity of order 1 in Proposition 3.2(ii) is satisfied for all directions (u, v, u, v, u, v, w), where $(u, v, u, v) \in \text{gr} DF((0, 0), (0, 0))$ and $(u, v, w) \in \text{gr} DG((0, 0), 0)$.

The assumption of Proposition 3.2(ii) is checked as follows: let $(u, v, u, v) \in \text{gr}\, DF((0, 0), (0, 0))$, $(u, v, u + v) \in \text{gr}\, DG((0, 0), 0)$ and $\lambda > 0$, it is enough to show that there exists $\mu > 0$ such that for all $t \in (0, \lambda)$, $(u_1, v_1, u_2, v_2) \in B_{\mathbb{R}^4}((u, v, u, v), \lambda)$, $(u', v', w') \in B_{\mathbb{R}^3}((u, v, u+v), \lambda)$ with

$$(0 + tu_1, 0 + tv_1, 0 + tu_2, 0 + tv_2, 0 + tu', 0 + tv', 0 + tw') \in \text{gr}\, F \times \text{gr}\, G,$$

then

$$\begin{aligned} &d((tu_1, tv_1, tu_2, tv_2, tu', tv', tw'), g_2^{-1}(0) \cap (\text{gr}\, F \times \text{gr}\, G)) \\ &\quad \leq \mu d(0, g_2((tu_1, tv_1, tu_2, tv_2, tu', tv', tw'))), \end{aligned}$$

where $g_2 : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}_+$ is defined by $g_2(x_1, y_1, x_2, y_2, x_3, y_3, z) := ||(x_2, y_2) - (x_3, y_3)||$.

Since $(tu_1, tv_1, tu_2, tv_2, tu', tv', tw') \in \text{gr}\, F \times \text{gr}\, G$, we get $u_1 = u_2$, $v_1 = v_2$, and $w' = u' + v'$. Thus, we need to find μ such that

$$\begin{aligned} &\inf_{\substack{(x,y)\in\mathbb{R}^2,\\ (x',y')\in F(x,y),\\ z'\in G(x',y')}} \left\{ ||t(u_1, v_1) - (x, y)|| + ||t(u_2, v_2) - (x', y')|| \right. \\ &\left. +||t(u', v') - (x', y')|| + |tw' - z'| \right\} \\ &\leq \mu t ||(u_2, v_2) - (u', v')||. \end{aligned} \tag{7}$$

It is obvious that

$$\begin{aligned} &\inf_{\substack{(x,y)\in\mathbb{R}^2,\\ (x',y')\in F(x,y),\\ z'\in G(x',y')}} \left\{ ||t(u_1, v_1) - (x, y)|| + \left|\left|t(u_2, v_2) - (x', y')\right|\right| \right. \\ &\left. +||t(u', v') - (x', y')|| + |tw' - z'| \right\} \\ = &\inf_{\substack{(x,y)\in\mathbb{R}^2,\\ (x',y')\in F(x,y),\\ z'\in G(x',y')}} \left\{ ||t(u_2, v_2) - (x, y)|| + \left|\left|t(u_2, v_2) - (x', y')\right|\right| \right. \\ &\left. +||t(u', v') - (x', y')|| + |tw' - z'| \right\} \\ = &\inf_{(x,y)\in\mathbb{R}^2} \{ 2||t(u_2, v_2) - (x, y)|| \\ &+ \left|\left|t(u', v') - (x, y)\right|\right| + |t(u' + v') - (x + y)| \} . \end{aligned}$$

Let $x := \dfrac{tu_2 + tu'}{2}$ and $y := \dfrac{tv_2 + tv'}{2}$, then

$$\begin{aligned} 2||t(u_2, v_2) - (x, y)|| &= t||(u_2, v_2) - (u', v')||, \\ ||t(u', v') - (x, y)|| &= (1/2)t||(u_2, v_2) - (u', v')||, \end{aligned}$$

and

$$\begin{aligned} |t(u' + v') - (x + y)| &= (1/2)t|(u_2 - u') + (v_2 - v')| \\ &\leq (1/2)t\sqrt{((u_2 - u')^2 + (v_2 - v')^2)(1^2 + 1^2)} \\ &\leq (\sqrt{2}/2)t||(u_2, v_2) - (u', v')||, \end{aligned}$$

which implies that

$$\inf_{\substack{(x,y)\in\mathbb{R}^2,\\ (x',y')\in F(x,y),\\ z'\in G(x',y')}} \left\{||t(u_1,v_1)-(x,y)|| + ||t(u_2,v_2)-(x',y')|| + ||t(u',v')-(x',y')|| + |tw'-z'|\right\}$$
$$\leq (1+1/2+\sqrt{2}/2)t||(u_2,v_2)-(u',v')||.$$

Thus, (7) is true for any $\mu \geq 1+1/2+\sqrt{2}/2$. Hence, by Proposition 3.2(ii), we get

$$DG((0,0),0)(DF((0,0),(0,0))(u,v)) \subseteq D(G\circ F)((0,0),0)(u,v) = u+v.$$

4 Sensitivity analysis of (CSOP)

4.1 Studniarski derivatives of weak perturbation maps

In this subsection, we first establish relationships between Studniarski derivatives of a set-valued map and its profile map. The following compactness notion is necessary for our later results.

