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The order continuity in ordered algebras

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Abstract Let A be an ordered algebra with a unit $\mathbf{e}$ and a cone A^+. The class of order continuous elements A_n of A is introduced and studied. If $A = L(E)$, where E is a Dedekind complete Riesz space, this class coincides with the band $L_\mathrm{n}(E)$ of all order continuous operators on E. Special subclasses of A_n are considered. Firstly, the order ideal $A_\mathbf{e}$ generated by $\mathbf{e}$. It is shown that $A_\mathbf{e}$ can be embedded into the algebra of continuous functions and, in particular, is a commutative subalgebra of A. If A is an ordered Banach algebra with normal cone A^+ then $A_\mathbf{e}$ is an AM-space and is closed in A. Secondly, the notion of an orthomorphism in the ordered algebra A is introduced. Among others, the conditions under which orthomorphisms are order continuous, are considered. In the second part, the main emphasis will be on the case of an ordered C^*-algebra A and, in particular, on the case of the algebra $B(H)$, where H is an ordered Hilbert space with self-adjoint cone H^+. If the cone A^+ is normal then every element of $A_\mathbf{e}$ is hermitian. In H the operations are introduced which coincide with the lattice ones when H is a Riesz space. It is shown that every regular $T \in B(H)$ is an order continuous element and operators $T \in (B(H))_I$ have properties which are analogous to the properties of orthomorphisms on Riesz spaces.

Keywords Ordered algebra · Order continuity · Center · Ordered C^*-algebra · Ordered Hilbert space · Orthomorphism

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1 Introduction

Let A be an (real or complex) algebra with an algebraic unit $\mathbf{e}$ and let A^+ be a (convex) cone in A. As usual, for elements $a, b \in A$, the symbol $a \geq b$ (or $b \leq a$) means $a - b \in A^+$. Under this ordering, A is an ordered linear space. From the definition of the cone, it follows that the inequalities $a \geq b$ and $b \geq a$ imply $a = b$ for all $a, b \in A$ and $\lambda x + \mu y \geq 0$ for all elements $x, y \in A^+$ and all scalars $\lambda, \mu \in \mathbb{R}^+$. The elements of A^+ are called *positive*. Throughout, we will tacitly assume $A \neq \{0\}$. If $\mathbf{e} \geq 0$ and the inequalities $a, b \geq 0$ imply $ab \geq 0$ then A is called an *ordered algebra*. If, in addition, an ordered algebra A is a (normed) Banach algebra with a closed cone A^+ then A is called an *ordered* (*normed*) *Banach algebra*. An important example of an ordered algebra is the algebra of (linear) operators on an ordered linear space. Namely, the algebra $L(E)$ of all operators on some ordered linear space E with a cone E^+ and the natural ordering induced by E^+ is an ordered algebra with unit I, where I is an identity operator, if and only if the cone E^+ is generating, i.e., the span of E^+ is equal to E. If E is an ordered (normed) Banach space with closed cone E^+ then the algebra $B(E)$ of all bounded operators on E is an ordered (normed) Banach algebra if and only if the span of E^+ is dense in E. In particular, if E is a Riesz space (Banach lattice) then the algebra $L(E)$ $(B(E))$ is an ordered (Banach) algebra.

The study of ordered Banach algebras was initiated in [8,9]. In these papers and in a number of subsequent ones the main emphasis was on the study of spectral properties of positive elements. Our paper extends this line of research. However, our emphasis will be on the notion of order continuity both in general ordered algebras and in some special subclasses of them.

The paper is organized as follows. In the second section elementary properties of order continuous elements in ordered algebras are considered. The third section is devoted to the investigation of special subclasses of order continuous elements, namely, the center $A_{\mathbf{e}}$ of the algebra A and the class of orthomorphisms of A. In the last section the results obtained in the preceding ones are employed to study ordered C^*-algebras and a special subclass of them, namely, the algebra $B(H)$, where H is an ordered Hilbert space.

For any unexplained terminology, notations, and elementary properties of cones and ordered linear spaces, we refer the reader to [5]. For information on the theory of Riesz spaces, Banach lattices, and operators on these spaces, we suggest [1,4] (see also [12,13]). More details on elementary properties of Banach algebras can be found in [10] (see also [6]).

2 The order continuous elements

First of all, we mention that below we will use the following definition of order convergence. The net $\{x_\alpha\}$ in an ordered linear space E is said to be *order convergent* to an element $x \in E$, in symbols, $x_\alpha \xrightarrow{o} x$, whenever there exist two nets $\{y_\alpha\}$ and $\{z_\alpha\}$ in E (with the same index set) satisfying $y_\alpha \uparrow 0$, $z_\alpha \downarrow 0$, and $y_\alpha \leq x - x_\alpha \leq z_\alpha$ for all α.

Let A be an ordered algebra. An element $p \in A$ is called an *order idempotent* (see [2]) if $0 \leq p \leq \mathbf{e}$ and $p^2 = p$. Under the partial ordering induced by A, the set of all order idempotents $\mathbf{OI}(A)$ of A is a Boolean algebra and its lattice operations satisfy the identities $p \wedge q = pq$ and $p \vee q = p + q - pq$ for all $p, q \in \mathbf{OI}(A)$ (see [2]). For $p \in \mathbf{OI}(A)$, we put $p^{\mathrm{d}} = \mathbf{e} - p$. Obviously, $p^{\mathrm{d}} \in \mathbf{OI}(A)$. If $a \in A$ and the modulus $|a|$ of a exists then $p|a| = |pa|$ and $|a|p = |ap|$ for $p \in \mathbf{OI}(A)$.

Recall that an ordered linear space E is said to be *Dedekind complete* whenever every nonempty subset that is bounded from above has a supremum. The next result will be needed later.

Lemma 1 *Let A be a Dedekind complete ordered algebra, let $\{p_\alpha\}$ be a net in $\mathbf{OI}(A)$, and let $p \in A$. Then $p_\alpha \downarrow p$ in A if and only if $p_\alpha \downarrow p$ in $\mathbf{OI}(A)$.*

Proof Necessity. The inclusion $p \in \mathbf{OI}(A)$ should be checked. Evidently, $0 \leq p \leq \mathbf{e}$. Fix an index α. For an arbitrary index $\beta \geq \alpha$, we have

$$0 \leq p - p_\alpha p \leq p - p_\beta p \leq p_\beta - p_\beta p \leq p_\beta (p_\beta - p) \leq p_\beta - p \downarrow_{\beta \geq \alpha} 0$$

in A, whence $p = p_\alpha p$. Consequently, $0 \leq p - p^2 = (p_\alpha - p)p \leq p_\alpha - p$. Thus, $p^2 = p$.

Sufficiency. Since A is Dedekind complete, we find an element $a \in A$ satisfying $p_\alpha \downarrow a$ in A. As shown above, $p_\alpha \downarrow a$ in $\mathbf{OI}(A)$ and so $a = p$. □

An element $a \in A$ is said to be a *regular element* whenever it can be written as a difference of two positive elements, i.e., if $a = a_1 - a_2$ with $a_1, a_2 \in A^+$. The collection of all regular elements of A will be denoted by A_{r}. Obviously, A_{r} is a *real* ordered algebra with cone A^+. An element $a \in A$ is said to be *left order continuous* (or *l-order continuous*) if $p_\alpha a \xrightarrow{o} 0$ in A whenever $p_\alpha \downarrow 0$ in $\mathbf{OI}(A)$; the notion of r-order continuity can be analogously defined. The collection of all l- and r-order continuous elements of A will be denoted by $A_{\mathrm{n_l}}$ and $A_{\mathrm{n_r}}$, respectively. Evidently, $A_{\mathrm{n_l}}$ and $A_{\mathrm{n_r}}$ are *real* linear spaces and, if A is real, are linear subspaces of A. The inclusion $A_{\mathrm{n_l}} \cup A_{\mathrm{n_r}} \subseteq A_{\mathrm{r}}$ holds. Indeed, let $a \in A_{\mathrm{n_l}} \cup A_{\mathrm{n_r}}$. We consider a sequence $\{p_n\}$ in $\mathbf{OI}(A)$ such that $p_1 = \mathbf{e}$ and $p_n = 0$ for all $n \geq 2$. Clearly, $p_n \downarrow 0$ and, hence, $a \leq b$ for some $b \in A^+$. Finally, $a = b - (b - a) \in A_{\mathrm{r}}$. Next, an element $a \in A$ which is both left and right order continuous is called *order continuous*. The collection of all order continuous elements of A will be denoted by A_{n}. Obviously, $A_{\mathrm{n}} = A_{\mathrm{n_l}} \cap A_{\mathrm{n_r}}$. The notion of order continuity for the case of positive elements in ordered Banach algebras was introduced in [2].

Example 2 Let E be a (Archimedean; real or complex) Riesz space. Recall that an operator T on E is said to be *order continuous* (see [4, p. 46]) whenever $x_\alpha \xrightarrow{o} 0$ in E implies $Tx_\alpha \xrightarrow{o} 0$. In this case, we refer to the order continuity of an *operator* T on E. The collection of all order continuous operators on E is denoted by $L_{\mathrm{n}}(E)$. Every order continuous operator is order bounded (see [4, p. 46]) and, hence, is norm continuous when E is a Banach lattice (see [1, p. 22]), i.e., $L_{\mathrm{n}}(E) \subseteq B(E)$. On the other hand, the algebra $L(E)$ of all operators on a Riesz space E is an ordered algebra. Consequently, in $L(E)$ order continuous elements can be considered in the sense defined above. In this case, we refer to the order continuity of an *element* T in the algebra $L(E)$. These two cases may differ.

Next, if we consider the Riesz space $E = C[0, 1]$ of all continuous functions on the interval $[0, 1]$ and the algebra $A = L(E)$ then, as is well known, the identity $\mathbf{OI}(A) = \{0, \mathbf{e}\}$ holds and, hence, $A_{\mathrm{n_r}} = A_{\mathrm{n_l}} = A_{\mathrm{n}} = A_{\mathrm{r}}$. However, since the band of all order continuous functionals $E_{\mathrm{n}}^{\sim} = \{0\}$, we have $L_{\mathrm{n}}(E) \subsetneq L_{\mathrm{r}}(E) = A_{\mathrm{r}}$. On the other hand, an order continuous operator $T \in L_{\mathrm{n}}(E)$ need not be an order continuous element of the algebra $L(E)$, i.e., the inclusion $L_{\mathrm{n}}(E) \subseteq (L(E))_{\mathrm{n}}$ does not hold in the general case. To see this, let us consider the subset

$$K = \{-\frac{1}{k} : k \in \mathbb{N}\} \cup [0, 1]$$

of the set $\mathbb{R}$ and the Riesz space $E = C(K)$. Let $A = L(E)$. Put $K_n = \{-\frac{1}{k} : k \geq n\} \cup [0, 1]$. Define the operators P_n on E via the formula $P_n x = \chi_{K_n} \cdot x$, where χ_{K_n} is the characteristic function of the set K_n. Obviously, $P_n \in \mathbf{OI}(A)$. We have $P_n \downarrow 0$ in $\mathbf{OI}(A)$. Indeed, if $P_n \geq P \in \mathbf{OI}(A)$ for all n then $(P\mathbb{1})(-\frac{1}{n}) = 0$ for all n, where $\mathbb{1}$ is the constant function one, and, hence, $(P\mathbb{1})(0) = 0$. Therefore, using the relations $P\mathbb{1} + (I - P)\mathbb{1} = \mathbb{1}$ and $P\mathbb{1} \perp (I - P)\mathbb{1}$ and the connectedness of the segment $[0, 1]$, we conclude $P\mathbb{1} = 0$. Finally, $P = 0$. On the other hand, as is easy to see, the relation $P_n \downarrow 0$ does not hold in $L(E)$. Consequently, the identity operator is not an r- or an l-order continuous element.

In the case of a Dedekind complete Riesz space E, the correlation between the notions of order continuity mentioned above can be made more precisely. Namely, the following two statements hold:

(a) *An operator T on E is an l-order continuous element in $L(E)$ if and only if T is a regular operator on E.*
(b) *For an operator T on E the following three statements are equivalent:*
  (i) *T is an r-order continuous element in$L(E)$;*
  (ii) *T is an order continuous element in $L(E)$;*
  (iii) *T is an order continuous operator.*

Thus, according to parts (a) and (b), we have the identities

$$(L(E))_{\mathrm{n_l}} = L_{\mathrm{r}}(E) = (L(E))_{\mathrm{r}} \quad \text{and} \quad (L(E))_{\mathrm{n_r}} = (L(E))_{\mathrm{n}} = L_{\mathrm{n}}(E).$$

For the proof of the sufficiency in (a), we consider a net $\{P_\alpha\}$ of order projections on E satisfying $P_\alpha \downarrow 0$ in $\mathbf{OI}(L(E))$. In view of Lemma 1, $P_\alpha \downarrow 0$ in $L(E)$ and, hence, $P_\alpha x \xrightarrow{o} 0$ for all $x \in E$. Therefore, $P_\alpha S \downarrow 0$ for every positive operator S on E. It remains to observe the validity of the inequalities $-P_\alpha|T| \leq P_\alpha T \leq P_\alpha|T|$, where $|T|$ is the modulus of T existing by the F. Riesz–Kantorovich theorem (see [4, p. 14]).

Now we will check (b). The implication (ii) $\Longrightarrow$ (i) is clear and the implication (iii) $\Longrightarrow$ (ii) can be analogously proven to the assertion (a) as for an arbitrary operator $S \in L(E)$ the inclusions $S \in L_{\mathrm{n}}(E)$ and $|S| \in L_{\mathrm{n}}(E)$ are equivalent. We shall show (i) $\Longrightarrow$ (iii). Assume that $T \in (L(E))_{\mathrm{n_r}}$. For every order projection P, we have the identity $|T|P = |TP|$, whence the relation $|T| \in (L(E))_{\mathrm{n_r}}$ at once follows. Therefore (see [2, Example 2.9(**a**)]), $|T| \in L_{\mathrm{n}}(E)$ and so $T \in L_{\mathrm{n}}(E)$.

Considering the algebra $A = L(E)$ with the multiplication $S \star T = TS$, we obtain the ordered algebra satisfying $A_{\mathrm{n_r}} = A_{\mathrm{r}}$ and $A_{\mathrm{n_l}} = A_{\mathrm{n}}$. □

Below, we will establish elementary properties of order continuous elements in ordered algebras. For the case of order continuous operators, the results which are analogous to those obtained below can be found, e.g., in [4, §1.4]. As the preceding example shows, the assumption about the Dedekind completeness which will be repeatedly used in this and next sections, is quite natural if we want to preserve nice properties of order continuous operators for the case of order continuous elements. For this reason, we recall that for an arbitrary ordered algebra A (or even an order linear space E) the following statements which will be tacitly used in the future hold: (a) A is a Dedekind complete if and only if A_{r} is Dedekind complete; (b) If A is Dedekind complete then A_{r} is a Riesz space; (c) If A is a Dedekind complete ordered Banach algebra then the cone A^+ is normal (see [5, p. 109, Exercise 10(b)]). We will restrict ourselves to the consideration of l-order continuity. However, the results obtained below can be extended without difficulty to the case of (r-)order continuity.

Recall that if E is an ordered linear space and an element $x \in E^+$ then the *ideal* E_x *generated by* x is the set

$$E_x = \{y \in E : -\lambda x \le y \le \lambda x \text{ for some } \lambda \in \mathbb{R}^+\}. \tag{1}$$

Under the algebraic operations and the ordering induced by E, E_x is a real ordered linear space and, if E is real, is a linear subspace of E. If A is an ordered algebra and $b \in A^+_{\mathrm{n_l}}$ then the ideal $A_b \subseteq A_{\mathrm{n_l}}$. Next, for arbitrary $a \in A$, we define the set

$$N^{\mathrm{l}}_a = \{p \in \mathbf{OI}(A) : pa = 0\}.$$

Clearly, N^{l}_a is a solid subset of $\mathbf{OI}(A)$, i.e., if $p \le q$ with $p \in \mathbf{OI}(A)$ and $q \in N^{\mathrm{l}}_a$ then $p \in N^{\mathrm{l}}_a$.

The following criterion of an l-order continuity holds.

Theorem 3 *Let A be an ordered algebra such that A_{r} is a Riesz space and let b be an element in A^+. The following statements are equivalent*:

(a) $b \in A_{\mathrm{n_l}}$;
(b) *For every $a \in A_b$ the set N^{l}_a is order closed in $\mathbf{OI}(A)$, i.e., the relations $p_\alpha \uparrow p$ in $\mathbf{OI}(A)$ and $p_\alpha \in N^{\mathrm{l}}_a$ for all α imply $p \in N^{\mathrm{l}}_a$*;
(c) *For every $a \in A_b$ and for every net $\{p_\alpha\}$ the relations $p_\alpha \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$ and $p_\alpha a = 0$ for all α imply $a = 0$*;
(d) *If $a \in A^+$ and there exists a net $\{p_\alpha\}$ satisfying $p_\alpha \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$ and $p_\alpha a = 0$ for all α then the infimum $a \wedge b = 0$.*

Proof (a) $\Longrightarrow$ (b) Let $-\lambda b \le a \le \lambda b$ for some $\lambda \ge 0$ and let $p_\alpha \uparrow p$ in $\mathbf{OI}(A)$ with $p_\alpha \in N^{\mathrm{l}}_a$. We have $pa = (p - p_\alpha)a \le \lambda(p - p_\alpha)b \downarrow 0$, whence $pa \le 0$. Similarly, $pa \ge 0$. Finally, $pa = 0$.

The implication (b) $\Longrightarrow$ (c) is obvious.

(c) $\Longrightarrow$ (a) Let $p_\alpha \downarrow 0$ in $\mathbf{OI}(A)$ and let $p_\alpha b \downarrow \geq c$. Since A_{r} is a Riesz space, we have $p_\alpha b \downarrow \geq c^+$. Clearly, $c^+ \in A_b$, $p_\alpha^{\mathrm{d}} c^+ = 0$, and $p_\alpha^{\mathrm{d}} \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$. Thus, $c^+ = 0$ and so $c \leq 0$.

(a) $\Longrightarrow$ (d) Let $c \in A$ and $a, b \geq c$. Then $0 = p_\alpha a \geq p_\alpha c$, whence $c \leq p_\alpha^{\mathrm{d}} c \leq p_\alpha^{\mathrm{d}} b \downarrow 0$ and so $c \leq 0$.