Definition 4.1 For $u \in X$, $F :\rightrightarrows Y$ is said to be compact at (x_0, y_0) in the direction u if for any $t_n \downarrow 0$, $u_n \to u$, and $y_n \in F(x_0+t_nu_n)$ for all n, then $\{y_n\}$ has a subsequence converging to y_0.

Remark 4.1 If F is mth-order u-directionally contingent compact at $(x_0, y_0) \in \text{gr}\, F$ (see Definition 4.1 in [9]) then F is compact at (x_0, y_0) in the direction u. However, the inverse statement is not true by Example 4.1 below. Thus, our concept of the relaxed compactness is weaker than that in [9].

Example 4.1 Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = \sqrt{x}$ for all $x \geq 0$. It is easy to check that F is compact at $(0, 0)$ in all directions $u \geq 0$. Nevertheless, F is not mth-order u-directionally contingent compact at $(0, 0)$ for any $m \in \mathbb{N}$, $u \geq 0$. Indeed, by choosing $t_n = 1/n^2$, $(u_n, v_n) = (1/n, n^{2m-(3/2)})$ for $u = 0$, and $(u_n, v_n) = (u, n^{2m-1}\sqrt{u})$ for $u > 0$, we get that $0 + t_n^m v_n \in F(0 + t_nu_n)$, but $\{v_n\}$ does not have a convergent subsequence.

Proposition 4.1 *Let* $F : X \rightrightarrows Y$, $(x_0, y_0) \in \text{gr}\, F$, *and* $u \in X$.

(i) $D^mF(x_0, y_0)(u) + C \subseteq D^m(F + C)(x_0, y_0)(u)$.
(ii) *Suppose that* Y *is finite dimensional,* F *is compact at* (x_0, y_0) *in the direction* u*, and* $D^mF(x_0, y_0)(0) \cap (-C) = \{0\}$. *Then,*

$$D^mF(x_0, y_0)(u) + C = D^m(F + C)(x_0, y_0)(u).$$

If, additionally, $\text{int}\, C \neq \emptyset$ *and* $\widetilde{C}$ *is a closed convex cone with* $\widetilde{C} \subseteq \text{int}\, C \cup \{0\}$*, then*

$$\text{WMin}_C\, D^mF(x_0, y_0)(u) = \text{WMin}_C\, D^m(F + \widetilde{C})(x_0, y_0)(u).$$

Proof (i) follows from Proposition 4.2 in [9].

(ii) It is enough to prove that $D^m(F+C)(x_0,y_0)(u) \subseteq D^m F(x_0,y_0)(u)+C$. Let $v \in D^m(F+C)(x_0,y_0)(u)$. If $(u,v)=(0,0)$, then $v \in D^m F(x_0,y_0)(u)+C$ (since $0 \in D^m F(x_0,y_0)(0)$). We suppose that $(u,v) \neq (0,0)$, then there exist $t_n \downarrow 0$, $(u_n,v_n) \to (u,v)$ such that $y_0 + t_n^m v_n \in (F+C)(x_0+t_n u_n)$, which implies the existence of $c_n \in C$ satisfying

$$y_0 + t_n^m \left(v_n - c_n/t_n^m\right) \in F(x_0+t_n u_n). \tag{8}$$

By setting $w_n := v_n - (c_n/t_n^m)$, if $\{w_n\}$ has a convergent subsequence, then $\{c_n/t_n^m\}$ (or its subsequence if necessary) has a limit point $\bar{c} \in C$ (since C is a closed convex cone). Thus, $v - \bar{c} \in D^m F(x_0,y_0)(u)$ and we are done. Suppose to the contrary, i.e., $||w_n|| \to +\infty$. It follows from the compactness of F that $y_n := y_0 + t_n^m w_n$ has a subsequence converging to y_0. Without loss of generality, we assume $y_n \to y_0$. Let $s_n := ||y_n - y_0||^{1/m}$, then

$$\frac{c_n}{t_n^m ||w_n||} = \frac{v_n}{||w_n||} - \frac{w_n}{||w_n||}.$$

Since Y is finite dimensional, $\{w_n/||w_n||\}$ (or its subsequence if necessary) converges to some $k \in Y$ with $||k|| = 1$, which implies that $\{c_n/(t_n^m ||w_n||)\} \to -k$. It follows from $c_n/(t_n^m||w_n||) \in C$ and the closeness of the cone C that $k \in -C$. Furthemore, with $k_n := w_n/||w_n|| = (y_n - y_0)/s_n^m$, one gets

$$y_0 + s_n^m k_n = y_n \in F\left(x_0 + s_n\left(\frac{t_n}{s_n}u_n\right)\right).$$