(d) $\Longrightarrow$ (c) Consider an element a and a net $\{p_\alpha\}$ satisfying $a \in A_b$, $p_\alpha \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$, and $p_\alpha a = 0$. Then $p_\alpha |a| = 0$, whence $|a| \wedge b = 0$. Since $|a| \leq \beta b$ for some $\beta \geq 0$, we have $|a| = 0$ and so $a = 0$. □

Theorem 4 *Let A be an ordered algebra. Then the subspace $A_{\mathrm{n_l}}$ is order closed in A_{r}. In particular, if A_{r} is a Riesz space then $A_{\mathrm{n_l}}$ is a band in A_{r}.*

Proof Consider the net $\{c_\beta\}$ in $A_{\mathrm{n_l}}$ satisfying in $0 \leq c_\beta \uparrow c$ in A_{r}. The *order closedness* means the validity of the inclusion $c \in A_{\mathrm{n_l}}$. To see this, let $d \leq p_\alpha c$ for all α, where $p_\alpha \downarrow 0$ in $\mathbf{OI}(A)$. We have

$$d \leq p_\alpha c = p_\alpha(c - c_\beta) + p_\alpha c_\beta \leq c - c_\beta + p_\alpha c_\beta,$$

whence $d \leq c - c_\beta$ and so $d \leq 0$. Finally, $p_\alpha c \downarrow 0$.

Now let A_{r} be a Riesz space and let $p_\alpha \downarrow 0$ in $\mathbf{OI}(A)$. If $|a| \leq |b|$ and $b \in A_{\mathrm{n_l}}$ then $|p_\alpha a| = p_\alpha |a| \leq p_\alpha |b| = |p_\alpha b| \xrightarrow{o} 0$ and, hence, $a \in A_{\mathrm{n_l}}$. Thus, $A_{\mathrm{n_l}}$ is an ideal. From the order closedness of $A_{\mathrm{n_l}}$, it follows that it is a band. □

The conditions which guarantee the order continuity of every regular element are discussed in the next theorem.

Theorem 5 *Let A be an ordered algebra such that A_{r} is a Riesz space. The following statements are equivalent*:

(a) $A_{\mathrm{r}} = A_{\mathrm{n_l}}$;
(b) *For every $a \in A_{\mathrm{r}}$ the equalities $p_\alpha a = 0$ for all α, where $p_\alpha \uparrow p$ in $\mathbf{OI}(A)$, imply $pa = 0$;*
(c) *For every non-zero element $a \in A_{\mathrm{r}}$ there exists a non-zero idempotent $p \in \mathbf{OI}(A)$ satisfying $qa \neq 0$ for $q \in \mathbf{OI}(A)$, $0 < q \leq p$.*

Proof The implication (a) $\Longrightarrow$ (b) is obvious.

(b) $\Longrightarrow$ (c) Proceeding by contradiction, for every non-zero $p \in \mathbf{OI}(A)$, we find an idempotent $q' \in \mathbf{OI}(A)$ satisfying the relations $q'a = 0$ and $0 < q' \leq p$. Consider the set $D = \{q \in \mathbf{OI}(A) : qa = 0\}$. We have $D \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$. Indeed, if $q \leq q_0 \in \mathbf{OI}(A)$ for all $q \in D$ and $q_0 < \mathbf{e}$ then for some $q_0' \in \mathbf{OI}(A)$ the relations $q_0' a = 0$ and $0 < q_0' \leq \mathbf{e} - q_0$ hold. Thus, $q_0' \in D$ and, hence, $q_0' \leq q_0$. Therefore, $q_0' = 0$, which is impossible. Hence, $D \uparrow \mathbf{e}$ and so $a = 0$, a contradiction.

(c) $\Longrightarrow$ (a) Let $a \in A^+$, let $p_\alpha \downarrow 0$ in $\mathbf{OI}(A)$, and let $p_\alpha a \geq c$ for all α. Since A_{r} is a Riesz space, we have $p_\alpha a \geq c^+$. Clearly, $p_\alpha^{\mathrm{d}} c^+ = 0$. If $c^+ > 0$ then there exists a non-zero idempotent $p \in \mathbf{OI}(A)$ satisfying $qc^+ > 0$ for all $q \in \mathbf{OI}(A)$, $0 < q \leq p$. Pick an index α_0 such that the inequalities $0 < p_{\alpha_0}^{\mathrm{d}} p \leq p$ hold. We have $p_{\alpha_0}^{\mathrm{d}} pc^+ > 0$, a contradiction. □

We mention at once the next result (see also, e.g., Corollary 23(b) and Proposition 32) which can be employed to ordered algebras that are isomorphic on the space of continuous functions $C(K)$ (see Theorems 15, 22, 44).

Proposition 6 *Let A be a Dedekind complete ordered algebra. If an element $a \in A^+$ is dominated by an invertible element b such that $b^{-1} \in A^+$ then a is order continuous.*

Proof Let $p_\alpha \downarrow 0$ in $\mathbf{OI}(A)$ and let $p_\alpha a \downarrow\geq c$ with $c \in A$. Obviously, $p_\alpha b \geq c$ and, hence, $p_\alpha \geq cb^{-1}$. By Lemma 1, $cb^{-1} \leq 0$ and so $c \leq 0$. Finally, $a \in A_{\mathrm{n_l}}$. Analogously, $a \in A_{\mathrm{n_r}}$. □

Let A be a Dedekind complete ordered algebra. Then the band $A_{\mathrm{n_l}}$ in A_{r} is a projection band. Therefore, the representation $A_{\mathrm{r}} = A_{\mathrm{n_l}} \oplus A_{\mathrm{s_l}}$ holds, where $A_{\mathrm{s_l}}$ is the band of all elements in A_{r} that are disjoint from $A_{\mathrm{n_l}}$, i.e., $A_{\mathrm{s_l}} = A^{\mathrm{d}}_{\mathrm{n_l}}$ and its members will be referred to as *l-singular elements*. Similarly, the bands $A_{\mathrm{s_r}}$ and A_{s} can be defined. In particular, each element $a \in A_{\mathrm{r}}$ has a unique decomposition $a = a_{\mathrm{n}} + a_{\mathrm{s}}$, where $a_{\mathrm{n}} \in A_{\mathrm{n}}$, $a_{\mathrm{s}} \in A_{\mathrm{s}}$, and $a_{\mathrm{n}} \perp a_{\mathrm{s}}$.

Lemma 7 *In a Dedekind complete ordered algebra A the set*

$$A_{\mathrm{s_l^\circ}} = \{a \in A_{\mathrm{r}} : p_\alpha a = 0 \textit{ for some } p_\alpha \uparrow \mathbf{e} \quad \textit{in} \quad \mathbf{OI}(A)\} \tag{2}$$

is an ideal in A_{r} which is order dense in $A_{\mathrm{s_l}}$.

In particular, if $A_{\mathrm{s_l^\circ}} = \{0\}$ then $A_{\mathrm{r}} = A_{\mathrm{n_l}}$.

Proof Let $a, b \in A_{\mathrm{s_l^\circ}}$ and let $\{p'_\alpha\}$ and $\{p''_\beta\}$ be two nets such that $p'_\alpha \uparrow \mathbf{e}$ and $p''_\beta \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$ and $p'_\alpha a = p''_\beta b = 0$. Then the net $\{p'_\alpha p''_\beta\}_{(\alpha,\beta)}$ satisfies the relations $p'_\alpha p''_\beta \uparrow \mathbf{e}$ and $p'_\alpha p''_\beta(\lambda a + \mu b) = 0$ for all scalars $\lambda, \mu \in \mathbb{R}$. Thus, $A_{\mathrm{s_l^\circ}}$ is a linear subspace of A_{r} and, hence, is an ideal in A_{r}. Now we shall establish the inclusion $A_{\mathrm{s_l^\circ}} \subseteq A_{\mathrm{s_l}}$. Indeed, for $a \in A_{\mathrm{s_l^\circ}}$ the equalities $0 = p'_\alpha|a| = p'_\alpha|a_{\mathrm{n_l}}| + p'_\alpha|a_{\mathrm{s_l}}|$ hold. Therefore, $p'_\alpha a_{\mathrm{n_l}} = 0$, whence $a_{\mathrm{n_l}} = 0$ and so $a \in A_{\mathrm{s_l}}$. Next, let $0 < c \in A_{\mathrm{s_l}}$. Taking into account Theorem 3(c), we find an element $d \in A_c$ and a net $\{p_\alpha\}$ satisfying $p_\alpha \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$, $p_\alpha d = 0$, and $0 < d \leq \xi c$ for some $\xi \geq 0$. We have $\frac{d}{\xi} \in A^+_{\mathrm{s_l^\circ}}$ and $\frac{d}{\xi} \leq c$. In view of the last two relations, the ideal $A_{\mathrm{s_l^\circ}}$ is order dense in $A_{\mathrm{s_l}}$. □

Recall (see [4, p. 57, Exercise 11]) that if $\{x_\alpha\}$ is an order bounded net in a Dedekind complete real Riesz space E then the lower limit and the upper limit of $\{x_\alpha\}$ are defined by

$$\liminf_\alpha x_\alpha = \bigvee_\alpha \bigwedge_{\beta \geq \alpha} x_\beta \quad \text{and} \quad \limsup_\alpha x_\alpha = \bigwedge_\alpha \bigvee_{\beta \geq \alpha} x_\beta,$$

respectively. The relation $x_\alpha \xrightarrow{o} x$ is equivalent to the equalities $x = \liminf_\alpha x_\alpha = \limsup_\alpha x_\alpha$.

Theorem 8 *Let A be a Dedekind complete ordered algebra and let $a \in A^+$. Then the next identities hold in A_{r}*

$$a_{\mathrm{nl}} = \inf\{\liminf_{\alpha} p_\alpha a\} = \inf\{\limsup_{\alpha} p_\alpha a\} = \inf\{\sup_{\alpha}\{p_\alpha a\}\}, \tag{3}$$

where the infimums were taken by all nets $\{p_\alpha\}$ such that $p_\alpha \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$. Moreover, if at least one of infimums is attained then the other two are also attained and this is equivalent to the existence of a net $\{q_\alpha\}$ satisfying $q_\alpha \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$ and $q_\alpha a_{\mathrm{sl}} = 0$.

Proof For an arbitrary element $b \in A^+$, we put $Tb = \inf\{\sup_\alpha\{p_\alpha b\}\}$. Obviously, the mapping $T : A^+ \to A^+$ is well defined, $Tb = b$ for $b \in A^+_{\mathrm{nl}}$, and $0 \le Tc \le c$ for all $c \in A^+$. We will show the additivity of T on A^+. To this end, let $b, c \in A^+$. Then

$$T(b + c) = \inf\{\sup_{\alpha}\{p_\alpha b + p_\alpha c\}\} = \inf\{\sup_{\alpha}\{p_\alpha b\} + \sup_{\alpha}\{p_\alpha c\}\} \ge Tb + Tc.$$

On the other hand, if $p'_\alpha \uparrow \mathbf{e}$ and $p''_\beta \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$ then the net $\{p'_\alpha p''_\beta\}_{(\alpha,\beta)}$ satisfying the relation $p'_\alpha p''_\beta \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$. Therefore,

$$\begin{aligned} T(b + c) &\le \sup_{\alpha,\beta}\{p'_\alpha p''_\beta (b + c)\} \\ &= \sup_{\alpha,\beta}\{p'_\alpha p''_\beta b\} + \sup_{\alpha,\beta}\{p'_\alpha p''_\beta c\} \le \sup_{\alpha}\{p'_\alpha b\} + \sup_{\beta}\{p''_\beta c\}. \end{aligned}$$

Since $\{p'_\alpha\}$ and $\{p''_\beta\}$ are arbitrary, $T(b+c) \le Tb+Tc$. So, $T(b+c) = Tb+Tc$. By the Kantorovich theorem (see [4, p. 9]), T extends uniquely to an operator (denoted by T again) from A_{r} into A_{r}. Evidently, $0 \le T \le I$ and $Tb = b$ for all $b \in A_{\mathrm{nl}}$.

For every element $b \in A_{\mathrm{s}^\circ_{\mathrm{l}}}$, we have $Tb = 0$. In view of the preceding lemma, the ideal $A_{\mathrm{s}^\circ_{\mathrm{l}}}$ is order dense in A_{sl}. Then, using the order continuity of the operator T, we infer $Tb = 0$ for all $b \in A_{\mathrm{sl}}$. Consequently, for arbitrary $a \in A^+$, we obtain

$$\begin{aligned} a_{\mathrm{nl}} = Ta_{\mathrm{nl}} = Ta = \inf\{\sup_{\alpha}\{p_\alpha a\}\} &\ge \inf\{\limsup_{\alpha} p_\alpha a\} \\ \ge \inf\{\liminf_{\alpha} p_\alpha a\} &\ge \inf\{\liminf_{\alpha} p_\alpha a_{\mathrm{nl}}\} = a_{\mathrm{nl}}, \end{aligned}$$

as required.

As follows at once from the last relations, if a_{nl} attains for at least one of infimums in (3) then $a_{\mathrm{nl}} = \liminf_\alpha p_\alpha a = \liminf_\alpha p_\alpha a_{\mathrm{nl}}$ with $p_\alpha \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$. Fix an index α_0 and consider a net $\{q_\alpha\}_{\alpha \ge \alpha_0}$ defined by $q_\alpha = p_\alpha$ for $\alpha \ge \alpha_0$. Then

$$\begin{aligned} a_{\mathrm{nl}} &= \liminf_{\alpha} p_\alpha a = \liminf_{\alpha} q_\alpha a = \liminf_{\alpha} (q_\alpha a_{\mathrm{nl}} + q_\alpha a_{\mathrm{sl}}) \\ &\ge \liminf_{\alpha} (q_\alpha a_{\mathrm{nl}} + q_{\alpha_0} a_{\mathrm{sl}}) = q_{\alpha_0} a_{\mathrm{sl}} + \liminf_{\alpha} q_\alpha a_{\mathrm{nl}} = q_{\alpha_0} a_{\mathrm{sl}} + a_{\mathrm{nl}} \end{aligned}$$

and so $p_{\alpha_0} a_{\mathrm{sl}} = 0$. For the converse, if $p_\alpha a_{\mathrm{sl}} = 0$ with $p_\alpha \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$ then we have $\sup_\alpha\{p_\alpha a\} = \sup_\alpha\{p_\alpha a_{\mathrm{nl}}\} = a_{\mathrm{nl}}$, and the proof is completed. □

Corollary 9 *Under the assumptions of Theorem 8, the following identities hold in* A_{r}

$$a_{\mathrm{s_l}} = \sup\{\inf_{\alpha}\{p_\alpha a\}\} = \sup\{\liminf_{\alpha} p_\alpha a\} = \sup\{\limsup_{\alpha} p_\alpha a\},$$

where the supremums were taken by all nets $\{p_\alpha\}$ *such that* $p_\alpha \downarrow 0$ *in* $\mathbf{OI}(A)$.

Proof It suffices to check the fist equality. We have

$$\begin{aligned} a_{\mathrm{s_l}} &= a - a_{\mathrm{n_l}} = a - \inf_{p_\alpha \uparrow \mathbf{e}}\{\sup_{\alpha}\{p_\alpha a\}\} = a + \sup_{p_\alpha \uparrow \mathbf{e}}\{\inf_{\alpha}\{-p_\alpha a\}\} \\ &= \sup_{p_\alpha \uparrow \mathbf{e}}\{\inf_{\alpha}\{a - p_\alpha a\}\} = \sup_{p_\alpha \uparrow \mathbf{e}}\{\inf_{\alpha}\{p_\alpha^{\mathrm{d}} a\}\} = \sup_{p_\alpha \downarrow 0}\{\inf_{\alpha}\{p_\alpha a\}\}, \end{aligned}$$

as desired. □

The next proposition deals with properties of the set $N^{\mathrm{l}}_{a_{\mathrm{s_l}}}$.

Proposition 10 *Let* A *be a Dedekind complete ordered algebra and let* $a \in A$. *Then* $N^{\mathrm{l}}_{a_{\mathrm{s_l}}}$ *is the largest solid subset of* $\mathbf{OI}(A)$ *on which the element* a *is* l*-order continuous.*

Proof Let J be a solid subset of $\mathbf{OI}(A)$ on which the element a is l-order continuous, i.e., if $p_\alpha \downarrow 0$ in J then $p_\alpha a \xrightarrow{o} 0$. It must be shown that $N^{\mathrm{l}}_{a_{\mathrm{s_l}}}$ satisfies this condition and the inclusion $J \subseteq N^{\mathrm{l}}_{a_{\mathrm{s_l}}}$ holds. For the check of the former, let $p_\alpha \downarrow 0$ in $N^{\mathrm{l}}_{a_{\mathrm{s_l}}}$. We have $p_\alpha \downarrow 0$ in $\mathbf{OI}(A)$ and $p_\alpha a = p_\alpha a_{\mathrm{n_l}} \xrightarrow{o} 0$. For the check of the second assertion, let $q \in J$ and $q a_{\mathrm{s_l}} \neq 0$. Using Theorem 3(c), we find an element b such that $0 < b \leq |a_{\mathrm{s_l}}| \leq |a|$, $qb > 0$, and $q_\alpha b = 0$ with $q_\alpha \uparrow q$ in $\mathbf{OI}(A)$. Obviously, b is l-order continuous on J and $q - q_\alpha \downarrow 0$ in J, whence $qb = (q - q_\alpha)b \xrightarrow{o} 0$ and so $qb = 0$, a contradiction. □

We shall close this section with the following two remarks. As in the case of operators on Riesz spaces, the notion of σ-(respectively σ_{l}, σ_{r}) order continuity can be introduced for elements of an arbitrary ordered algebra A. For instance, an element $a \in A$ is said to be σ_{l}*-order continuous* if $p_n a \xrightarrow{o} 0$ in A whenever the sequence $\{p_n\}$ satisfies $p_n \downarrow 0$ in $\mathbf{OI}(A)$. As in the case of operators, for a wide class of ordered algebras the notions of order and σ-order continuity coincide (e.g., if A_{r} is a Dedekind complete Riesz space with the countable sup property (see [4, p. 56])). Keeping the last remark in mind, many results obtained above can be extended to the case of σ-order continuous elements.

Next, let A be a real ordered algebra and let $A_{\mathbb{C}}$ be a complexification of A. Obviously, $A_{\mathbb{C}}$ equipped with a cone A^+ is also an ordered algebra. By analogy to the case of operators on the complexification $E_{\mathbb{C}}$ of the real Riesz space E, we can define the notion of order continuity of an element $a \in A_{\mathbb{C}}$ as follows. An element a is said to be *order continuous* whenever it is (uniquely) represented as $a = b + ic$, where $b, c \in A$ and are order continuous (in A) in the sense defined above. Again, many results above can be extended to this case.

3 Special classes of order continuous elements

3.1 The center of an ordered algebra

Let A be an ordered algebra. If the unit $\mathbf{e} \in A_{\mathrm{n}}$ then for the ideal $A_{\mathbf{e}}$ generated by $\mathbf{e}$ (see (1)) the inclusion $A_{\mathbf{e}} \subseteq A_{\mathrm{n}}$ holds. Moreover, under the algebraic operations and the ordering induced by A, $A_{\mathbf{e}}$ is a real ordered algebra. It should be at once mentioned that, in view of Lemma 1, we have the inclusion $\mathbf{e} \in A_{\mathrm{n}}$ when A is Dedekind complete while, in general, it does not hold (see Example 2). In the case of the ordered algebra $A = L(E)$, where E is a real Riesz space, the ideal $A_{\mathbf{e}}$ coincides with the *center* $\mathcal{Z}(E)$ of E (see [1, § 3.3]), i.e.,

$$A_{\mathbf{e}} = (L(E))_I = \mathcal{Z}(E) = \{T \in L(E) : |Tx| \leq \lambda|x| \text{ for all } x \in E \text{ and some } \lambda \in \mathbb{R}^+\}.$$

For this reason, the ideal $A_{\mathbf{e}}$, where A is an arbitrary ordered algebra, will be referred to as the *center* of A. The principal purpose of this subsection is to study of properties of $A_{\mathbf{e}}$.

On any ideal E_x in an ordered linear space E, we can define the Minkowski seminorm $\|\cdot\|_x$ (see [5, p. 103]). In particular, for the ideal $A_{\mathbf{e}}$ the seminorm $\|\cdot\|_{\mathbf{e}}$ is equal to

$$\|a\|_{\mathbf{e}} = \inf\{\lambda \in \mathbb{R}^+ : -\lambda\mathbf{e} \leq a \leq \lambda\mathbf{e}\},$$

where $a \in A_{\mathbf{e}}$. If $A_{\mathbf{e}}$ is Archimedean then $\|\cdot\|_{\mathbf{e}}$ is a norm on $A_{\mathbf{e}}$, the cone $A^+_{\mathbf{e}}$ is closed, and the closed unit ball $B_{A_{\mathbf{e}}}$ of $A_{\mathbf{e}}$ coincides with the order interval $[-\mathbf{e}, \mathbf{e}]$ (see [5, p. 104]). Under the norm $\|\cdot\|_{\mathbf{e}}$, the center $A_{\mathbf{e}}$ is an ordered normed algebra. Below, unless otherwise stated, we shall consider $A_{\mathbf{e}}$ with the norm $\|\cdot\|_{\mathbf{e}}$, and, hence $A_{\mathbf{e}}$ will be assumed to be Archimedean. Moreover, if A is an ordered normed algebra equipped with some norm $\|\cdot\|_A$ then this norm and the norm $\|\cdot\|_{\mathbf{e}}$ may be differ.

Lemma 11 *Let A be an ordered algebra (not necessarily commutative) and let $f \in A'_{\mathbf{e}}$ be a non-zero functional. The following statements are equivalent:*

(a) *f is an extreme point of the positive part of the unit ball of $A'_{\mathbf{e}}$, i.e., $f \in \operatorname{ext} B^+_{A'_{\mathbf{e}}}$;*
(b) *f is an extremal vector of the cone $(A'_{\mathbf{e}})^+$ and $\|f\|_{A'_{\mathbf{e}}} = 1$;*
(c) *f is multiplicative on $A_{\mathbf{e}}$.*

Proof (a) $\Longrightarrow$ (b) Let g also belong to the dual space $A'_{\mathbf{e}}$ of the space $A_{\mathbf{e}}$ equipped with the norm $\|\cdot\|_{\mathbf{e}}$ and let $0 \leq g \leq f$. We must show that f and g are linearly dependent. In view of the identity $\|h\|_{A'_{\mathbf{e}}} = h(\mathbf{e})$ for every $h \in (A'_{\mathbf{e}})^+$, we can suppose $0 < g(\mathbf{e}) < f(\mathbf{e}) = 1$. We have the equality $f = (1 - g(\mathbf{e}))\frac{f-g}{1-g(\mathbf{e})} + g(\mathbf{e})\frac{g}{g(\mathbf{e})}$ with $\frac{f-g}{1-g(\mathbf{e})}, \frac{g}{g(\mathbf{e})} \in B^+_{A'_{\mathbf{e}}}$ and, hence, $f = \frac{g}{g(\mathbf{e})}$, as required.

(b) $\Longrightarrow$ (a) If $f = \frac{f_1+f_2}{2}$, where $f_i \in B^+_{A'_{\mathbf{e}}}$ for $i = 1, 2$, then $f_i \leq 2f$. Thus, $f_i = f_i(\mathbf{e})f$. Therefore, $1 = f(\mathbf{e}) = \frac{f_1(\mathbf{e})+f_2(\mathbf{e})}{2} \leq \frac{1+1}{2} = 1$. Therefore, $f_i(\mathbf{e}) = 1$ and so $f_1 = f_2$.

(b) $\Longrightarrow$ (c) For an arbitrary element $a \in A^+_{\mathbf{e}}$ satisfying the inequality $f(a) > 0$, we define the functional f_a on $A_{\mathbf{e}}$ via the formula $f_a(b) = \frac{f(ab)}{f(a)}$. As is easy to see,

$f_a(\mathbf{e}) = 1$ and $0 \le f_a \le \frac{\|a\|_{\mathbf{e}}}{f(a)} f$. Therefore, $f_a = f$ and so $f(ab) = f(a)f(b)$. Obviously, the last identity holds if $f(a) = 0$ and, hence, for every $a \in A_{\mathbf{e}}^{+}$. Next, for arbitrary elements $a, b \in A_{\mathbf{e}}$, using the inequality $a + \|a\|_{\mathbf{e}}\mathbf{e} \ge 0$, we have

$$f((a + \|a\|_{\mathbf{e}}\mathbf{e})b) = f(a + \|a\|_{\mathbf{e}}\mathbf{e}) f(b).$$

Finally, $f(ab) = f(a)f(b)$, i.e., f is multiplicative.

The proof of (c) $\Longrightarrow$ (a) will be given below (see the remarks after Theorem 15). □

The collection of all non-zero, continuous, positive, multiplicative (linear) functionals from $A_{\mathbf{e}}$ onto $\mathbb{R}$ will be denoted by $\mathcal{M}_{A_{\mathbf{e}}}$.

Lemma 12 *The set $\mathcal{M}_{A_{\mathbf{e}}}$ is $\sigma(A'_{\mathbf{e}}, A_{\mathbf{e}})$-compact and separates points of $A_{\mathbf{e}}$. Moreover, if $a \in A_{\mathbf{e}}$ then $a \in A_{\mathbf{e}}^{+}$ if and only if $f(a) \ge 0$ for all $f \in \mathcal{M}_{A_{\mathbf{e}}}$.*

Proof Evidently, we have the inclusion $\mathcal{M}_{A_{\mathbf{e}}} \subseteq B^{+}_{A'_{\mathbf{e}}}$ (and even the inclusion $\mathcal{M}_{A_{\mathbf{e}}} \subseteq S^{+}_{A'_{\mathbf{e}}}$, where $S_{A'_{\mathbf{e}}}$ is the unit sphere of $A'_{\mathbf{e}}$) and $\mathcal{M}_{A_{\mathbf{e}}}$ is a $\sigma(A'_{\mathbf{e}}, A_{\mathbf{e}})$-closed subset of $B_{A'_{\mathbf{e}}}$. Consequently, by the Alaoglu theorem (see [4, p. 148]), $\mathcal{M}_{A_{\mathbf{e}}}$ is $\sigma(A'_{\mathbf{e}}, A_{\mathbf{e}})$-compact. Since the unit $\mathbf{e}$ is an interior point of the cone $A_{\mathbf{e}}^{+}$, the cone $A_{\mathbf{e}}^{+}$ is generating. Moreover, in view of the monotonicity of the norm $\|\cdot\|_{\mathbf{e}}$, $A_{\mathbf{e}}^{+}$ is normal. Therefore, by the M. G. Krein theorem (see [5, p. 89]), the dual wedge $(A_{\mathbf{e}}^{+})'$ is a generating cone; in particular, we have the identity

$$(A_{\mathbf{e}}^{+})' - (A_{\mathbf{e}}^{+})' = A'_{\mathbf{e}}. \tag{4}$$

On the other hand, by the Krein–Milman theorem (see [4, p. 144]), the relation $\operatorname{ext} B^{+}_{A'_{\mathbf{e}}} \ne \emptyset$ holds and, moreover,

$$\overline{\operatorname{co}}(\operatorname{ext} B^{+}_{A'_{\mathbf{e}}}) = B^{+}_{A'_{\mathbf{e}}}, \tag{5}$$

where the closure of the convex hull was taken in the $\sigma(A'_{\mathbf{e}}, A_{\mathbf{e}})$-topology. Taking into account the last equality and the identity (4), we obtain that the set $\operatorname{ext} B^{+}_{A'_{\mathbf{e}}}$ separates points of $A_{\mathbf{e}}$. Therefore, in view of the validity of the implication (a) $\Longrightarrow$ (c) of the preceding lemma, $\mathcal{M}_{A_{\mathbf{e}}}$ also separates the points of $A_{\mathbf{e}}$. If $f(a) \ge 0$ for every $f \in \mathcal{M}_{A_{\mathbf{e}}}$ then, according to (5), $f(a) \ge 0$ for all $f \in B^{+}_{A'_{\mathbf{e}}}$. As is well known, the latter implies $a \in A_{\mathbf{e}}^{+}$. □

Corollary 13 *The center $A_{\mathbf{e}}$ is a commutative algebra.*

Proof For every $a, b \in A_{\mathbf{e}}$ and $f \in \mathcal{M}_{A_{\mathbf{e}}}$, the identities $f(ab) = f(a)f(b) = f(ba)$ hold. In view of Lemma 12, $ab = ba$. □

The preceding corollary is not valid without the assumption that the algebra $A_{\mathbf{e}}$ is Archimedean. Indeed, let B be an arbitrary noncommutative algebra. Consider the unitization $A = B \otimes \mathbb{R}$ of B, which is an algebra obtained from B by adjoining a unit. Obviously, A is noncommutative and $\mathbf{e} = (0, 1)$. On the other hand, under the ordering defined by the cone $A^+ = \{(b, \lambda) : \lambda > 0, b \in B\} \cup \{0\}$, the algebra A is an ordered algebra and $A_\mathbf{e} = A$.

Corollary 14 *If A is an ordered algebra then for $a \in A_\mathbf{e}$ the following statements hold*:

(a) *For a polynomial φ (with real coefficients) the inequality $\varphi(a) \geq 0$ is valid if and only if $\varphi(\lambda) \geq 0$ for all $\lambda \in \{f(a) : f \in \mathcal{M}_{A_\mathbf{e}}\}$. In particular, $a^k \geq 0$ for every even $k \in \mathbb{N}$*;

(b) *If a is an idempotent then $a \in \mathbf{OI}(A)$ if and only if $a \in A_\mathbf{e}$.*

Now, using the Gelfand's idea about an embedding of a commutative Banach algebra into an algebra of continuous functions, for an arbitrary element $a \in A_\mathbf{e}$, we define the function $\widehat{a}$ on $\mathcal{M}_{A_\mathbf{e}}$ via the formula $\widehat{a} f = f(a)$, where $f \in \mathcal{M}_{A_\mathbf{e}}$. Evidently, $\widehat{a}$ is a continuous function on the (Hausdorff) compact space $\mathcal{M}_{A_\mathbf{e}}$ with the $\sigma(A'_\mathbf{e}, A_\mathbf{e})$-topology (see Lemma 12), i.e., $\widehat{a} \in C(\mathcal{M}_{A_\mathbf{e}})$, and the mapping $\Psi_A : a \to \widehat{a}$ from $A_\mathbf{e}$ into $C(\mathcal{M}_{A_\mathbf{e}})$ is a positive, algebraic homomorphism. Moreover, Ψ_A is an isometry. Indeed, $\|\widehat{a}\|_{C(\mathcal{M}_{A_\mathbf{e}})} = \sup_{f \in \mathcal{M}_{A_\mathbf{e}}} |f(a)| \leq \|a\|_\mathbf{e}$. On the other hand, the inequalities

$$-\|\widehat{a}\|_{C(\mathcal{M}_{A_\mathbf{e}})} \mathbb{1} \leq \widehat{a} \leq \|\widehat{a}\|_{C(\mathcal{M}_{A_\mathbf{e}})} \mathbb{1}$$

are equivalent to the inequalities

$$-\|\widehat{a}\|_{C(\mathcal{M}_{A_\mathbf{e}})} f(\mathbf{e}) \leq f(a) \leq \|\widehat{a}\|_{C(\mathcal{M}_{A_\mathbf{e}})} f(\mathbf{e})$$

for all $f \in \mathcal{M}_{A_\mathbf{e}}$. Whence, using Lemma 12 once more, we infer

$$-\|\widehat{a}\|_{C(\mathcal{M}_{A_\mathbf{e}})} \mathbf{e} \leq a \leq \|\widehat{a}\|_{C(\mathcal{M}_{A_\mathbf{e}})} \mathbf{e}$$

and so $\|a\|_\mathbf{e} \leq \|\widehat{a}\|_{C(\mathcal{M}_{A_\mathbf{e}})}$. So, $\|a\|_\mathbf{e} = \|\widehat{a}\|_{C(\mathcal{M}_{A_\mathbf{e}})}$. Next, the range $R(\Psi_A)$ of the mapping Ψ_A is a subalgebra of the algebra $C(\mathcal{M}_{A_\mathbf{e}})$. Therefore, the Stone-Weierstrass theorem (see [10, p. 124]) guarantees that $R(\Psi_A)$ is dense in $C(\mathcal{M}_{A_\mathbf{e}})$. The relation $R(\Psi_A) \neq C(\mathcal{M}_{A_\mathbf{e}})$ is possible. To see this, we consider the algebra $A = C^1[0, 1]$ of all continuously differentiable functions on $[0, 1]$ equipped with the cone A^+ of all nonnegative functions. Clearly, $A_\mathbf{e} = A$ and $\|a\|_\mathbf{e} = \|a\|_{C[0,1]}$. As is well known, every non-zero multiplicative functional f on the algebra $C(K)$ of all continuous functions on some compact K can be represented in the form of $f(x) = x(t)$ for some $t \in K$. Therefore, in our case, $\mathcal{M}_{A_\mathbf{e}}$ can be identified with $[0, 1]$.

Thus, we have established the following result.

Theorem 15 *The center $A_\mathbf{e}$ of an ordered algebra A is isometrically, algebraically, and order-embeddable into a dense subalgebra of the real algebra $C(M)$, where M is some compact space. Moreover, if $A_\mathbf{e}$ is Dedekind complete then this isomorphism is a bijection from $A_\mathbf{e}$ onto $C(M)$.*

Now we shall mention some important consequences of the preceding result. First of all, we shall **prove the implication** (c) $\Longrightarrow$ (a) of Lemma 11. To this end, consider a non-zero multiplicative functional $f \in A'_{\mathbf{e}}$. We claim first that $f \in B^+_{A'_{\mathbf{e}}}$, in particular, f is *positive*. Indeed, consider the embedding $\Psi_A : A_{\mathbf{e}} \to C(M)$ and define the functional g on $R(\Psi_A)$ via the formula $g(\Psi_A(a)) = f(a)$. Obviously, $\|g\|_{(R(\Psi_A))'} = \|f\|_{A'_{\mathbf{e}}}$ and g is multiplicative on $R(\Psi_A)$. Therefore, g extends uniquely to all of $C(M)$. These extensions will be denoted by g again. Clearly, g is multiplicative on $C(M)$. Thus, $g \geq 0$, whence $f \geq 0$ and so $f \in B^+_{A'_{\mathbf{e}}}$. Let $f = \frac{f_1+f_2}{2}$ with $f_i \in B^+_{A'_{\mathbf{e}}}$. For $i = 1, 2$, we put $g_i(\Psi_A(a)) = f_i(a)$. Functionals g_i extend uniquely to all of $C(M)$. Clearly, $g = \frac{g_1+g_2}{2}$. As is well known, every multiplicative functional on $C(M)$ is an extreme point of $B^+_{(C(M))'}$. Consequently, $g_1 = g_2$ and so $f_1 = f_2$.

Next, we mention the validity of the next assertion: *for a multiplicative functional f on $A_{\mathbf{e}}$ the continuity is equivalent to the positivity*. Indeed, the necessity was established below in the proof of the implication (c) $\Longrightarrow$ (a) of Lemma 11 and the sufficiency follows at once from the boundedness of the set $f(B_{A_{\mathbf{e}}}) = f([-\mathbf{e}, \mathbf{e}])$ (without the assumption about the multiplicativity of f). It should be noticed that the multiplicativity of some functional on $A_{\mathbf{e}}$ does not imply the positivity. To see this, we consider the algebra A of all polynomials on the segment [0, 1] under the natural algebraic operations and the order. Fix an arbitrary number $t \notin [0, 1]$. Every polynomial φ on [0, 1] extends uniquely to a polynomial $\widetilde{\varphi}$ from $\mathbb{R}$ to $\mathbb{R}$. Obviously, the mapping $\varphi \to \widetilde{\varphi}$ is an algebraic homomorphism. We define the functional g on A by $g(\varphi) = \widetilde{\varphi}(t)$. Then the functional g is multiplicative while, as is easy to see, it is not positive.