On the other hand, one has

$$\begin{aligned}\frac{t_n}{s_n}u_n &= \frac{t_n}{s_n}||w_n||^{1/m}\frac{u_n}{||w_n||^{1/m}} = \\ &= \left(\frac{t_n}{s_n}\right)\left(\frac{||y_n-y_0||^{1/m}}{t_n}\right)\frac{u_n}{||w_n||^{1/m}} = \frac{u_n}{||w_n||^{1/m}} \to 0 \text{ (since } ||w_n|| \to +\infty).\end{aligned}$$

Hence, $k \in D^m F(x_0,y_0)(0)$, which contradicts the fact that $D^m F(x_0,y_0)(0) \cap (-C) = \{0\}$.

The rest of the proof follows from Proposition 4.3(i)(d) in [9]. □

The following example illustrates the advantage of Proposition 4.1 over Proposition 4.2 in [9] and Proposition 2.3 in [5].

Example 4.2 Let $C = \mathbb{R}_+$ and $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined by

$$F(x) := \begin{cases} \{-1\}, & \text{if } x < 0, \\ \{\sqrt{x}, x^2\}, & \text{if } x \geq 0. \end{cases}$$

By calculating, we get

$$D^2F(0,0)(u) = \begin{cases} \emptyset, & \text{if } u < 0, \\ \{u^2\}, & \text{if } u \geq 0, \end{cases} \qquad D_S^2F(0,0)(u) = \begin{cases} \emptyset, & \text{if } u < 0, \\ -\mathbb{R}_+, & \text{if } u = 0, \\ \{u^2\}, & \text{if } u > 0. \end{cases}$$

We can see that F is compact at $(0,0)$ in the direction u for all $u > 0$ and $D^2F(0,0)(0) \cap (-C) = \{0\}$. Then, it follows from Proposition 4.1(ii) that for all $u > 0$,

$$D^2F(0,0)(u) + C = D^2(F+C)(0,0)(u) = \{v \in \mathbb{R} | v \geq u^2\}. \tag{9}$$

On the other hand, it is easy to check that F is not locally pseudo-Hölder calm of order 2 at $(0,0)$ (see Definition 1.1(iii) in [5]) and $D_S^2F(0,0)(0) \cap (-C) = -\mathbb{R}_+$. Moreover, F is not second-order u-directionally contingent compact at $(0,0)$ for any $u \in \mathbb{R}$ (see Example 4.1). Thus, Proposition 4.2 in [9] and Proposition 2.3 in [5] cannot be employed to get (9).

Let U be a normed space of perturbation parameters, Y be an objective (normed) space ordered partially by a closed convex cone C with $\text{int}\, C \neq \emptyset$, and $F : U \rightrightarrows Y$ be the feasible-objective map (the term "feasible-objective" was proposed by Diem et al. in [9]). We define a set-valued map W from U to Y by $W(u) := \text{WMin}_C F(u)$ for $u \in U$. The map W is called the weak perturbation map.

In the rest of this subsection, we apply the above-mentioned results to investigate relationships between Studniarski derivatives of F and that of W.

Let $\widetilde{C}$ be a closed convex cone with $\widetilde{C} \subseteq \text{int}\, C \cup \{0\}$. Recall that F is said to be $\widetilde{C}$-dominated by W near u_0 (see [15]) iff there exists a neighborhood V of u_0 such that $F(u) \subseteq W(u) + \widetilde{C}$ for all $u \in V$. If F is $\widetilde{C}$-dominated by W near u_0, then $D^m(W+\widetilde{C})(u_0, y_0)(u) = D^m(F+\widetilde{C})(u_0, y_0)(u)$ for any $(u_0, y_0) \in \text{gr}\, W$ and $u \in U$ (see Remark 5.1 in [9]).

Proposition 4.2 *$F : X \rightrightarrows Y$, $(u_0, y_0) \in \text{gr}\, W$, and $u \in X$. Suppose that the assumptions in Proposition 4.1(ii) are satisfied wrt (u_0, y_0) and F is $\widetilde{C}$-dominated by W near u_0. Then,*

$$\text{WMin}_C\, D^mF(u_0, y_0)(u) \subseteq D^mW(u_0, y_0)(u). \tag{10}$$

If, additionally, F has the mth-order proto-Studniarski derivative at (u_0, y_0) in the direction u and the map $g : (X \times Y)^2 \to \mathbb{R}_+$ defined as in Proposition 3.1 is directionally metric subregular of order m in $Y \times Y$ at $((u_0, y_0, u_0, y_0), 0)$ in the direction $(u, \overline{v}, u, \hat{v})$ wrt $\text{gr}\, W \times \text{gr}\, F$, for all $\overline{v} \in D^mW(u_0, y_0)(u)$ and $\hat{v} \in D^mF(u_0, y_0)(u)$, then (10) becomes an equality for this u.

Proof We can check that all assumption in Proposition 4.1(ii) are also fulfilled for W wrt (u_0, y_0). Then, one has

$$\begin{aligned} \text{WMin}_C\, D^mF(u_0, y_0)(u) &= \text{WMin}_C\, D^m(F+\widetilde{C})(u_0, y_0)(u) \text{ (by Proposition 4.1(ii))} \\ &= \text{WMin}_C\, D^m(W+\widetilde{C})(u_0, y_0)(u) \\ &\subseteq D^m(W+\widetilde{C})(u_0, y_0)(u). \end{aligned}$$

For the inverse inclusion, let $v \in D^m W(u_0, y_0)(u)$, then there are $t_n \downarrow 0$, $(u_n, v_n) \to (u, v)$ such that $y_0 + t_n^m v_n \in W(u_0 + t_n u_n)$. Suppose that $v \notin \mathrm{WMin}_C\, D^m F(u_0, y_0)(u)$, i.e., there exists $\widetilde{v} \in D^m F(u_0, y_0)(u)$ with $\widetilde{v} - v \in -\mathrm{int}\, C$. Since F has the mth-order proto-Studniarski derivative at (u_0, y_0) in the direction u, with t_n above, there is $(\widetilde{u}_n, \widetilde{v}_n) \to (u, \widetilde{v})$ satisfying $y_0 + t_n^m \widetilde{v}_n \in F(u_0 + t_n \widetilde{u}_n)$.

It follows from the directionally metric subregularity assumption and the proof similar to that of Proposition 3.1 that there exist $\hat{u}_n \to u$ and $(v_n^1, v_n^2) \to (v, \widetilde{v})$ such that

$$y_0 + t_n^m v_n^1 \in W(u_0 + t_n \hat{u}_n) = \mathrm{WMin}_C\, F(u_0 + t_n \hat{u}_n), \quad y_0 + t_n^m v_n^2 \in F(u_0 + t_n \hat{u}_n).$$

Thus, for n large enough, one has $(y_0 + t_n^m v_n^2) - (y_0 + t_n^m v_n^1) = t_n^m (v_n^2 - v_n^1) \in -\mathrm{int}\, C$, which contradicts the fact that $y_0 + t_n^m v_n^1 \in \mathrm{WMin}_C\, F(u_0 + t_n \hat{u}_n)$. □

The inverse inclusion of (10) was discussed in Proposition 5.2 in [9] under the mth-order semi-Studniarski derivative property of F. However, Example 3.1 provides a case where it does not work, while Proposition 4.2 can be used. Let $U = \mathbb{R}^2$ and $Y = \mathbb{R}$. Consider the map G as in Example 3.1. By Example 3.1, G has the first order proto-Studniarski derivative at $(0, 0)$, but does not have the first order semi-Studniarski derivative at $(0, 0)$. Thus, Proposition 5.2 in [9] cannot be employed. However, by the same method in Example 3.1, we can check that the directionally metric subregularity of order 1 of G in Proposition 4.2 is satisfied at $(u_0, y_0) = (0, 0)$. Thus, by Proposition 4.2, we get $DW(u_0, y_0)(u, v) \subseteq \mathrm{WMin}_C\, DG(u_0, y_0)(u, v)$ for all $(u, v) \in U$, where $DW(u_0, y_0)(u, v) = u + v$.

4.2 Sensitivity analysis of (CSOP)

Let U, W, Y be normed spaces, C is a closed convex ordering cone in Y, $X : U \rightrightarrows W$ and $F : U \times W \rightrightarrows Y$. We consider the following constrained set-valued optimization problem

$$\mathrm{WMin}_C\ F(u, x), \quad \text{subject to } x \in X(u). \tag{11}$$

Define a set-valued map H from U to Y by

$$H(u) := F(u, X(u)) = \{y \in Y | y \in F(u, x), x \in X(u)\}.$$

$H(u)$ is the parameterized feasible set in the objective space, called the feasible-objective map in [9]. The solution set in Y to problem (11) is denoted by $S(u) := \mathrm{WMin}_K\, H(u)$.