Corollary 16 *Let A be an ordered algebra, let $a \in A_{\mathbf{e}}$, and let φ be a polynomial. Then $\varphi(a) = 0$ if and only if $\varphi(\lambda) = 0$ for all $\lambda \in \{f(a) : f \in \mathcal{M}_{A_{\mathbf{e}}}\}$. In particular, $a^k \neq 0$ if $a \neq 0$ and $k \in \mathbb{N}$.*

We now turn our attention to the case of the center of an ordered normed algebra.

Before, we need some preliminary discussion. First of all, recall that (see [5, pp. 104, 105]) the sequence $\{z_n\}$ in an (Archimedean) ordered linear space E is said to be x-*uniformly convergent* to an element z, where $x, z \in E$ and $x > 0$, if for each $\epsilon > 0$ there exists an index n_0 such that $-\epsilon x \leq z - z_n \leq \epsilon x$ for all $n \geq n_0$ and is said to be x-*uniformly Cauchy* if for each $\epsilon > 0$ there exists an index n_0 such that $-\epsilon x \leq z_m - z_n \leq \epsilon x$ for all $m, n \geq n_0$. A space E is said to be x-*uniformly complete* if every x-uniformly Cauchy sequence of E is x-uniformly convergent. The space E_x equipped with the norm $\|\cdot\|_x$ is a Banach space if and only if E is x-uniformly complete. Moreover, by the Andô theorem (see [5, p. 106]), the (closed) cone E^+ in an ordered Banach space E is normal if and only if E is x-uniformly complete for every $x > 0$.

On the other hand, as is well known (see [5, p. 87]), the cone E^+ in an ordered Banach space E is normal if and only if the order intervals of E are norm bounded. Now let E be an ordered normed space with a (closed) cone E^+. We shall say that the cone E^+ is x-*normal*, where $x \in E$ and $x > 0$, whenever order intervals of E_x are bounded with respect of the norm induced by E. Obviously, every normal cone is

x-normal for all $x > 0$ and if E is an ordered Banach space then the converse holds. The proof of the following assertion is elementary and will be omitted.

Lemma 17 *For an ordered normed space E and a non-zero element $x \in E^+$ the following statements are equivalent*:

(a) *The cone E^+ is x-normal*;
(b) *The order interval $[0, x]$ is norm bounded in E*;
(c) *The relations $0 \le x_n \le \lambda_n x$ for all n and $\lambda_n \to 0$ as $n \to \infty$ imply $x_n \to 0$ in E*;
(d) *The embedding $(E_x, \|\cdot\|_x) \to (E, \|\cdot\|_E)$ is continuous.*

Corollary 18 *Let E be an ordered Banach space and let $x \in E^+$ be a non-zero element. Then E is x-uniformly complete if and only if E^+ is x-normal.*

Proof If E is x-uniformly complete then the embedding of part (d) of the preceding lemma is automatically continuous and, hence, E^+ is x-normal. For the converse, let the sequence $\{z_n\}$ in E be x-uniformly Cauchy. Fix $\epsilon > 0$. There exists an index n_0 such that

$$-\epsilon x \le z_m - z_n \le \epsilon x \tag{6}$$

for all $m, n \ge n_0$. We have $\|z_m - z_n\| \le \epsilon C$, where the constant $C = \sup_{y \in [-x,x]} \|y\|_E < \infty$ as E^+ is x-normal. Therefore, $\{z_n\}$ is a norm Cauchy sequence and, hence, $z_n \to z$ in E. Letting in (6) $m \to \infty$, we get $-\epsilon x \le z - z_n \le \epsilon x$, as required. □

The example of the space $C^1[0, 1]$ with the sup norm shows that in the case of an ordered norm space E the x-normality of E^+ does not imply the x-uniform completeness of E. Now we want to discuss the validity of the converse implication. To this end, we recall first the following Kaplansky theorem (see [10, p. 176]) which will be employed later on (see, in particular, Theorems 22, 24).

Theorem 19 *Any norm under which $C(K)$ is a normed algebra majorizes the sup norm.*

The preceding theorem leads at once to the next problem which was formulated in 1948 and is sometimes called the *Kaplansky conjecture*.

Hypothesis 20 *Every norm $\|\cdot\|$ under which $C(K)$ is a normed algebra is equivalent to the sup norm.*

As is easy to see, this hypothesis is valid if and only if every multiplicative homomorphism from $C(K)$ into an arbitrary Banach algebra is automatically continuous. The validity of Hypothesis 20 was a long standing problem. First Dales and Esterle proved in 1976 that it is false if the continuum hypothesis is assumed. Shortly afterwards, Solovay and Woodin proved that Hypothesis 20 is true in some model of ZFC (the Zermelo–Fraenkel set theory with the axiom of choice). Thus, the validity of it turned out to be independent of ZFC (see [6, Chapter 5] for the detailed discussion).[1] Next, as is easy to see, the answer to this hypothesis is affirmative if the cone $(C(K))^+$ is $\mathbb{1}$-normal in the space $C(K)$ equipped with the norm $\|\cdot\|$ (see Lemma 17(d)).

[1] The author wishes to thank Professor A. R. Schep for bringing these results to his attention.

Proposition 21 *Hypothesis 20 is valid if and only if for an arbitrary ordered normed algebra A the* **e**-*uniform completeness of A implies the* **e**-*normality of A^+.*

Proof Necessity. Let A be an **e**-uniformly complete ordered normed algebra. In view of the remarks above, under the norm $\|\cdot\|_{\mathbf{e}}$, the center $A_{\mathbf{e}}$ is a Banach space. Putting $M = \mathcal{M}_{A_{\mathbf{e}}}$, we have the identity $R(\Psi_A) = C(M)$ as Ψ_A is an isometry from $A_{\mathbf{e}}$ into $C(M)$ and $R(\Psi_A)$ is dense in $C(M)$ (see Theorem 15 and the remarks before it). For arbitrary $x \in C(M)$, we put $\|x\|_0 = \|\Psi_A^{-1}(x)\|_A$. Obviously, $\|\cdot\|_0$ is a norm and, under this norm, $C(M)$ is a normed algebra. In view of our hypothesis, the norm $\|\cdot\|_0$ is equivalent to the sup norm $\|\cdot\|_{C(M)}$, whence $\|x\|_0 \leq R\|x\|_{C(M)}$ for all $x \in C(M)$ and some constant $R > 0$. Consequently, $\|a\|_A \leq R\|a\|_{\mathbf{e}}$ for all $a \in A_{\mathbf{e}}$. Now it remains to remember part (d) of Lemma 17.

Sufficiency. Let the space $C(K)$ be a normed algebra under some norm $\|\cdot\|$. In view of Theorem 19, $C(K)$ is an ordered normed algebra under the natural ordering and, obviously, is $\mathbb{I}$-uniformly complete. Therefore, the embedding $((C(K))_{\mathbb{I}}, \|\cdot\|_{C(K)}) \to (C(K), \|\cdot\|)$ is continuous and, hence, $\|x\| \leq r\|x\|_{C(K)}$ for all real-valued functions $x \in C(K)$ and some constant $r > 0$. Now in the case of the complex space $C(K)$, we have $\|x\| \leq 2r\|x\|_{C(K)}$ for all $x \in C(K)$. □

Theorem 22 *Let A be an* **e**-*uniformly complete ordered normed algebra. Then the center $A_{\mathbf{e}}$ is an AM-space with unit* **e** *and for every element $a \in A_{\mathbf{e}}$ the relations*

$$\||a|\|_{\mathbf{e}} = \|a\|_{\mathbf{e}} \leq \|a\|_A \tag{7}$$

hold. Moreover, $A_{\mathbf{e}}$ is isometrically, algebraically, and order-embeddable onto the algebra $C(M)$ for some compact space M. If, in addition, the norm of A is monotone on $A_{\mathbf{e}}$ and $\|\mathbf{e}\|_A = 1$ then

$$\|a\|_{\mathbf{e}} = \||a|\|_A \leq \|a\|_A \leq 2\|a\|_{\mathbf{e}} \tag{8}$$

for all $a \in A_{\mathbf{e}}$.

Proof We have the identity $R(\Psi_A) = C(M)$ for some compact space M (see the proof of Proposition 21). Therefore, $A_{\mathbf{e}}$ is isometrically, algebraically, and order-embeddable onto the AM-space $C(M)$. Consequently, under the norm $\|\cdot\|_{\mathbf{e}}$ and the order induced by A, $A_{\mathbf{e}}$ is an AM-space with unit **e**. In particular, $A_{\mathbf{e}}$ *is a Riesz space, i.e., for every $a, b \in A_{\mathbf{e}}$ there exist the supremum $a \vee b$ and the infimum $a \wedge b$ in $A_{\mathbf{e}}$ and, moreover,* $\|a\|_{\mathbf{e}} = \||a|\|_{\mathbf{e}}$, where $|a|$ is a modulus of a in $A_{\mathbf{e}}$ (we don't assert that the modulus of a exists in A). Again, for an arbitrary function $x \in C(M)$, we put $\|x\|_0 = \|\Psi_A^{-1}(x)\|_A$. By Theorem 19, the inequality $\|x\|_{C(M)} \leq \|x\|_0$ holds for all $x \in C(M)$ and so $\|a\|_{\mathbf{e}} \leq \|a\|_A$ for all $a \in A_{\mathbf{e}}$.

Now let the norm of A be monotone on $A_{\mathbf{e}}$ and let $\|\mathbf{e}\|_A = 1$. Then, for an arbitrary element $a \in A_{\mathbf{e}}$, using the inequality $|a| \leq \|a\|_{\mathbf{e}}\mathbf{e}$, we have $\|a\|_{\mathbf{e}} = \||a|\|_{\mathbf{e}} \leq \||a|\|_A \leq \|a\|_{\mathbf{e}}$, whence $\|a\|_{\mathbf{e}} = \||a|\|_A$. □

Both inequalities in (8) can be strict. Indeed, let us consider the real space $C[0, 1]$ equipped with the norm $\|x\| = \|x^+\|_{C[0,1]} + \|x^-\|_{C[0,1]}$. As is easy to see, under this norm, the space $C[0, 1]$ is an ordered Banach algebra satisfying all assumptions of the preceding theorem. Nevertheless, for the function $x(t) = -\frac{3}{2}t + 1$, we have $\|x\|_{C[0,1]} = 1$ and $\|x\| = \frac{3}{2}$.

Corollary 23 *Let A be an $\mathbf{e}$-uniformly complete ordered normed algebra such that the cone A^+ is normal. The following statements hold*:

(a) *The center $A_\mathbf{e}$ is order closed in A. In particular, if A_r is a Riesz space then $A_\mathbf{e}$ coincides with the band generated by $\mathbf{e}$ in A_r and if, in addition, A is Dedekind complete then the representation $A_\mathrm{r} = A_\mathbf{e} \oplus A_\mathbf{e}^\mathrm{d}$ holds*;

(b) *If A is Dedekind complete, $a \in A_\mathbf{e}$, and a is invertible in A_r then $a^{-1} \in A_\mathbf{e}$.*

Proof (a) Let $\{a_\alpha\}$ be a net in $A_\mathbf{e}$ such that $a_\alpha \xrightarrow{o} a$ in A. We must check the inclusion $a \in A_\mathbf{e}$. There exist two nets $\{b_\alpha\}$ and $\{c_\alpha\}$ satisfying $b_\alpha \le a - a_\alpha \le c_\alpha$ and $b_\alpha \uparrow 0$, $c_\alpha \downarrow 0$ in A. Fix an index α_0. Then $b_{\alpha_0} \le a - a_\alpha \le c_{\alpha_0}$ for all $\alpha \ge \alpha_0$. In view of the normality of A^+, the net $\{a_\alpha\}_{\alpha \ge \alpha_0}$ is norm bounded in A and, hence, in view of (7), is $\|\cdot\|_\mathbf{e}$-bounded. Therefore, for $\alpha \ge \alpha_0$, we have $-C\mathbf{e} \le a_\alpha \le C\mathbf{e}$, where $C = \sup_{\alpha \ge \alpha_0} \|a_\alpha\|_\mathbf{e}$, and so $b_\alpha - C\mathbf{e} \le a \le c_\alpha + C\mathbf{e}$. Finally, $-C\mathbf{e} \le a \le C\mathbf{e}$, i.e., $a \in A_\mathbf{e}$. Now the rest of the conclusion is obvious.

(b) According to part (a), we write $a^{-1} = b_1 + b_2$ with $b_1 \in A_\mathbf{e}$ and $b_2 \in A_\mathbf{e}^\mathrm{d}$. Obviously, $\mathbf{e} = ab_1 + ab_2$ and, hence, $ab_2 = \mathbf{e} - ab_1 \in A_\mathbf{e}$. On the other hand, in view of the relation $|ab_2| \le \|a\|_\mathbf{e}|b_2|$, we have $ab_2 \in A_\mathbf{e}^\mathrm{d}$. Hence, $ab_2 = 0$, i.e., $b_2 = 0$ and so $a^{-1} \in A_\mathbf{e}$. □

Let E be a Banach lattice and let an operator T belong to the center $\mathcal{Z}(E)$ of E. Then, under the order induced by $B(E)$, $\mathcal{Z}(E)$ is a Riesz space and so T has a modulus $|T| \in \mathcal{Z}(E)$ (in fact, as is well known (see [4, p. 114]), for every $T \in \mathcal{Z}(E)$ its modulus exists in $L(E)$). Obviously, $\pm T \le |T|$ in $B(E)$ and, hence, $\|T\|_{B(E)} \le \||T|\|_{B(E)}$. Taking into account this observation, as a consequence of Theorem 22 and Corollary 23, we see that the center $\mathcal{Z}(E)$ of E equipped with the operator norm is an AM-space with unit I and if, in addition, E is Dedekind complete then $L_\mathrm{r}(E) = \mathcal{Z}(E) \oplus I^\mathrm{d}$. These are famous Wickstead's results (see [1, p. 113]).

For the case of an $\mathbf{e}$-normal cone A^+, we have the following result.

Theorem 24 *Let A be an ordered normed algebra with an $\mathbf{e}$-normal cone A^+. Then for every element $a \in A_\mathbf{e}$ the next inequality holds*

$$\|a\|_\mathbf{e} \le \|a\|_A \tag{9}$$

Proof In view of Theorem 15 and the remarks before it, for some compact space M the range $R(\Psi_A)$ of the mapping Ψ_A from the center $A_\mathbf{e}$ into the space $C(M)$ is a dense subalgebra of $C(M)$. For an arbitrary function $x \in C(M)$ there exists a sequence $\{y_n\}$ in $R(\Psi_A)$ satisfying $y_n \to x$ in $C(M)$ and $y_n = \Psi_A(a_n)$ for some $a_n \in A_\mathbf{e}$. Obviously, $\|a_m - a_n\|_\mathbf{e} = \|y_m - y_n\|_{C(M)} \to 0$ as $m, n \to \infty$. Since the cone A^+ is $\mathbf{e}$-normal, we have $\|a_m - a_n\|_A \to 0$ and, hence, the sequence $\{\|a_n\|_A\}$ is converging. Put $\|x\| = \lim_{n\to\infty} \|a_n\|_A$. If $\{z_n\}$ is another sequence in $R(\Psi_A)$ satisfying $z_n \to x$ in $C(M)$ and $z_n = \Psi_A(b_n)$ for some $b_n \in A_\mathbf{e}$ then $\|a_n - b_n\|_\mathbf{e} = \|y_n - z_n\|_{C(M)} \to 0$

and so $\|a_n - b_n\|_A \to 0$. Consequently, $\lim_{n\to\infty} \|a_n\|_A = \lim_{n\to\infty} \|b_n\|_A$. Thus, the function $\|\cdot\|$ is well defined.

If $\|w\| = 0$ for some $w \in C(M)$ then there exists a sequence $\{w_n\}$ in $R(\Psi_A)$ satisfying $w_n \to w$ in $C(M)$, $w_n = \Psi_A(c_n)$, and $c_n \to 0$ in A as $n \to \infty$. Fix $\epsilon > 0$ and find an index n_0 such that $-\epsilon\mathbf{e} \leq c_m - c_n \leq \epsilon\mathbf{e}$ for all $m, n \geq n_0$. Letting $m \to \infty$, we have $-\epsilon\mathbf{e} \leq c_n \leq \epsilon\mathbf{e}$ and so $\epsilon \geq \|c_n\|_{\mathbf{e}} = \|w_n\|_{C(M)}$ for all $n \geq n_0$. Therefore, $w_n \to 0$ and, hence, $w = 0$. Now, as is easy to see, $\|\cdot\|$ is a norm and $C(K)$ equipped with this norm is a normed algebra. Using Theorem 19, we have the inequality $\|x\|_{C(K)} \leq \|x\|$ for all $x \in C(K)$ and, in particular, $\|a\|_{\mathbf{e}} = \|\Psi_A(a)\|_{C(K)} \leq \|\Psi_A(a)\| = \|a\|_A$ for all $a \in A_{\mathbf{e}}$, as required. □

In the case of an ordered C^*-algebra the relations (7), (8), and (9) can be made more precisely (see (14)).

The rest of this subsection is devoted to the conditions which guarantee that the center $A_{\mathbf{e}}$ is closed in an ordered normed algebra A. We start our discussion with the next auxiliary result.