We assume that there is $(u_0, y_0) \in U \times Y$ such that $y_0 \in H(u_0)$. Then, there exists $x_0 \in X(u_0)$ satisfying $y_0 \in F(x_0, u_0)$.

Relationships of Studniarki derivatives of F and X to the corresponding that of H are given as follows.

Proposition 4.3 *Let $u \in U$. Suppose that W is finite dimensional, X is compact at (u_0, x_0) in the direction u, and $DX(u_0, x_0)(0) = \{0\}$. Then,*

$$D^m H(u_0, y_0)(u) \subseteq \bigcup_{x \in DX(u_0,x_0)(u)} D^m F((u_0, x_0), y_0)(u, x) \tag{12}$$

If, additionally, X has the first-order proto-Studniarski derivative at (u_0, x_0) in the direction u and the map $g : (U \times W)^2 \times Y \to \mathbb{R}_+$ defined by $g(\alpha, \beta, \gamma, \delta, \zeta) := (||\alpha - \gamma|| + ||\beta - \delta||)^m$ is directionally metric subregular of order m in Y at $((u_0, x_0, u_0, y_0, 0), 0)$ in the direction (u, x, u, x, y) wrt gr $X \times$ gr F*, for all $x \in DX(u_0, x_0)(u)$ and $y \in D^m F((u_0, x_0), y_0)(u, x)$, then (12) becomes an equality for such u.*

Proof Let $v \in D^m H(u_0, y_0)(u)$, then there exist $t_n \downarrow 0$ and $(u_n, v_n) \to (u, v)$ such that $y_0 + t_n^m v_n \in H(u_0 + t_n u_n)$. By the definition of H, there is $x_n \in X(u_0 + t_n u_n)$ satisfying $y_0 + t_n^m v_n \in F(u_0 + t_n u_n, x_n)$. Setting $w_n := \dfrac{x_n - x_0}{t_n}$, then one has

$$x_0 + t_n w_n \in X(u_0 + t_n u_n), \quad y_0 + t_n^m v_n \in F(u_0 + t_n u_n, x_0 + t_n w_n). \tag{13}$$

Suppose $||w_n|| \to +\infty$, it follows from (13) that

$$x_0 + t_n ||w_n|| \left(\frac{w_n}{||w_n||} \right) \in X \left(u_0 + t_n ||w_n|| \left(\frac{u_n}{||w_n||} \right) \right).$$

Since W is finite dimensional, $\{w_n/||w_n||\}$ has a subsequence converging to w with $||w|| = 1$. Moreover, $t_n||w_n|| = ||x_n - x_0||$ tends to 0 (by the compactness of X), then we get $w \in DX(u_0, x_0)(0)$, which contradicts the assumption. Thus, without loss of generality, we assume that w_n converges to some $\overline{w} \in W$. From (13), one obtains $\overline{w} \in DX(u_0, x_0)(u)$ and $v \in D^m F((u_0, x_0), y_0)(u, \overline{w})$.

For the inverse inclusion of (12), let v belongs to the right-hand side of (12), i.e., there exists $x \in DX(u_0, x_0)(u)$ such that $v \in D^m F((u_0, x_0), y_0)(u, x)$. Then, there are $t_n \downarrow 0$ and $(\hat{u}_n, \hat{x}_n, \hat{v}_n) \to (u, x, v)$ satisfying $y_0 + t_n^m \hat{v}_n \in F(u_0 + t_n \hat{u}_n, x_0 + t_n \hat{x}_n)$. Since X has the first order proto-Studniarski derivative at (u_0, x_0), with t_n above, we get $(\overline{u}_n, \overline{x}_n) \to (u, x)$ with $x_0 + t_n \overline{x}_n \in X(u_0 + t_n \overline{u}_n)$.

By the directionally metric subregularity assumption, there exist $\mu > 0$ and $\lambda > 0$ such that for every $t \in (0, \lambda)$ and $(u_1, x_1, u_2, x_2, v_2) \in B_{U \times W}((u, x), \lambda) \times B_{U \times W \times Y}((u, x, v), \lambda)$ with $(u_0 + tu_1, x_0 + tx_1, u_0 + tu_2, x_0 + tx_2, y_0 + t^m v_2) \in$ gr $X \times$ gr F,

$$\begin{aligned} d &\Big((u_0 + tu_1, x_0 + tx_1, u_0 + tu_2, x_0 + tx_2, y_0 + t^m v_2), g^{-1}(0) \cap (\text{gr } X \times \text{gr } F \Big) \le \\ &\mu d \left(0, g\left(u_0 + tu_1, x_0 + tx_1, u_0 + tu_2, x_0 + tx_2, y_0 + t^m v_2\right)\right) \end{aligned} \tag{14}$$

For n large enough, we have $(\overline{u}_n, \overline{x}_n, \hat{u}_n, \hat{x}_n, \hat{v}_n) \in B_{U \times W}((u, x), \lambda) \times B_{U \times W \times Y}((u, x, v), \lambda)$. Thus, from (14), there exist $(\overline{u}'_n, \overline{x}'_n, \hat{u}''_n, \hat{x}''_n, \hat{v}'_n) \in$ gr $X \times$ gr F with $\overline{u}'_n = \hat{u}''_n$ and $\overline{x}'_n = \hat{x}''_n$ for all n such that

$$\begin{aligned} &\left\| \left(u_0 + t_n \overline{u}_n, x_0 + t_n \overline{x}_n, u_0 + t_n \hat{u}_n, x_0 + t_n \hat{x}_n, y_0 + t_n^m \hat{v}_n\right) - \left(\overline{u}'_n, \overline{x}'_n, \hat{u}''_n, \hat{x}''_n, \hat{v}'_n\right) \right\| \\ &\quad \le \mu t_n^m (||\overline{u}_n - \hat{u}_n|| + ||\overline{x}_n - \hat{x}_n||)^m, \end{aligned}$$

which implies

$$\left\|u_0 + t_n\overline{u}_n - \overline{u}'_n\right\| \le \mu t_n^m \left(||\overline{u}_n - \hat{u}_n|| + ||\overline{x}_n - \hat{x}_n||\right)^m,$$
$$\left\|x_0 + t_n\overline{x}_n - \overline{x}'_n\right\| \le \mu t_n^m \left(||\overline{u}_n - \hat{u}_n|| + ||\overline{x}_n - \hat{x}_n||\right)^m,$$

$$\left\|y_0 + t_n^m\hat{x}_n - \hat{v}'_n\right\| \le \mu t_n^m \left(||\overline{u}_n - \hat{u}_n|| + ||\overline{x}_n - \hat{x}_n||\right)^m.$$

Thus,

$$\left\|\frac{\overline{u}'_n - u_0}{t_n} - \overline{u}_n\right\| \le \mu t_n^{m-1} \left(||\overline{u}_n - \hat{u}_n|| + ||\overline{x}_n - \hat{x}_n||\right)^m, \left\|\frac{\overline{x}'_n - x_0}{t_n} - \overline{x}_n\right\| \le \mu t_n^{m-1} \left(||\overline{u}_n - \hat{u}_n|| + ||\overline{x}_n - \hat{x}_n||\right)^m,$$
$$\left\|\frac{\hat{v}'_n - y_0}{t_n^m} - \hat{v}_n\right\| \le \mu \left(||\overline{u}_n - \hat{u}_n|| + ||\overline{x}_n - \hat{x}_n||\right)^m.$$

By setting $\widetilde{u}_n := \frac{\overline{u}'_n - u_0}{t_n}$, $\widetilde{x}_n := \frac{\overline{x}'_n - x_0}{t_n}$, and $\widetilde{v}_n := \frac{\hat{v}'_n - y_0}{t_n^m}$, then $\widetilde{u}_n \to u$, $\widetilde{x}_n \to x$, $\widetilde{v}_n \to v$ and one gets

$$x_0 + t_n\widetilde{x}_n = \overline{x}'_n \in X(\overline{u}'_n) = X(u_0 + t_n\widetilde{u}_n),$$

$$y_0 + t_n^m\widetilde{v}_n = \hat{v}'_n \in F(\hat{u}''_n, \hat{x}''_n) = F(\overline{u}'_n, \overline{x}'_n) = F(u_0 + t_n\widetilde{u}_n, x_0 + t_n\widetilde{x}_n).$$

Hence, $v \in D^m H(u_0, y_0)(u)$ and the proof is completed. □

Theorem 4.1 *Let $(u_0, y_0) \in \mathrm{gr}\, S$ and $u \in X$. Assume that the assumptions in Propostion 4.3 are satisfied. If all conditions in Proposition 4.2 are fulfilled wrt the maps H and S. Then,*

$$D^m S(u_0, y_0)(u) = \mathrm{WMin}_C \left(\bigcup_{x \in DX(u_0, x_0)(u)} D^m F\left((u_0, x_0), y_0\right)(x, u) \right).$$

Proof It follows from Propositions 4.2 and 4.3. □

By virtue of Theorem 4.1, sensitivity analysis of the multiobjective optimization problem mentioned in [9] is obtained in forms of Studniarski derivatives.