Lemma 25 *Let Z be a normed space equipped with the norm $\|\cdot\|_Z$ and let Z_0 be a linear subspace of Z equipped with the norm $\|\cdot\|_{Z_0}$. If the embedding $(Z_0, \|\cdot\|_Z) \to (Z_0, \|\cdot\|_{Z_0})$ is continuous and the unit ball B_{Z_0} of $(Z_0, \|\cdot\|_{Z_0})$ is norm closed in Z then Z_0 is a norm closed in Z.*

Proof Let $\{z_n\}$ be a sequence in Z_0 satisfying $z_n \to z$ in Z, where $z \in Z$. Clearly, z_n is norm bounded in Z and, hence, is $\|\cdot\|_{Z_0}$-bounded in Z_0. Therefore, for some scalar $r > 0$, we have the inclusion $z_n \in rB_{Z_0}$ for all n. Finally, $z \in rB_{Z_0}$ and so $z \in Z_0$. □

Theorem 26 *Let A be an ordered normed algebra. Each of the following conditions ensures that the center $A_{\mathbf{e}}$ is norm closed in A:*

(a) *The algebra A is* **e***-uniformly complete*;
(b) *The cone A^+ is* **e***-normal.*

Proof In view of the relations (7) and (9), the embedding

$$(A_{\mathbf{e}}, \|\cdot\|_A) \to (A_{\mathbf{e}}, \|\cdot\|_{\mathbf{e}}) \tag{10}$$

is continuous. On the other hand, the unit ball $B_{A_{\mathbf{e}}}$ of $(A_{\mathbf{e}}, \|\cdot\|_{\mathbf{e}})$ is equal to the order interval $[-\mathbf{e}, \mathbf{e}]$ which is norm closed in A. It remains to remember the preceding lemma. □

The author does not know an example of an ordered normed algebra A such that the center $A_{\mathbf{e}}$ of A is not norm closed in A. The proof above is valid for every ordered normed algebra A such that the embedding (10) is continuous (in particular, the latter holds when the center $A_{\mathbf{e}}$ satisfies the inequality (9)). As follows at once from the Baire category theorem and the equality $A_{\mathbf{e}} = \bigcup_{n=1}^{\infty} n[-\mathbf{e}, \mathbf{e}]$, in the case of an ordered Banach algebra A the closedness of $A_{\mathbf{e}}$ in A is equivalent to the continuity of the embedding (10). The preceding theorem reduces to the well-known result asserting that the center $\mathcal{Z}(E)$ of a Banach lattice E is closed in the algebra $B(E)$.

Corollary 27 *Let A be a normed algebra which is an ordered normed algebra both with the cone A_1^+ and with the cone A_2^+. If the algebra A with the cone A_1^+ satisfies one of the conditions (a) or (b) of Theorem 26 and with the cone A_2^+ satisfies both these conditions then the inclusion $A_{\mathbf{e},1} \cap A_{\mathbf{e},2}^+ \subseteq A_{\mathbf{e},1}^+$ holds, where $A_{\mathbf{e},i}$ is the center of A under the order induced by A_i^+.*

Proof Consider a non-zero element $a \in A_{\mathbf{e},1} \cap A_{\mathbf{e},2}^+$. There exists (see the proof of Proposition 21) an isometric order isomorphism Ψ from $A_{\mathbf{e},2}$ onto $C(M)$, where M is some compact space depending upon A_2^+. If $\{\varphi_n\}$ is a sequence polynomials which converges uniformly on the segment $[0, \|a\|_{A_{\mathbf{e},2}}]$ to the function $\sqrt{\lambda}$ then $\varphi_n(\Psi(a)) \to \sqrt{\Psi(a)}$ in $C(M)$ as $n \to \infty$ and so $\varphi_n^2(\Psi(a)) \to \Psi(a)$. Therefore, $\varphi_n^2(a) = \Psi^{-1}(\varphi_n^2(\Psi(a))) \to a$ in $(A_{\mathbf{e},2}, \|\cdot\|_{A_{\mathbf{e},2}})$. Whence, using the $\mathbf{e}$-normality of A_2^+, we have $\varphi_n^2(a) \to a$ in A. On the other hand, a glance at Corollary 14(a) yields $\varphi_n^2(a) \in A_{\mathbf{e},1}^+$. Consequently, in view of Theorem 26, $a \in A_{\mathbf{e},1}^+$. □

If, under the assumptions of the preceding corollary, $A_{\mathbf{e},1} = A$ (as can be shown, the last identity is impossible if A is complex) and $A_{\mathbf{e},2}^+ = A_2^+$ then we have the inclusion $A_2^+ \subseteq A_1^+$. In particular, if K is a cone in the space $C(S)$ with the natural order which is $\mathbb{1}$-normal, $\mathbb{1}$ is an interior point of K, and $C(S)$ equipped with the cone K is an ordered normed algebra then the inclusion $K \subseteq C(S)^+$ holds and, in particular, K is normal. Nevertheless, the author does not know an example of a space $C(S)$ with a cone K such that under the ordering induced by K this space is an ordered Banach algebra while K is not normal.

Finally, some remarks about spectral properties of elements of an ordered normed algebra $A_{\mathbf{e}}$ are in order (we mention once more that $A_{\mathbf{e}}$ is real). The following assertion holds: *for an arbitrary element $a \in A_{\mathbf{e}}$ the spectral radius $r(a) = \|a\|_{\mathbf{e}}$*. Indeed, the *spectral radius* $r(a)$ of a defined by $r(a) = \lim_{n\to\infty} \|a^n\|_{\mathbf{e}}^{\frac{1}{n}}$ (see [10, pp. 10, 30]). Now it remains only to observe the validity of the identities $\|a\|_{\mathbf{e}} = \|\Psi_A(a)\|_{C(\mathcal{M}_{A_{\mathbf{e}}})}$ and $\|\Psi_A(a^n)\|_{C(\mathcal{M}_{A_{\mathbf{e}}})} = \|\Psi_A(a)\|^n_{C(\mathcal{M}_{A_{\mathbf{e}}})}$. The next version of the classical Gelfand's result holds: *for $a \in A_{\mathbf{e}}$ the spectrum $\sigma(a) = \{1\}$ if and only if $a = \mathbf{e}$*. Indeed,

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible in } (A_{\mathbf{e}})_{\mathbb{C}}\},$$

where $(A_{\mathbf{e}})_{\mathbb{C}}$ is the complexification of $A_{\mathbf{e}}$. The conclusion now follows immediately from the relation $\{f(a) : f \in \mathcal{M}_{A_{\mathbf{e}}}\} \subseteq \sigma(a)$. Next, since the algebra $A_{\mathbf{e}}$ is a subalgebra of $C(\mathcal{M}_{A_{\mathbf{e}}})$, $A_{\mathbf{e}}$ *is semi-simple*, i.e., the radical rad $A_{\mathbf{e}} = \{0\}$ (see [10, p. 57]).

3.2 Orthomorphisms

An operator T on a Riesz space E is said to be *band preserving* (see [4, p. 112]) whenever T leaves all bands of E invariant. If E is Dedekind complete then the latter is equivalent to the identity

$$(I - P)TP = 0 \tag{11}$$

for all order projections P on E. An operator T on an arbitrary Riesz space E is said to be *orthomorphism* (see [4, p. 115]) whenever T is a band preserving operator that is also order bounded. If E is Dedekind complete then the operator T on E is an orthomorphism if and only if the identity (11) holds and $T \in L_{\mathrm{r}}(E)$. Moreover, if E is an arbitrary Banach lattice then, by the Wickstead theorem (see [4, p. 258]), the operator T on E is an orthomorphism if and only if T belongs to the center $\mathcal{Z}(E)$ of E and, by the Abramovich–Veksler–Koldunov theorem (see [4, p. 256]), every band preserving operator on E is an orthomorphism.

Now let A be an ordered algebra. Using (11), we say that an element $a \in A$ is *order idempotent preserving* whenever $p^{\mathrm{d}}ap = 0$ for all $p \in \mathbf{OI}(A)$. An element $a \in A$ is said to be an *orthomorphism* whenever a is an order idempotent preserving element that is also regular. The collection of all orthomorphisms of an ordered algebra A will be denoted by $\mathrm{Orth}(A)$. The purpose of this section is to make a number of observations about order idempotent preserving elements and about orthomorphisms in ordered algebras and, thus, to break some ground for further research. For the case of orthomorphisms on Riesz spaces, the results which are analogous to obtained below can be found, e.g., in [4, § 2.3] and [13, Chapter 20].

First of all, we mention the next. Let A be an ordered algebra with the trivial set of order idempotents, i.e., $\mathbf{OI}(A) = \{0, \mathbf{e}\}$. Then every element a in A is order idempotent preserving and, in particular, $\mathrm{Orth}(A) = A_{\mathrm{r}}$. Thus, in this case, one cannot expect any distinctive properties of ortomorphisms. Moreover, this circumstance complicates researching the properties of orthomorphisms in arbitrary ordered algebras. That is how the matter stands if, for instance, A is the unitization of an arbitrary ordered algebra A_0, i.e., $A = A_0 \otimes \mathbb{R}$ or $A = A_0 \otimes \mathbb{C}$, under the order $(a, \lambda) \geq 0$ whenever $a \geq 0$ and $\lambda \geq 0$. Obviously, the analogous situation, namely, the identity $\mathrm{Orth}(A) = A_{\mathrm{r}}$, also holds when A is commutative.

Proposition 28 *For an element a in an ordered algebra A the following two statements are equivalent*:

(a) *a is order idempotent preserving*;
(b) *a commutes with every order idempotent.*
If, *in additional*, *a has a modulus then (a) and (b) are equivalent*:
(c) *The relation $p_1 \perp p_2$ in $\mathbf{OI}(A)$ implies $p_1 a \perp a p_2$ in A.*

Proof (a) $\Longrightarrow$ (b) Obviously, $p^{\mathrm{d}}ap = 0$ and so $ap = pap$, where $p \in \mathbf{OI}(A)$. On the other hand, $q^{\mathrm{d}}aq = 0$ with $q = p^{\mathrm{d}}$ and so $pa = pap$. Finally, $ap = pa$.

(b) $\Longrightarrow$ (a) If $ap = pa$ for all $p \in \mathbf{OI}(A)$ then $pap = ap^2 = ap$ and, hence, $p^{\mathrm{d}}ap = 0$.

(a) $\Longrightarrow$ (c) If $p_1|a| \geq b$ and $|a|p_2 \geq b$ with $b \in A$ then, taking into account the identity $p_1 p_2 = 0$, we get $0 \geq p_1^{\mathrm{d}} b$ and $0 \geq p_1 b$. Therefore, $0 \geq (p_1^{\mathrm{d}} + p_1)b = b$.

(c) $\Longrightarrow$ (a) In view of the relation $p^{\mathrm{d}} \perp p$ with $p \in \mathbf{OI}(A)$, we have $p^{\mathrm{d}}a \perp ap$, whence $|p^{\mathrm{d}}ap| \leq (p^{\mathrm{d}}|a|) \wedge (|a|p) = 0$ and so $p^{\mathrm{d}}ap = 0$. □

From part (b) of the preceding proposition, the next result follows: *if an element a is order idempotent preserving and invertible then a^{-1} is also order idempotent preserving*. Indeed, put $p^{\mathrm{d}}a^{-1}p = b$, where $p \in \mathbf{OI}(A)$. Then $ab = ap^{\mathrm{d}}a^{-1}p = p^{\mathrm{d}}aa^{-1}p = 0$ and, hence, $b = 0$.

As is easy to see, for an arbitrary ordered algebra A the inclusion $A_{\mathbf{e}} \subseteq \mathrm{Orth}(A)$ holds. On the other hand, if $A = L(E)$, where E is a Dedekind complete Riesz space, then $\mathrm{Orth}(A)$ coincides with the band B_I generated by the identity operator I in A_{r} (see [4, p. 118]). In view of the remarks done above, the equality $B_{\mathbf{e}} = \mathrm{Orth}(A)$ does not, in general, hold in the case of an arbitrary ordered algebra A. We can only assert the following.

Proposition 29 *Let A be an ordered algebra such that A_{r} is a Riesz space. Then for the band $B_{\mathbf{e}}$ generated by $\mathbf{e}$ in A_{r} the inclusion $B_{\mathbf{e}} \subseteq \mathrm{Orth}(A)$ holds.*

Proof Consider an element $a \in B_{\mathbf{e}}^{+}$. There exists a net $\{a_\alpha\}$ in $A_{\mathbf{e}}$ satisfying the relations $0 \le a_\alpha \uparrow a$ in A_{r}. For an arbitrary idempotent $p \in \mathbf{OI}(A)$, we have

$$0 \le p^{\mathrm{d}} a p = p^{\mathrm{d}}(a - a_\alpha) p + p^{\mathrm{d}} a_\alpha p = p^{\mathrm{d}}(a - a_\alpha) p \le a - a_\alpha \downarrow 0$$

and so $p^{\mathrm{d}} a p = 0$. □

In the theory of operators on a Riesz space E the important property of orthomorphisms is what every orthomorphism $T : E \to E$ is an order continuous operator (see [4, p. 117] or the remarks after the next theorem). Therefore, if E is Dedekind complete and $A = L(E)$ then $\mathrm{Orth}(A) \subseteq A_{\mathrm{n}}$. In the case of an arbitrary ordered algebra A the last inclusion does not, in general, hold. Indeed, there exists (see Example 2) a Riesz space E such that the identity operator I is not an order continuous element in the algebra $A = L(E)$, i.e., $I \notin A_{\mathrm{n}}$, while $I \in \mathrm{Orth}(A)$ (see also Example 31). Obviously, for $a \in \mathrm{Orth}(A)$, where A is an arbitrary ordered algebra, the inclusions $a \in A_{\mathrm{n_l}}$ and $a \in A_{\mathrm{n_r}}$ are equivalent.

Theorem 30 *Let A be a Dedekind complete ordered algebra such that $A_{\mathrm{s_l^\circ}} \cap A_{\mathrm{s_r^\circ}} = \{0\}$ (see (2)). Then the inclusion $\mathrm{Orth}(A) \subseteq A_{\mathrm{n}}$ holds.*

Proof Consider an element $a \in \mathrm{Orth}(A)$. Evidently, we can assume $a \ge 0$. Let $p_\alpha \downarrow 0$ in $\mathbf{OI}(A)$ and let $a p_\alpha \downarrow b \ge 0$ in A. Clearly, $p_\alpha^{\mathrm{d}} \uparrow \mathbf{e}$ in $\mathbf{OI}(A)$ and $p_\alpha^{\mathrm{d}} b = b p_\alpha^{\mathrm{d}} = 0$. According to our condition, we infer $b = 0$ and so $a \in A_{\mathrm{n}}$. □

If $A = L(E)$, where E is a Dedekind complete Riesz space, then, as is easy to see, we have $A_{\mathrm{s_l^\circ}} = \{0\}$ and, hence, in this case the preceding theorem can be applied. As the next example shows, the inclusion $\mathrm{Orth}(A) \subseteq A_{\mathrm{n}}$ does not even hold in the case of a *Dedekind complete* ordered algebra.

Example 31 Let B be a real Dedekind complete ordered Banach algebra with a unit $\mathbf{e}_0 \ne 0$ which admits a non-zero, positive, multiplicative functional f and a net $\{p_\alpha\}$ satisfying $p_\alpha \uparrow \mathbf{e}_0$ in $\mathbf{OI}(B)$ and $f(p_\alpha) = 0$ for all α. First of all, we mention that such an algebra exists. Indeed, consider a Dedekind complete ordered Banach algebra ℓ_∞ with the natural order, norm, and multiplication. Put $p_n = (\underbrace{1, \ldots, 1}_{n}, 0, 0, \ldots)$. Obviously, $p_n \uparrow (1, 1, \ldots)$. On the other hand, as is well known, ℓ_∞ admits a non-zero, positive, multiplicative functional f satisfying $f(c_0) = \{0\}$ (if $f_n(x) = x_n$ for $x = (x_1, x_2, \ldots) \in \ell_\infty$ then f can be taken as a cluster point of $\{f_n\}$ in the $\sigma(\ell'_\infty, \ell_\infty)$-topology). Next, consider the linear space $A = B \oplus \mathbb{R}$. Under the multiplication

$$(b_1, \lambda_1) \star (b_2, \lambda_2) = (b_1 b_2, \lambda_1 f(b_2) + \lambda_2 f(b_1) + \lambda_1 \lambda_2),$$

the norm

$$\|(b, \lambda)\|_A = \|b\|_B + |\lambda|,$$

and the order induced by the cone

$$A^+ = \{(b, \lambda) : b \in B^+ \text{ and } \lambda \in \mathbb{R}^+\},$$

the algebra A is a Dedekind complete ordered Banach algebra with unit $\mathbf{e} = (\mathbf{e}_0, 0)$. Evidently, $\mathbf{OI}(A) = \{(p, 0) : p \in \mathbf{OI}(B)\}$ and, hence, $\mathrm{Orth}(A) = \{(b, \lambda) : b \in \mathrm{Orth}(B)\}$. In particular, $(0, 1) \in \mathrm{Orth}(A)$. However, we claim that $(0, 1) \notin A_\mathrm{n}$. Indeed, on the one hand $(\mathbf{e}_0 - p_\alpha, 0) \downarrow 0$ in $\mathbf{OI}(A)$ and, on the other hand,

$$(\mathbf{e}_0 - p_\alpha, 0) \star (0, 1) = (0, 1) \star (\mathbf{e}_0 - p_\alpha, 0) = (0, 1),$$

and we are done. □

In the case of a Riesz space E, every positive orthomorphism T on E is a lattice homomorphism. On the other hand, if S is an order continuous lattice homomorphism on a Dedekind complete Riesz space E then the operator $U \to SU$ from $L_\mathrm{r}(E)$ into $L_\mathrm{r}(E)$ is also a lattice homomorphism and if the order dual $E^\sim \neq \{0\}$ then S is a lattice homomorphism (see [4, p. 96]). These results justify the following definition (see also Sect. 4.3). Let A be an ordered algebra such that A_r is a Riesz space. An element $a \in A$ is said to be a *lattice homomorphism* (of the algebra A) whenever the operator $L_a : A_\mathrm{r} \to A_\mathrm{r}$ defined by $L_a b = ab$ is a lattice homomorphism on A_r. Obviously, if a is a lattice homomorphism of A then $a \geq 0$.

Proposition 32 *Let A be an ordered algebra such that A_r is a Riesz space and let $b \in A$ be invertible. Then the elements b and b^{-1} are lattice homomorphisms of A if and only if $b, b^{-1} \in A^+$.*

Proof The necessity is obvious. We shall prove the sufficiency. Fix $a \in A_\mathrm{r}$. Clearly, $|L_b a| = |ba| \leq b|a|$, whence $|a| \geq b^{-1}|ba| \geq |a|$. Therefore, $|a| = b^{-1}|ba|$ and so $b|a| = |ba|$, i.e., L_b is a lattice homomorphism on A_r. □

For the case of operators on ordered Hilbert spaces the notions which are close to ones considered in this subsection, will be discussed in Sect. 4.3.