To illustrate Theorem 4.1, we provide the the following example.

Example 4.3 Let $U = \mathbb{R}^2$, $W = Y = \mathbb{R}$, $K = \mathbb{R}^2_+ \subseteq U$, and $C = \mathbb{R}_+$Consider two set-valued maps $X : U \rightrightarrows W$ and $F : U \times W \rightrightarrows Y$ defined by $X(u) := \{x \in W | -(u_1+u_2) \leq x \leq u_1+u_2\}$ for all $u = (u_1, u_2) \in K$ and

$$F(u,x) := \begin{cases} \emptyset, & \text{if } u_1, u_2 \in \left\{\frac{1}{n^2} | n \in \mathbb{N}\right\} \\ (u_1+u_2)^2 + x^2, & \text{otherwise.} \end{cases}$$

Then,

$$H(u) := \{y \in Y | y \in F(u,x), x \in X(u)\} =$$

$$= \begin{cases} \emptyset, & \text{if } u_1, u_2 \in \left\{\frac{1}{n^2} | n \in \mathbb{N}\right\} \\ \{y \in Y | (u_1+u_2)^2 \leq y \leq 2(u_1+u_2)^2\}, & \text{otherwise} \end{cases}$$

and

$$S(u) := \mathrm{WMin}_C\, H(u) = \begin{cases} \emptyset, & \text{if } u_1, u_2 \in \left\{\frac{1}{n^2} | n \in \mathbb{N}\right\} \\ (u_1+u_2)^2, & \text{otherwise.} \end{cases}$$

Let $(u_0, x_0, y_0) = (0_{\mathbb{R}^2}, 0, 0)$. We can check that all conditions in Theorem 4.1 are satisfied. By calculating, we have for all $(u, x) \in K \times W$,

$$\begin{aligned} DX(u_0, x_0)(u) &= \{x \in W | -(u_1+u_2) \leq x \leq (u_1+u_2)\}, \\ D^2F((u_0, x_0), y_0)(u, x) &= (u_1+u_2)^2 + x^2, \\ D^2H(u_0, y_0)(u) &= \{y \in Y | (u_1+u_2)^2 \leq y \leq 2(u_1+u_2)^2\}, \\ D^2S(u_0, y_0)(u) &= (u_1+u_2)^2. \end{aligned}$$

Thus, ones get for all $u \in K$,

$$\begin{aligned} D^2S(u_0, y_0)(u) &= \mathrm{WMin}_C\, D^2H(u_0, y_0)(u) \\ &= \mathrm{WMin}_C \left(\bigcup_{x \in DX(u_0,x_0)(u)} D^m F\left((u_0, x_0), y_0\right)(x, u) \right). \end{aligned}$$

On the other hand, H does not have the second-order semi-Studniarski derivative at (u_0, y_0). So, Proposition 5.3 in [9] cannot be used for this case.

5 Perspectives

For further works, we think that Sect. 4 can be considered for other kinds of solutions of (CSOP), for example, quasi efficient solutions or quasi-relative efficent solutions in case of the ordering cone C being nonsolid (such as the positive cones in the spaces l^p and $L^p(\Omega)$). On the other hand, one can apply these results to other models, e.g., parametric vector variational inequalities and parametric vector equilibria (see [14]). Besides, discussions on relationships between our results and those developed and summarized in the recent book [16], which are closely related to the topics investigated in the paper, may be interesting problems.

Along with the derivative approach to multiobjective optimization and sensitivity analysis, there is well-developed coderivative approach. The reader is referred to [22] for its applications to set-valued optimization in the paper. Thus, for another possible development, we can discuss some relationships between the primal derivative approach used in this paper and the dual coderivative in set-valued optimization.

Acknowledgements This research was supported by Vietnam National University Hochiminh City (VNU-HCM) under grant number B2015-28-03. The author is grateful to an anonymous referee for his valuable comments.