4 Ordered C^*-algebras

4.1 A general case

An ordered Banach algebra A is called an *ordered* C^*-algebra if it is equipped with an involution $a \to a^*$ that maps the cone A^+ into itself and under which A is a C^*-algebra, i.e., $\|a^* a\|_A = \|a\|_A^2$ for all $a \in A$. The most important example of C^*-algebra is the algebra $B(H)$ of all bounded operators on some Hilbert space H. If a Hilbert space H is an ordered Banach space with a cone H^+ then (see [3]) the algebra $B(H)$ is an ordered C^*-algebra with the cone

$$(B(H))^+ = \{T \in B(H) : T(H^+) \subseteq H^+\} \tag{12}$$

if and only if H^+ is a *self-adjoint* cone, i.e., $H^+ = (H^+)'$, where $(H^+)'$ is a dual cone of H^+. A Hilbert space H with a self-adjoint cone H^+ is called an *ordered Hilbert space*. The study of ordered C^*-algebras and, in particular, of ordered Hilbert spaces was initiated in [3]. This section is a continuation of that research. Of course, we shall keep in mind the results obtained above. The main emphasis will be on the case of an ordered C^*-algebra $B(H)$ (see Sect. 4.3). Before, in Sect. 4.2, we shall consider some properties of ordered Hilbert spaces.

In view of Corollary 13, the center $A_{\mathbf{e}}$ of an ordered C^*-algebra A is a real commutative $\star$-algebra. On the other hand (see [3]), if an element a belongs to $A_{\mathbf{e}}$ and $a \wedge a^*$ exists then a is hermitian, i.e., $a = a^*$. As will be shown in the next theorem, there is a wide class of ordered C^*-algebras such that the center of these algebras consists only of hermitian elements.

Theorem 33 *Let A be an ordered C^*-algebra and let one of the assumptions* (*a*) *or* (*b*) *of Theorem 26 be satisfied. Then every element $a \in A_{\mathbf{e}}$ is hermitian.*

Proof In view of Corollary 18, the **e**-uniform completeness of A and the **e**-normality of A^+ are equivalent. Nevertheless, below we shall give the proof which does not use the norm completeness of A.

If A is **e**-uniformly complete then, by Theorem 22, the center $A_{\mathbf{e}}$ is an AM-space and, in particular, for every $a \in A_{\mathbf{e}}$ an element $a \wedge a^*$ exists in $A_{\mathbf{e}}$. Then, as was mentioned above, $a = a^*$.

Now we shall assume the **e**-normality of A^+. Putting $M = \mathcal{M}_{A_{\mathbf{e}}}$, we consider an isomorphism $\Psi_A : A_{\mathbf{e}} \to C(M)$ (see Theorem 15 and the remarks before it) and define the operator $T : R(\Psi_A) \to R(\Psi_A)$ by $T(\Psi_A(a)) = \Psi_A(a^*)$. As is easy to see, T is an algebraic homomorphism. Moreover, in view of the identity $\|a\|_{\mathbf{e}} = \|a^*\|_{\mathbf{e}}$ for all $a \in A_{\mathbf{e}}$, the operator T is continuous. Therefore, since $R(\Psi_A)$ is dense in $C(M)$, T extends to all of $C(M)$ as a (algebraic and lattice) homomorphism (denoted by T again). There exists (see [1, p. 145]) a continuous mapping $\xi : M \to M$ satisfying $(Tx)(s) = x(\xi(s))$ for all $x \in C(M)$ and $s \in M$.

We shall check the identity $\xi(s) = s$ for all $s \in M$. To this end, proceeding by contradiction, we find a point $s \in M$ and two neighborhoods $\mathcal{U}_s$ and $\mathcal{V}_{\xi(s)}$ of the points s and $\xi(s)$, respectively, such that

$$\mathcal{U}_s \cap \mathcal{V}_{\xi(s)} = \emptyset \quad \text{and} \quad \xi(\mathcal{U}_s) \subseteq \mathcal{V}_{\xi(s)}. \tag{13}$$

Taking into account Urysohn's lemma, we pick a non-zero function $x \in C(M)$ satisfying the relation $x(M \backslash \mathcal{U}_s) = \{0\}$. Therefore, taking into account (13), we have $x(t)x(\xi(t)) = 0$ for all $t \in M$ and so $x \cdot Tx = 0$. On the other hand, there exists a sequence $\{a_n\}$ in $A_{\mathbf{e}}$ such that $\Psi_A(a_n) \to x$ as $n \to \infty$. Then $\Psi_A(a_n)\Psi_A(a_n^*) \to x \cdot Tx = 0$ and, hence, $\Psi_A(a_n a_n^*) \to 0$. The latter implies the relation $a_n a_n^* \to 0$ in $A_{\mathbf{e}}$. Using the **e**-normality of A^+, we obtain $a_n a_n^* \to 0$ in A. Since A is a C^*-algebra, $a_n \to 0$ in A and so $x = 0$, a contradiction. Finally, $Tx = x$ for all x. Consequently, $\Psi_A(a) = \Psi_A(a^*)$ or $a = a^*$ for all $a \in A_{\mathbf{e}}$. □

It is not known if the preceding theorem is valid for an arbitrary ordered C^*-algebra. Moreover, the author does not know an example of an ordered C^*-algebra with an unnormal cone A^+ (see also the remarks after Corollary 27).

Our next aim is to make more precisely the relations (7), (8), and (9).

Theorem 34 *Let A be an ordered C^*-algebra and let one of the following assumptions be satisfied*:

(a) *A is* **e**-*uniformly complete and the norm of A is monotone on $A_{\mathbf{e}}$*;
(b) *A^+ is* **e**-*normal.*

Then for every element $a \in A_{\mathbf{e}}$ the next identity holds

$$\|a\|_{\mathbf{e}} = \|a\|_A. \tag{14}$$

Proof Again, we shall give the proof which does not use the norm completeness of A.

Let the algebra A be **e**-uniformly complete and let the norm of A be monotone on $A_{\mathbf{e}}$. In view of the preceding theorem, a is hermitian. Then, using Corollary 14(a) and the relations (8), we have

$$\|a\|_A^2 = \|a^2\|_A = \||a^2|\|_A = \|a^2\|_{\mathbf{e}} = \|a\|_{\mathbf{e}}^2$$

and, hence, $\|a\|_{\mathbf{e}} = \|a\|_A$.

Let the cone A^+ be **e**-normal. Assume first that $0 \le a \le \mathbf{e}$. Obviously, $0 \le a^n \le \mathbf{e}$ for all $n \in \mathbb{N}$. Taking into account the **e**-normality of A^+, we find a constant $C > 0$ satisfying $\|a\|_A^{2^n} = \|a^{2^n}\|_A \le C$ for all n and, hence, $\|a\|_A \le C^{\frac{1}{2^n}}$. Letting $n \to \infty$, we infer $\|a\|_A \le 1$. Then, using the inequalities $0 \le a \le \|a\|_{\mathbf{e}}\mathbf{e}$, we have $\|a\|_A \le \|a\|_{\mathbf{e}}$. On the other hand, a glance at (9) yields $\|a\|_{\mathbf{e}} \le \|a\|_A$. Now the identity (14) is obvious. □

4.2 An ordered Hilbert space

First of all, we mention some properties of an ordered Hilbert space H which will be needed later on (see [3]). Every ordered Hilbert space H is *real* and, hence, the space $B(H)$ is also real (the approach to the complex case is discussed in [3]). The cone H^+ is generating. Moreover, the norm on H is strictly monotone, i.e., the inequalities $0 \le x < y$ imply $\|x\|_H < \|y\|_H$ and, in particular, the cone H^+ is normal. An ordered Hilbert space H is a Riesz space (under the order induced by H^+) if and only if H^+ has the Riesz decomposition property; in this case, the Riesz space H is Dedekind complete. However, there exists an ordered Hilbert space H which is not a Riesz space (see Example 38(b)). Nevertheless, as will be shown below (see (19) and Corollary 40), every ordered Hilbert space H admits operations which are "close" to the lattice ones. Below, unless otherwise stated, H will denote an ordered Hilbert space with a cone H^+ and the norm $\|\cdot\|$. We assume $H \neq \{0\}$.

We start our discussion with the following auxiliary result. Recall that if N is nonempty, closed, convex subset of an arbitrary Hilbert space H then for each point x of H, there is a unique nearest point of N to x. This point is called the *projection* of x onto N and is denoted by $P_N x$. Obviously, if $x \in N$ then $P_N x = x$. As is well known, a point $z \in N$ is the projection of x onto N if and only if the inequality

$\langle x - z, n - z\rangle \le 0$ holds for all $n \in N$. Moreover, the mapping $P_N : H \to N$ is continuous.

Lemma 35 *Let H be an ordered Hilbert space, $x \in H$. The following statements hold*:

(a) *A point $z \in H^+$ is the projection of x onto H^+ if and only if $z \ge x$ and $\langle x - z, z\rangle = 0$. In particular, if $x \le 0$ then $P_{H^+}x = 0$;*
(b) *The decomposition $x = P_{H^+}x + P_{-H^+}x$ holds, moreover, $\langle P_{H^+}x, P_{-H^+}x\rangle = 0$ and $P_{-H^+}x = -P_{H^+}(-x)$.*

Proof (a) Let z be the projection of x onto H^+. As was mentioned above, the latter is equivalent the validity of the inequality

$$\langle x - z, k - z\rangle \le 0 \tag{15}$$

for all $k \in H^+$. Fix $k \in H^+$. For an arbitrary scalar $\lambda \ge 0$, we have $\langle x - z, \lambda k - z\rangle \le 0$ and so $\lambda\langle x - z, k\rangle \le \langle x - z, z\rangle$. The last relation implies at once $\langle x - z, k\rangle \le 0$ for all $k \in H^+$. Whence, we infer $z \ge x$ as H^+ is self-adjoint. On the other hand, using (15) once more, we obtain $\langle x - z, \lambda z - z\rangle \le 0$ for all $\lambda \ge 0$. Therefore, $(\lambda - 1)\langle x - z, z\rangle \le 0$. Hence, $\langle x - z, z\rangle = 0$.

For the converse, for arbitrary $k \in H^+$, we have $\langle x - z, k - z\rangle = \langle x - z, k\rangle \le 0$. Thus, (15) holds and so $z = P_{H^+}x$.

(b) Obviously, for all $k \in H^+$ the inequality $\langle z, k\rangle \ge 0$ is valid with $z = P_{H^+}x$. Then, in view of part (a), $\langle z, k + x - z\rangle \ge 0$ and, hence, $\langle x - (x - z), -k - (x - z)\rangle \le 0$. The latter is equivalent to the relation $x - z = P_{-H^+}x$. Now the equality $P_{H^+}(-x) = -P_{-H^+}x$ follows at once from part (a). □

Corollary 36 *Assume that an ordered Hilbert space H is a Riesz space. Then for an arbitrary element $x \in H$ the identities $x^+ = P_{H^+}x$ and $x^- = P_{H^+}(-x)$ hold.*

In particular, $\langle x^+, x^-\rangle = 0$.

Proof Since every element $x \in H$ has a unique decomposition into the difference of two nonnegative, disjoint elements, it suffices to check the relation $P_{H^+}x \wedge P_{H^+}(-x) = 0$. Indeed, if the inequalities $z \le P_{H^+}x$ and $z \le P_{H^+}(-x)$ hold for some $z \in H^+$ then, in view of the relations $0 \le \langle z, z\rangle \le \langle P_{H^+}x, P_{H^+}(-x)\rangle = 0$, we have $z = 0$. □

Corollary 37 *Let $x, y \in H^+$. Then $\langle x, y\rangle = 0$ if and only if $P_{H^+}(x - y) = x$.*

In view of Corollary 36, the study of the projection operator P_{H^+} is of special interest when H^+ is not a lattice cone. Before proceeding further, we consider an important example of such a cone.

Example 38 Let H be an arbitrary Hilbert space with $\dim H \ge 2$. Let $z \in H$ with $\|z\| = 1$ and let $\epsilon > 0$. Then (see [5, § 2.6]), the *ice cream cone* is the cone

$$K_{z,\epsilon} = \{x \in H : \langle x, z\rangle \ge \epsilon\|x\|\}.$$

The proof of the following two statements can be found in [3]:

(a) In a real space H the ice cream cone $K_{z,\epsilon}$ is self-adjoint if and only if $\epsilon = \frac{1}{\sqrt{2}}$.

(b) If $\dim H \geq 3$ and H equipped with the cone $K_{z,\epsilon}$, where $\epsilon \in (0, 1)$, then the center $(B(H))_I$ of the ordered Banach algebra $B(H)$ is trivial, i.e.,

$$(B(H))_I = \{\lambda I : \lambda \in \mathbb{R}\}.$$

As follows at once from part (b), if H is real and $\dim H \geq 3$ then the ordered Hilbert space H with the cone $K_{z,\frac{1}{\sqrt{2}}}$ is not a Riesz space.

(c) In a real space H with the ice cream cone $K = K_{z,\frac{1}{\sqrt{2}}}$ for an element $x \notin \pm K$ the identity

$$P_K x = \frac{1}{2}\left(1 + \frac{\langle x, z\rangle}{l(x, z)}\right)(x + (l(x, z) - \langle x, z\rangle)z) \tag{16}$$

holds, where $l(x, z) = (\|x\|^2 - \langle x, z\rangle^2)^{\frac{1}{2}}$.

To see this, we mention first that if $l(x, z) = 0$ then $\|x\| = |\langle x, z\rangle|$, whence x and z are linearly dependent. The latter contradicts the condition $x \notin \pm K$. Thus, the element w in the right part of (16) is well defined. Moreover, the last relation is equivalent to the inequality $|\langle x, z\rangle| < \frac{1}{\sqrt{2}}\|x\|$ and so $|\langle x, z\rangle| < l(x, z)$. Using elementary calculations, it is easy to check the equality $\|w\| = \frac{1}{\sqrt{2}}(l(x, z) + \langle x, z\rangle)$. Whence, we obtain the identity $\langle w, z\rangle = \frac{1}{\sqrt{2}}\|w\|$ which implies $w \in K$ and the identity

$$\langle x - w, w\rangle = 0. \tag{17}$$

Next, we claim that the relation

$$\|w - x\|^2 = \frac{1}{2}(l(x, z) - \langle x, z\rangle)^2 \tag{18}$$

holds. Indeed,

$$\begin{aligned}
\|w - x\|^2 &= \|w\|^2 - 2\langle w, x\rangle + \|x\|^2 \\
&= \|w\|^2 - \left(1 + \frac{\langle x, z\rangle}{l(x, z)}\right)\left(\|x\|^2 - \langle x, z\rangle^2 + l(x, z)\langle x, z\rangle\right) + \|x\|^2 \\
&= \|w\|^2 - \left(1 + \frac{\langle x, z\rangle}{l(x, z)}\right)^2 l(x, z)^2 + \|x\|^2 = \|x\|^2 - \|w\|^2 \\
&= \|x\|^2 - \frac{1}{2}\left(1 + 2\frac{\langle x, z\rangle}{l(x, z)} + \frac{\langle x, z\rangle^2}{l(x, z)^2}\right) l(x, z)^2 = \frac{\|x\|^2}{2} - \langle x, z\rangle l(x, z) \\
&= \frac{1}{2}(l(x, z)^2 - 2\langle x, z\rangle l(x, z) + \langle x, z\rangle^2) = \frac{1}{2}(l(x, z) - \langle x, z\rangle)^2.
\end{aligned}$$

Whence, using the relations $\langle w - x, z\rangle = \frac{1}{2}(l(x, z) - \langle x, z\rangle) > 0$, we obtain

$$\langle w - x, z\rangle = \frac{1}{\sqrt{2}}\|w - x\|$$

and so $w - x \in K$. Finally, according to Lemma 35(a) and the equality (17), we have $w = P_K x$.

The next formula for the distance function $d_K(x) = \|x - P_K x\|$, which associates with each point x of H its distance from K, follows from (18).

(d) If $x \notin -K$ then $d_K(x) = \frac{1}{\sqrt{2}}(l(x,z) - \langle x, z\rangle)^+$ and if $x \in -K$ then $d_K(x) = \|x\|$. □

Corollary 36 suggests as, for the case of an arbitrary ordered Hilbert space, operations which are "close" to lattice ones can be defined. Namely, for an arbitrary element $x \in H$, we put $|x|_{H^+} = P_{H^+}x + P_{H^+}(-x)$. Now for $x, y \in H$, we can define the next operations

$$x \underset{H^+}{\vee} y = \frac{1}{2}(x + y + |x - y|_{H^+}) \quad \text{and} \quad x \underset{H^+}{\wedge} y = \frac{1}{2}(x + y - |x - y|_{H^+}). \tag{19}$$

In view of Lemma 35(a), $|x|_{H^+} \geq \pm x$, whence $x \vee_{H^+} y \geq x, y$ and $x \wedge_{H^+} y \leq x, y$. Moreover, $|x|_{H^+} = |-x|_{H^+}$ and the identities

$$x + y = x \underset{H^+}{\vee} y + x \underset{H^+}{\wedge} y \quad \text{and} \quad |x - y|_{H^+} = x \underset{H^+}{\vee} y - x \underset{H^+}{\wedge} y$$

hold. The operations defined above are continuous in the norm topology. It follows at once from Corollary 36, if H is a Riesz space then the operations $\vee_{H^+}$ and $\wedge_{H^+}$ coincide with the usual supremum and infimum. In fact, this coincidence holds in a more general case (see Corollary 40).

Lemma 39 *If for $x, y \in H$ the infimum $x \wedge y$ exists in H (under the order induced by H^+) and $x \wedge y = 0$ then $\langle x, y\rangle = 0$.*

Proof A glance at Lemma 35(a) yields $P_{H^+}(x - y) \geq 0$ and $P_{H^+}(x - y) \geq x - y$. Whence, $x \geq x - P_{H^+}(x - y)$ and $y \geq x - P_{H^+}(x - y)$. Therefore, using our condition, we conclude $P_{H^+}(x - y) \geq x$. The last relation and Lemma 35(a) once more imply

$$0 = \langle P_{H^+}(x - y), P_{H^+}(x - y) - x + y\rangle \geq \langle P_{H^+}(x - y), y\rangle \geq \langle x, y\rangle \geq 0.$$

Finally, $\langle x, y\rangle = 0$. □

Corollary 40 *Let $x, y \in H$. If x has a modulus $|x|$ then $|x| = |x|_{H^+}$. Analogously, if either $x \vee y$ or $x \wedge y$ exists, then either $x \vee y = x \vee_{H^+} y$ or $x \wedge y = x \wedge_{H^+} y$, respectively.*

Proof First of all, we recall that in an ordered linear space E the modulus $|v|$ of an element $v \in E$ exists if and only if its positive part v^+ (its negative part v^-) exists. Moreover, if $u, w \in E$ then the modulus $|u - w|$ exists if and only if $u \vee w$ $(u \wedge w)$ exists; in this case, e.g., $u \vee w = \frac{1}{2}(u + w + |u - w|)$.

Now assume that $|x|$ exists. Then $x^+ \wedge x^- = 0$. A glance at the preceding lemma yields $\langle x^+, x^- \rangle = 0$. Whence, using Corollary 37, we obtain

$$|x|_{H^+} = P_{H^+}(x^+ - x^-) + P_{H^+}(x^- - x^+) = x^+ + x^- = |x|.$$

Therefore, taking into account the remarks above, we easily obtain the required equalities for $x \vee y$ and $x \wedge y$. □

As follows at once from the preceding corollary, the operations $\vee_{H^+}$ and $\wedge_{H^+}$ defined above coincide with the usual lattice operations $\vee$ and $\wedge$, when the latter ones are well defined. Later on, we shall simply write $x \vee y$ instead of $x \vee_{H^+} y$; analogously, for the cases of $x \wedge_{H^+} y$ and $|x|_{H^+}$. Thus, now the expression $x \vee y$ means some element in H which can be defined for every pair $\{x, y\}$ and coincides with the usual least upper bound $\sup\{x, y\}$ if the latter exists. Moreover, now we have a right to put $x^+ = P_{H^+}x$ and $x^- = P_{H^+}(-x)$.

The next proposition contains a list of several elementary properties of operations introduced above.

Proposition 41 *For any elements x, y, and z in H the following statements hold*:

(a) $|\alpha x| = |\alpha||x|$ *for all* $\alpha \in \mathbb{R}$ *and* $(\alpha x)^{\pm} = \alpha x^{\pm}$ *for all* $\alpha \in \mathbb{R}^+$;
(b) $x^+ = x \vee 0$ *and* $x^- = (-x) \vee 0$;
(c) $\alpha(x \vee y) = (\alpha x) \vee (\alpha y)$ *for all* $\alpha \in \mathbb{R}^+$, $x \vee y = -((-x) \wedge (-y)) = (x - y)^+ + y$, *and* $z + (x \vee y) = (z + x) \vee (z + y)$;
(d) $|x| = x^+ \vee x^-$ *and* $x^+ \wedge x^- = 0$;
(e) *The inequality* $x \geq y$ *holds if and only if* $x \vee y = x$;
(f) *If* $x, y \geq 0$ *then* $x \wedge y = 0$ *if and only if* $\langle x, y \rangle = 0$;
(g) $\langle x, y \rangle = \langle x \vee y, x + y \rangle - \|x \vee y\|^2 = \langle x \wedge y, x + y \rangle - \|x \wedge y\|^2$;
(h) *The norm on* H *is lattice*, *i.e.*, *the inequality* $|x| \leq |y|$ *implies* $\|x\| \leq \|y\|$.

Proof (a) Taking into account Lemma 35(a), it suffices to observe the validity of the equality $P_{H^+}(\alpha x) = \alpha P_{H^+}x$ for all $x \in H$ and $\alpha \in \mathbb{R}^+$.

(b) We have

$$2(x \vee 0) = x + |x| = P_{H^+}x - P_{H^+}(-x) + P_{H^+}x + P_{H^+}(-x) = 2P_{H^+}x = 2x^+.$$

The proof of the second identity is similar.

(c) We have

$$2\alpha(x \vee y) = \alpha(x + y + |x - y|) = \alpha x + \alpha y + |\alpha(x - y)| = 2((\alpha x) \vee (\alpha y)),$$

$$-2((-x) \wedge (-y)) = -(-x - y - |x - y|) = 2(x \vee y),$$

$$2(z + (x \vee y)) = z + x + z + y + |(z + x) - (z + y)| = 2((z + x) \vee (z + y)),$$

and

$$(x - y)^+ + y = (x - y) \vee 0 + y = x \vee y.$$

(d) According to Lemma 35(b) and Corollary 37, we have

$$\begin{aligned}2(x^+ \vee x^-) &= P_{H^+}x + P_{H^+}(-x) + P_{H^+}(P_{H^+}x - P_{H^+}(-x)) \\ &\quad + P_{H^+}(P_{H^+}(-x) - P_{H^+}x) \\ &= P_{H^+}x + P_{H^+}(-x) + P_{H^+}x + P_{H^+}(-x) = 2|x|.\end{aligned}$$

(e) The sufficiency is clear. We shall check the necessity. Indeed, $x = \frac{1}{2}(x + y + |x - y|)$, whence $x - y = |x - y| \geq 0$.

(f) The relation $x \wedge y = 0$ is equivalent to the equalities

$$x + y = |x - y| = P_{H^+}(x - y) + P_{H^+}(y - x).$$

The latter, in view of the identity $x - y = P_{H^+}(x - y) - P_{H^+}(y - x)$, is equivalent to the relation $x = P_{H^+}(x - y)$. It remains to remember Corollary 37.

(g) According to (c), we have $(x \vee y - x) \wedge (x \vee y - y) = 0$. Whence, in view of (f), we obtain $\langle x \vee y - x, x \vee y - y \rangle = 0$, as required.

(h) Using Lemma 35(b), it suffices to observe the validity of the identity $\| |x| \| = \|x\|$ and to use the monotonicity of the norm on H. □

Below, for any *positive* elements x and y in an ordered Hilbert space H, we shall write $x \perp y$ whenever x and y are either orthogonal in H, i.e., $\langle x, y \rangle = 0$, or disjoint in H, i.e., $x \wedge y = 0$. In view of part (f) of the preceding proposition, these two notions are equivalent and, hence, this notation cannot lead to an ambiguity.

Unfortunately, a number of other properties of lattice operations which hold in the case of Riesz spaces, are not valid in the case of operations $\vee$ and $\wedge$ introduced above. For instance, the inequalities $x, y \geq 0$ do not imply $x \wedge y \geq 0$. Otherwise, for elements $x, y, z \in H$, from the relations $x - z, y - z \in H^+$, it follows $(x - z) \wedge (y - z) \geq 0$ and so $x \wedge y \geq z$. Consequently, there exists the infimum of the set $\{x, y\}$ in H. Therefore, since x and y are arbitrary, H is a Riesz space. The latter does not hold for an arbitrary ordered Hilbert space. Thus, the inequality $x + y \geq |x - y|$ is not valid for all $x, y \geq 0$. Consequently, the *modulus does not satisfy the triangle inequality* and so, that is equivalent, the operations $x \to x^{\pm}$ do not satisfy it. Moreover, in a general case, the inequality

$$(x + y)^+ \leq x^+ + y \tag{20}$$

with $x \in H$ and $y \in H^+$ does not also hold. This, in turn, means that the operation $\vee$ is *not associative*. Otherwise, for arbitrary $x, y \in H$, we have $(x \vee (-y)) \vee 0 = x \vee y^-$. Whence, assuming $y \geq 0$, we obtain

$$(x + y)^+ - y \leq (x \vee (-y))^+ = x \vee 0 = x^+$$

and so (20) holds, a contradiction.

Example 38′ We shall use the notations and the results of Example 38. Let us consider the self-adjoint ice cream cone $K = K_{z, \frac{1}{\sqrt{2}}}$. As is easy to see, for arbitrary $x \notin \pm K$, we have the identity

$$|x| = l(x, z)^{-1}((\|x\|^2 - 2\langle x, z \rangle^2)z + \langle x, z \rangle x);$$

in particular, $\langle x, z\rangle = 0$ if and only if $|x| = \|x\|z$.

Now let $\dim H \geq 3$ and let u and v be two elements in H such that the system $\{u, v, z\}$ is orthonormal. Clearly, $y = z + v \geq 0$. On the other hand, as is easy to check, $\langle u^-, (u-y)^+\rangle > 0$ and, hence, $u^- \wedge (u-y)^+ \neq 0$ (the last relation does not hold in the case of a Riesz space). □

The next lemma is useful for the construction of self-adjoint cones with special properties.

Lemma 42 *Let K_0 be a (not necessarily closed) cone in an arbitrary real Hilbert space H such that $K_0 \subseteq K_0'$. Then there exists a self-adjoint cone K satisfying $K_0 \subseteq K$.*

Proof Consider the collection $\mathcal{K}$ of all cones C satisfying the relations $K_0 \subseteq C \subseteq C'$, ordered by inclusion. Under this ordering, $\mathcal{K}$ is a partially ordered set. Let $\{C_\alpha\}$ be a chain in $\mathcal{K}$. Put $C_0 = \bigcup_\alpha C_\alpha$. Evidently, C_0 is a cone and, moreover, $C_0 \subseteq C_0'$. By Zorn's lemma, $\mathcal{K}$ has a maximal element, say K. The cone K is self-adjoint. To see this, proceeding by contradiction, we find $x \in K'\backslash K$ and put $K_1 = K + \{\lambda x : \lambda \geq 0\}$. As is easy to see, K_1 is a cone, $K \subsetneq K_1$, and if $y, z \in K_1$ then $\langle y, z\rangle \geq 0$. Thus, $K_1 \subseteq K_1'$ and so $K_1 \in \mathcal{K}$, which is impossible in view of the maximality of K. Finally, $K = K'$. □

Example 43 (a) As follows from Proposition 41(f), if $x, y \in H$ and the greatest lower bound $\inf\{x, y\} = 0$ then $x \perp y$, i.e., $\langle x, y\rangle = 0$. Nevertheless, the converse is not valid. To see this, we consider the Euclidean space $H = \mathbb{R}^3$ and the five elements $x_\pm = (1, \pm 1, 0)$, $y = (1, 0, -\lambda)$, and $z_\pm = (1, \pm 1, \lambda^{-1})$, where $\lambda > 0$, and a wedge K_0 generated by these elements. Then $K_0 \subseteq K_0'$. Moreover, the wedge K_0 is closed (see [5, p. 126]) and is a cone. According to the preceding lemma, there exists a self-adjoint cone K satisfying $K_0 \subseteq K$ (in fact, as can be shown, the cone K_0 is self-adjoint and so $K_0 = K$). Clearly, $x_+ \perp x_-$. On the other hand, we consider the element $w = (0, 0, \alpha)$ with $\alpha \in (0, \lambda^{-1}]$. Then $w \notin K$ while $x_\pm \geq -w$ and, hence, the infimum of the subset $\{x_+, x_-\}$ of the space H with the cone K does not exist.

(b) Let $H = \mathbb{R}^3$ and let us consider the elements $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $z = (1, 1, \frac{1}{2})$, and $w = (1, 1, -\frac{1}{2})$. According to the preceding lemma once more, there exists a self-adjoint cone K which contains these four elements and also the elements $z - \mathbf{e}_1$ and $z - \mathbf{e}_2$. Then, under the order induced by the cone K in H, the following relations $\mathbf{e}_1 \perp \mathbf{e}_2$, $0 \leq \mathbf{e}_1 \leq z$, and $0 \leq \mathbf{e}_2 \leq z$ hold. Nevertheless, the inequality $\mathbf{e}_1 + \mathbf{e}_2 \leq z$ does not hold (in particular, H is not a Riesz space). □

In conclusion of this subsection, we mention the paper [7], where the related concept of ordered Banach spaces which admit a quasi-lattice structure, was considered, but from another viewpoint. In particular, as was shown in this paper, every strictly convex reflexive ordered Banach space E with a generating cone E^+ admits such a structure.

4.3 The ordered C^*-algebra $B(H)$

Let H be an ordered Hilbert space. We mention at once that below, unless otherwise stated, to avoid ambiguity, an operator $T \in B(H)$ will be called *positive* if it maps H^+

into itself, i.e., if T is positive in the sense of the theory of ordered linear spaces. In this case, we write $T \geq 0$. Nevertheless, as will be shown later on (see Theorem 50), for operators belonging to $(B(H))_I$ both possible notions of positivity coincide. Since the cone H^+ has the strong Levi property (see [3]), for every net $\{T_\alpha\}$ in $B(H)$ satisfying $T_\alpha \uparrow\leq S$ in $B(H)$ and every $x \in H$, the net $\{T_\alpha x\}$ in H is norm convergent in H and $T_\alpha \uparrow T$ in $B(H)$, where $Tx = \lim_\alpha T_\alpha x$. Next, as was noticed in the preceding subsection, the cone H^+ is generating and normal. Therefore, the cone $(B(H))^+$ (see (12)) is also normal.

Now, using Theorems 22, 26, 33, 34, Corollary 23, and the remarks above, we are in a position to state the next important result.

Theorem 44 *Let H be an ordered Hilbert space. Then the center $(B(H))_I$ is a commutative, norm closed, and order closed subalgebra of $B(H)$, every element of $(B(H))_I$ is a hermitian operator, and, under the order induced by $B(H)$, $(B(H))_I$ is a Dedekind complete AM-space with unit I such that $\|T\|_I = \|T\|_{B(H)}$ for every $T \in (B(H))_I$.*

Moreover, $(B(H))_I$ is isometrically, algebraically, and order-embeddable onto the algebra $C(M)$ for some extremally disconnected compact space M.

By analogy with the case of a Riesz space, the projection P on H satisfying the inequalities $0 \leq P \leq I$ is called an *order projection*.

Corollary 45 *Every order projection P on H is an orthogonal projection.*

Our next goal is to study the class $(B(H))_{\mathrm{n}}$ of order continuous elements in the ordered C^*-algebra $B(H)$.

Lemma 46 *Let $\{P_\alpha\}$ be a decreasing net of order projections on H. The following statements are equivalent*:

(a) $P_\alpha \downarrow 0$ *in* $B(H)$;
(b) $P_\alpha \downarrow 0$ *in* $\mathbf{OI}(B(H))$;
(c) $P_\alpha x \downarrow 0$ *in* H *for all* $x \in H^+$;
(d) $\|P_\alpha x\| \to 0$ *for all* $x \in H$.

Proof The implications (d) $\Longrightarrow$ (c) $\Longrightarrow$ (a) $\Longrightarrow$ (b) are obvious.

(b) $\Longrightarrow$ (d) Evidently, $P_\alpha \downarrow$ in $B(H)$, whence, for some $P \in B(H)$, we have $P_\alpha \downarrow P$ in $B(H)$ and $P_\alpha x \to Px$ for all $x \in H$. Moreover, for arbitrary $y \in H$ the relations

$$P_\alpha Py - Py = P_\alpha(P - P_\alpha)y + P_\alpha y - Py \to 0$$

hold and so $P^2 = P$. Thus, P is an order projector on H and $P_\alpha \geq P$ for all α. Therefore, $P = 0$. Finally, $P_\alpha x$ is norm convergent to zero, as required. □

Theorem 47 *Every regular operator T on H is an order continuous element of $B(H)$, i.e., $(B(H))_{\mathrm{n}} = (B(H))_{\mathrm{r}}$.*

Proof Let $P_\alpha \downarrow 0$ in $\mathbf{OI}(B(H))$. If $0 \leq S \in B(H)$ then, as follows at once from the preceding lemma, $SP_\alpha \downarrow 0$ and $P_\alpha S \downarrow 0$. The operator T can be written as a difference of two positive operators T_1 and T_2. Evidently, $-T_2P_\alpha \leq TP_\alpha \leq T_1P_\alpha$ and, hence, $TP_\alpha \xrightarrow{o} 0$. Analogously, $P_\alpha T \xrightarrow{o} 0$. Finally, $T \in (B(H))_{\mathrm{n}}$. □

Let A be an arbitrary ordered algebra. An algebra A is said to have a (*strongly*) *disjunctive product* if for any $a, b \in A_{\mathrm{n}}^{+}$ ($a, b \in A^{+}$) with $ab = 0$ there exists an order idempotent p satisfying $ap = p^{\mathrm{d}}b = 0$. Now let E be a Dedekind complete Riesz space. Then the following statements hold (see [2, Example 3.3]): (a) The algebra $L(E)$ has a disjunctive product; (b) If E is a Banach lattice then the algebra $B(E)$ has a strongly disjunctive product if and only if E has order continuous norm. On the other hand, if F is a Banach lattice then the identity $L_{\mathrm{n}}(F) = L_{\mathrm{r}}(F)$ holds if and only if F has order continuous norm. In view of the preceding theorem, the analogous identity $(B(H))_{\mathrm{n}} = (B(H))_{\mathrm{r}}$ is valid in the case of an ordered Hilbert space H. Nevertheless, as the following example shows, the ordered C^{*}-algebra $B(H)$ does not have a disjunctive product in general. Before, we mention that if H is a Riesz space then, as follows at once from Proposition 41(h), H is a reflexive Banach lattice, in particular, it has order continuous norm and, hence, $B(H)$ has a strongly disjunctive product.

Example 38″ We shall use the notations and the results of Example 38. Let us consider the self-adjoint ice cream cone $K = K_{z,\frac{1}{\sqrt{2}}}$ and let $\dim H \geq 3$. Fix an element $v \in H$ satisfying the relations $\|v\| = 1$ and $\langle v, z\rangle = 0$. Obviously, $z \pm v \in K$. Define the operator T on H via the formula $Ty = (z - v) \otimes (z + v)y = \langle y, z - v\rangle(z + v)$. As is easy to see, $T^2 = 0$. On the other hand, $\mathbf{OI}(B(H)) = \{0, I\}$ and, hence, $B(H)$ does not have a disjunctive product. □

As follows at once from Theorem 47, for an arbitrary ordered Hilbert space H, we have the inclusion $\mathrm{Orth}(B(H)) \subseteq (B(H))_{\mathrm{n}}$, i.e., every orthomorphism of the algebra $B(H)$ is an order continuous element. Moreover, when the center of $B(H)$ is trivial (see Example 38(b)), this inclusion becomes an equality.

Corollary 36 and Proposition 41 suggest the following definition of a lattice homomorphism on an arbitrary ordered Hilbert space (this definition and the definition of a lattice homomorphism for the case of an element of an arbitrary ordered algebra given in Sect. 3.2 may differ). Namely, an operator T on H is said to be a *lattice homomorphism* whenever the identity $T(x \vee y) = (Tx) \vee (Ty)$ holds for all $x, y \in H$. Obviously, every lattice homomorphism T is necessarily a positive operator. The proof of the next proposition is analogous to the proof of Theorem 2.14 in [4] and will be omitted.