References

1. Anh, N.L.H.: Higher-order optimaltity conditions in set-valued optimization using Studniarski derivatives and applications to duality. Positivity **18**, 449–473 (2014)
2. Anh, N.L.H., Khanh, P.Q., Tung, L.T.: Higher-order radial derivatives and optimality conditions in nonsmooth vector optimization. Nonlinear Anal. TMA **74**, 7365–7379 (2011)
3. Anh, N.L.H., Khanh, P.Q.: Higher-order optimality conditions in set-valued optimization using radial sets and radial derivatives. J. Global Optim. **56**, 519–536 (2013)
4. Anh, N.L.H., Khanh, P.Q.: Higher-order optimality conditions for proper efficiency in nonsmooth vector optimization using radial sets and radial derivatives. J. Global Optim. **58**, 693–709 (2014)
5. Anh, N.L.H., Khanh, P.Q.: Calculus and applications of Studniarski's derivatives to sensitivity and implicit function theorems. Control Cybern. **43**, 33–57 (2014)
6. Anh, N.L.H., Khanh, P.Q., Tung, L.T.: Variational sets: calculus and applications to nonsmooth vector optimization. Nonlinear Anal. **74**, 2358–2379 (2011)
7. Aubin, J.-P., Frankowska, H.: Set-Valued Analysis. Birkhauser, Boston (1990)
8. Cominetti, R.: Metric regularity, tangent sets, and second-order optimality conditions. Appl. Math. Optim. **21**, 265–287 (1990)
9. Diem, H.T.H., Khanh, P.Q., Tung, L.T.: On higher-order sensitivity analysis in nonsmooth vector optimization. J. Optim. Theory Appl. **162**, 463–488 (2014)
10. Durea, M., Strugariu, R.: Calculus of tangent sets and derivatives of set-valued maps under metric subregularity conditions. J. Glob. Optim. **56**, 587–603 (2013)
11. Fiacco, A.V.: Introduction to sensitivity and stability analysis in nonlinear programming. Academic Press, New York (1983)
12. Gfrerer, H.: On directional metric regularity, subregularity and optimality conditions for nonsmooth mathematical programs. Set-Valued Var. Anal. **21**, 151–176 (2013)
13. Gfrerer, H.: On directional metric subregularity and second-order optimality conditions for a class of nonsmooth mathematical programs. SIAM J. Optim. **23**, 632–665 (2013)
14. Giannessi, F.: Vector Variational Inequalities and Vector Equilibria: Mathematical Theories. Kluwer Academic, Dordrecht (2000)
15. Jahn, J., Khan, A.A.: Some calculus rules for contingent epiderivatives. Optimization **52**, 113–125 (2003)
16. Khan, A.A., Tammer, C., Zălinescu, C.: Set-Valued Optimization: An Introdution with Applications. Springer, Berlin (2015)
17. Khanh, P.Q., Tuan, N.D.: Variational sets of multivalued mappings and a unified study of optimality conditions. J. Optim. Theory Appl. **139**, 45–67 (2008)
18. Kuk, H., Tanino, T., Tanaka, M.: Sensitivity analysis in vector optimization. J. Optim. Theory Appl. **89**, 713–730 (1996)

19. Li, S.J., Chen, C.R.: Higher-order optimality conditions for Henig efficient solutions in set-valued optimization. J. Math. Anal. Appl. **323**, 1184–1200 (2006)
20. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, vol.I: Basic Theory. Springer, Berlin (2006)
21. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, vol.II: Applications. Springer, Berlin (2006)
22. Mordukhovich, B.S.: Multiobjective optimization with equilibrium constraints. Math. Program. **117**, 331–354 (2009)
23. Penot, J.P.: Differentiability of relations and differential stability of perturbed optimization problems. SIAM J. Control Optim. **22**, 529–551 (1984)
24. Rockafellar, R.T.: Proto-differentiability of set-valued mapping and its applications in optimization. Ann. Inst. H. Poincaré **6**, 449–482 (1989)
25. Rockafellar, R.T., Wets, R.J.B.: Variational Analysis, 3rdedn. Springer, Berlin (2009)
26. Shi, D.S.: Sensitivity analysis in convex vector optimization. J. Optim. Theory Appl. **77**, 145–159 (1993)
27. Studniarski, M.: Necessary and sufficient conditions for isolated local minima of nonsmooth functions. SIAM J. Control Optim. **25**, 1044–1049 (1986)
28. Sun, X.K., Li, S.J.: Lower Studniarski derivative of the perturbation map in parametrized vector optimization. Optim. Lett. **5**, 601–614 (2011)
29. Sun, X.K., Li, S.J.: Weak lower Studniarski derivative in set-valued optimization. Pacific J. Optim. **8**, 307–320 (2012)
30. Tanino, T.: Sensitivity analysis in multiobjective optimization. J. Optim. Theory Appl. **56**, 479–499 (1988)
31. Tanino, T.: Stability and sensitivity analysis in convex vector optimization. SIAM J. Control Optim. **26**, 521–536 (1988)
32. Wang, Q.L., Li, S.J., Teo, K.L.: Higher-order optimality conditions for weakly efficient solutions in nonconvex set-valued optimiaztion. Optim. Lett. **4**, 425–437 (2010)