Proposition 48 *For an operator T on H the following statements are equivalent*:

(a) *T is a lattice homomorphism*;
(b) $T(x^{+}) = (Tx)^{+}$ *for all* $x \in H$;
(c) $T(x \wedge y) = (Tx) \wedge (Ty)$ *for all* $x, y \in H$;
(d) *If* $x \perp y$ *in* H *then* $Tx \perp Ty$ *in* H;
(e) $T|x| = |Tx|$ *for all* $x \in H$.

As follows at once from part (d) of the preceding proposition, every positive operator T belonging to the center $(B(H))_I$ of $B(H)$ is a lattice homomorphism. On the other hand, if H is a Riesz space and $T \in B(H)$ then $T \in (B(H))_I$ if and only if the operator T is an orthomorphism on H (see Sect. 3.2). As the next theorem shows, in the case of an arbitrary ordered Hilbert space H, an operator $T \in (B(H))_I$ also possesses many nice properties of orthomorphisms on Riesz spaces (see [4, § 2.3] and [13, Chapter 20]). Analogous to the case of a Riesz space, a linear subspace J of H is called an *ideal* whenever $|x| \leq |y|$ and $y \in J$ imply $x \in J$.

Theorem 49 *For operators $S, T \in (B(H))_I$ the following statements hold*:

(a) $|Tx| = |T||x| = |T|x||$ *for all* $x \in H$ *and* $T^{\pm}y = (Ty)^{\pm}$ *for all* $y \in H^+$;
(b) $(S \vee T)x = Sx \vee Tx$ *and* $(S \wedge T)x = Sx \wedge Tx$ *for all* $x \in H^+$;
(c) *The null space* $N(T)$ *is an ideal*, $N(T) = N(|T|) = N(T^+) \cap N(T^-)$, *and if* $S, T \geq 0$ *then* $N(S \vee T) = N(S + T) = N(S) \cap N(T)$;
(d) *The inclusion* $N(S) \subseteq N(T)$ *holds if and only if* T *belongs to the band* B_S *generated by* S *in* $(B(H))_I$;
(e) *The relation* $\langle Sx, Ty\rangle = 0$ *holds for all* $x, y \in H$ *if and only if* $S \perp T$ *in* $(B(H))_I$;
(f) *The range* $R(T)$ *and its closure* $\overline{R(T)}$ *are ideals and* $R(T) = R(|T|)$.

Proof (a) Taking into account Theorem 44, we have the decomposition $T = T^+ - T^-$ with $T^{\pm} \in (B(H))_I^+$ and $T^+T^- = 0$. Then for an arbitrary element $y \in H^+$, the relation $T^+y \perp T^-y$ is valid. Indeed,

$$\langle T^+y, T^-y\rangle = \langle y, (T^+)^*T^-y, \rangle = \langle y, T^+T^-y\rangle = 0.$$

Using Corollary 37, we get $(Ty)^+ = (T^+y - T^-y)^+ = T^+y$ and, analogously, $(Ty)^- = T^-y$. Therefore, for every $x \in H$, we have

$$|T||x| = T^+|x| + T^-|x| = (T|x|)^+ + (T|x|)^- = |T|x||.$$

On the other hand, $T^+x^+ \perp T^+x^-$ and $T^-x^+ \perp T^-x^-$ as $T^{\pm}$ are lattice homomorphisms. Consequently,

$$\begin{aligned} |T||x| &= |T^+x| + |T^-x| = |T^+x^+ - T^+x^-| + |T^-x^+ - T^-x^-| \\ &= T^+x^+ + T^+x^- + T^-x^+ + T^-x^- \\ &= |(T^+x^+ + T^-x^-) - (T^+x^- + T^-x^+)| \\ &= |T^+x - T^-x| = |Tx|. \end{aligned}$$

The statement (b) follows easily from (a) and the statement (c) from (a) and (b).

(d) We can assume $S, T \geq 0$. For the proof of the necessity, proceeding by contradiction and taking into account the identity $B_S = \{S\}^{\text{dd}}$ in the Riesz space $(B(H))_I$, we find $W \in (B(H))_I$ satisfying

$$0 < W \leq T \quad \text{and} \quad W \perp S. \tag{21}$$

The second relation in (21) implies $SW = 0$, whence $R(W) \subseteq N(S) \subseteq N(T)$. Then, using the first one in (21), for arbitrary $z \in H^+$, we have $\|Wz\|^2 \leq \langle Wz, Tz\rangle \leq \langle TWz, z\rangle = 0$ and so $W = 0$, a contradiction.

For the converse, taking into account the relation $T \wedge (nS) \uparrow T$ in $(B(H))_I$ and the order closedness $(B(H))_I$ in $B(H)$, we have $T \wedge (nS) \uparrow T$ in $B(H)$ and so $(T \wedge (nS))x \to Tx$ for all $x \in H$. It remains to remember parts (b) and (c).

(e) For the check of the necessity, we mention first that our condition is equivalent to the equality $ST = 0$ and, hence, to the equality $|S||T| = 0$. Thus, $S \perp T$. For the converse, $|\langle Sx, Ty\rangle| \leq \langle |S||x|, |T||y|\rangle = 0$.

(f) Let us verify that $R(T)$ is an ideal. First of all, we consider the case of $T \geq 0$. Let $x \leq Ty$ for some $x, y \in H^+$. According to Theorem 44, for every $n \in \mathbb{N}$ the operator $\frac{1}{n}I + T$ is invertible and $(\frac{1}{n}I + T)^{-1} \geq 0$. Define the sequence $\{y_n\}$ in H via the formula $y_n = (\frac{1}{n}I + T)^{-1}x$. As is easy to see, the relations $y_n \uparrow \leq (I + T)^{-1}(x + y)$ and $0 \leq x - Ty_n \leq \frac{1}{n}y$ hold. The former implies that the sequence $\{y_n\}$ is norm convergent to some $y_0 \geq 0$. Then, using the second one, we obtain $Ty_0 = x$.

In general case, let $0 \leq x \leq |Ty|$. In view of part (a), $x \leq |T||y|$. Therefore, as shown above, $x = |T|z$ with $z \geq 0$. Since $Tz \in R(|T|)$, there exists an element $w \in H$ satisfying $Tz = |T|w$ and so $T^+(z - w) = T^-(z + w)$. Using the relation $T^+ \perp T^-$ and part (e), we have $T^+(z - w) = T^-(z + w) = 0$. Whence,

$$x = |T|z = T^+z + T^-z = T^+w - T^-w = Tw.$$

Thus, $x \in R(T)$. Finally, $R(T)$ is an ideal. Now the identity $R(T) = R(|T|)$ can be checked without difficulty.

Let us show that $\overline{R(T)}$ is an ideal. According to Theorem 44 once more, the operator T is hermitian. Therefore, as is well know, the next orthogonal decomposition $H = N(T) \oplus \overline{R(T)}$ holds. Fix $x' \in N(T)$ and $x'' \in \overline{R(T)}$. Then $|x'| \perp |x''|$. Indeed, there exists a sequence $\{z_n\}$ in H satisfying $Tz_n \to x''$, whence, using part (a), we get $\langle |x'|, |x''|\rangle = \lim_{n\to\infty}\langle |x'|, |T||z_n|\rangle = 0$. Next, *the orthogonal projections on* $N(T)$ *and on* $\overline{R(T)}$ *are positive operators*. To see this, let $x \geq 0$ and let $x = x_1 + x_2$ with $x_1 \in N(T)$ and $x_2 \in \overline{R(T)}$. Since $|x_1| \perp |x_2|$, we have $0 \leq \langle x, x_1^-\rangle = -\|x_1^-\|^2$. Thus, $x_1^- = 0$ and so $x_1 \geq 0$. Analogously, $x_2 \geq 0$.

Consider an element $y \in \overline{R(T)}$ and pick a sequence $\{w_n\}$ in H satisfying $Tw_n \to y$. Whence, $|T||w_n| \to |y|$ and so $|y| \in \overline{R(T)}$. Now let $0 \leq x \leq y \in \overline{R(T)}$. It suffices to check the inclusion $x \in \overline{R(T)}$. As shown above, $x = x_1 + x_2$ with $x_1 \in N(T)^+$ and $x_2 \in (\overline{R(T)})^+$. Then $0 \leq x_1 \leq y - x_2 \in \overline{R(T)}$ and so $x_1 = 0$. Finally, $x \in \overline{R(T)}$. Thus, $\overline{R(T)}$ is an ideal. □

The rest of this section is devoted to the connection of the theory of operators on an ordered Hilbert spaces with the general theory of operators on arbitrary Hilbert spaces. In particular, some remarks about the spectral resolution of operators from $(B(H))_I$ will be given.

Theorem 50 *For an operator* $T \in (B(H))_I$ *the following statements are equivalent*:

(a) *The operator* T *is positive in the sense of the theory of ordered linear space*, *i.e.*, $Tx \in H^+$ *for all* $x \in H^+$;
(b) *The operator* T *is positive in the sense of the theory of Hilbert space*, *i.e.*, $\langle Tx, x\rangle \geq 0$ *for all* $x \in H$;
(c) $\langle Tx, x\rangle \geq 0$ *for all* $x \in H^+$.

Proof (a) $\Longrightarrow$ (b) For every $x \in H$, using the relation $x^+ \perp x^-$, we have

$$\langle Tx, x\rangle = \langle T(x^+ - x^-), x^+ - x^-\rangle = \langle T(x^+ + x^-), x^+ + x^-\rangle = \langle T|x|, |x|\rangle \geq 0.$$

The implication (b) $\Longrightarrow$ (c) is obvious.

(c) $\Longrightarrow$ (a) The inequalities

$$\langle T^{+}x, x\rangle \geq \langle T^{-}x, x\rangle \geq 0 \tag{22}$$

hold for all $x \in H^{+}$. According to Theorem 44, there exists an operator $S \in (B(H))_I^{+}$ satisfying $S^2 = T^{-}$ and $S \perp T^{+}$. Then, taking into account (22), for every $z \in H^{+}$, we get $0 \leq \langle T^{-}Sz, Sz\rangle \leq \langle T^{+}Sz, Sz\rangle = 0$, whence $0 = \langle T^{-}Sz, Sz\rangle = \langle S^3 z, Sz\rangle = \|S^2 z\|^2$ and so $T^{-}z = 0$. Finally, $T^{-} = 0$ or $T \geq 0$. □

As follows from the preceding theorem, the order on $(B(H))_I$ induced by the ordered C^*-algebra $B(H)$ coincides with the order considering in the general theory of Hilbert spaces. In particular, the operators $|T|$ and $T^{\pm}$ in the Riesz space $(B(H))_I$ coincide with analogous operators defined in the case of an arbitrary Hilbert space (see [11, Section 108]).

Let A be an ordered algebra. For an element $a \in A_{\mathbf{e}}$, we define the *lower bound* and an *upper bound* of a by $m_a = \max\{\lambda \in \mathbb{R} : \lambda\mathbf{e} \leq a\}$ and $M_a = \min\{\lambda \in \mathbb{R} : a \leq \lambda\mathbf{e}\}$, respectively. If the mapping $\Psi_A : A_{\mathbf{e}} \to C(\mathcal{M}_{A_{\mathbf{e}}})$ is a bijection (see Theorem 15 and the proof of Proposition 21) then for every function $\varphi \in C[m_a, M_a]$, we can put $\varphi(a) = \Psi_A^{-1}(\varphi(\Psi_A(a)))$. Obviously, the mapping $\varphi \to \varphi(a)$ from $C[m_a, M_a]$ into $A_{\mathbf{e}}$ is a positive algebraic homomorphism. Next, as follows from Theorem 50, for the case of an operator $T \in (B(H))_I$ the values m_T and M_T coincide with "usual" lower and upper bounds of a hermitian operator. Namely, we have the following.

Corollary 51 *For an operator $T \in (B(H))_I$ the next identities*

$$m_T = \inf_{\|x\|=1} \langle Tx, x\rangle \text{ and } M_T = \sup_{\|x\|=1} \langle Tx, x\rangle$$

hold.

The next result will be needed later.

Lemma 52 *Let A be an ordered algebra. For an element $p \in A$ the following statements are equivalent*:

(a) $p \in \mathbf{OI}(A)$;
(b) $p \wedge (\mathbf{e} - p) = 0$ *in* $\mathbf{OI}(A)$;
(c) $p \wedge (\mathbf{e} - p) = 0$ *in* $A_{\mathbf{e}}$;
(d) $p \wedge (\mathbf{e} - p) = 0$ *in* A;
(e) *p is an extreme point of the order interval* $[0, \mathbf{e}]$.

Proof The implications (b) $\Longrightarrow$ (a) and (d) $\Longrightarrow$ (c) are obvious.

(c) $\Longrightarrow$ (b) Clearly, $0 \leq p \leq \mathbf{e}$. Therefore, we have the equalities

$$0 = p^2 \wedge ((\mathbf{e} - p)^2) = p^2 \wedge (\mathbf{e} - 2p + p^2) = p \wedge (\mathbf{e} - p) + p^2 - p = p^2 - p$$

in $A_{\mathbf{e}}$ and so $p^2 = p$. Thus, $p \in \mathbf{OI}(A)$ and the validity of (b) follows.

(a) $\Longrightarrow$ (d) If $a \leq p$ and $a \leq \mathbf{e} - p$ for some $a \in A$ then $(\mathbf{e} - p)a \leq 0$ and $pa \leq 0$. Whence, $a \leq 0$.

(d) $\Longrightarrow$ (e) Let $p = (1 - \lambda)x + \lambda y$ with $x, y \in [0, \mathbf{e}]$ and $\lambda \in (0, 1)$. Then

$$0 \leq (1 - \lambda)(x \wedge (\mathbf{e} - p)) \leq ((1 - \lambda)x + \lambda y) \wedge (\mathbf{e} - p) = p \wedge (\mathbf{e} - p) = 0$$

and so $x \wedge (\mathbf{e} - p) = 0$. Consequently,

$$x = x \wedge \mathbf{e} = (x - p) \wedge (\mathbf{e} - p) + p \leq x \wedge (\mathbf{e} - p) + p = p$$

and, hence, $x \leq p$. Analogously, $y \leq p$. Finally, $x = y = p$.

(e) $\Longrightarrow$ (a) Put $z = p(\mathbf{e} - p)$. Evidently, $p \pm z \in [0, \mathbf{e}]$ and $p = \frac{(p+z)+(p-z)}{2}$. Therefore, $z = 0$ and so $p^2 = p$. □

Before we discuss the spectral resolution of an operator $T \in (B(H))_I$, we shall make some remarks for the general case of an ordered algebra. To this end, let A be an ordered algebra and let its center $A_{\mathbf{e}}$ be Dedekind complete. Then (see [12, Section IV.10]) for arbitrary $a \in A_{\mathbf{e}}$ there exists a spectral family $\{\mathbf{e}_\lambda^a : \lambda \in \mathbb{R}\}$ (or the resolution of the identity corresponding to a) with the following properties:

(a) $\mathbf{e}_\lambda^a \leq \mathbf{e}_\mu^a$ for $\lambda \leq \mu$;
(b) $\mathbf{e}_\lambda^a = 0$ for $\lambda \leq m_a$ and $\mathbf{e}_\lambda^a = \mathbf{e}$ for $\lambda > M_a$;
(c) The family $\{\mathbf{e}_\lambda^a\}$ is left continuous, i.e., $\sup_{\mu<\lambda} \mathbf{e}_\mu^a = \mathbf{e}_\lambda^a$ for all λ;
(d) If $\lambda_1 \leq \lambda_2 \leq \mu_1 \leq \mu_2$ then $\mathbf{e}_{\lambda_2}^a - \mathbf{e}_{\lambda_1}^a \perp \mathbf{e}_{\mu_2}^a - \mathbf{e}_{\mu_1}^a$ in $A_{\mathbf{e}}$.

Moreover, $\mathbf{e}_\lambda^a$ are components of the unit $\mathbf{e}$, i.e., $\mathbf{e}_\lambda^a \perp \mathbf{e} - \mathbf{e}_\lambda^a$ in $A_{\mathbf{e}}$ for all λ. Whence, taking into account the preceding lemma, we obtain the next property of $\{\mathbf{e}_\lambda^a\}$:

(e) The elements $\mathbf{e}_\lambda^a$ are order idempotents, i.e., $\mathbf{e}_\lambda^a \in \mathbf{OI}(A)$ for all λ.

By the Freudenthal's Spectral Theorem, the integral representation

$$a = \int_{m_a}^{M_a+0} \lambda d\mathbf{e}_\lambda^a$$

holds, where the integral means the $\mathbf{e}$-uniformly limit in $A_{\mathbf{e}}$ of integral sums $\sum_{i=0}^{n-1} l_i(\mathbf{e}_{\lambda_{i+1}}^a - \mathbf{e}_{\lambda_i}^a)$ with $n \in \mathbb{N}$, $m_a = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} \leq M_a < \lambda_n$, and $\lambda_i \leq l_i \leq \lambda_{i+1}$ for $i = 0, 1, \ldots, n-1$ as $\sup_{0\leq i\leq n-1}(\lambda_{i+1} - \lambda_i) \to 0$. Thus, if $a \in A_{\mathbf{e}}$, we obtain the integral representation with help of a family of *order idempotents*.

Now we consider the case of an operator $T \in (B(H))_I$. In view of Theorem 44, T is hermitian. In the general case of an arbitrary Hilbert space, there exist several ways of definitions of a spectral family of projections $\{P_\lambda^T\}$ under which T admits the integral representation. For example, one can simply use the integral representation of elements of an ordered algebra suggested above. We shall use the construction from [11, Sections 107, 108]. Namely, for every $\lambda \in \mathbb{R}$, let us consider the operator $(T - \lambda I)^+$ and let P_λ^T be the orthogonal projection onto the null space $N((T - \lambda I)^+)$. Then

$\{P_\lambda^T\}$ is a spectral family and the properties of it is connected closely with the spectral properties of the operator T. Moreover, in this case, since $(T - \lambda I)^+ \in (B(H))_I$, the family $\{P_\lambda^T\}$ consists of (see the proof of part (f) of Theorem 49) *order projections*.

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