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Associate space with respect to a locally σ-finite measure on a δ-ring and applications to spaces of integrable functions defined by a vector measure

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Abstract We show that for a locally σ-finite measure μ defined on a δ-ring, the associate space theory can be developed as in the σ-finite case, and corresponding properties are obtained. Given a saturated σ-order continuous μ-Banach function space E, we prove that its dual space can be identified with the associate space $E^\times$ if, and only if, $E^\times$ has the Fatou property. Applying the theory to the spaces $L^p(\nu)$ and $L^p_w(\nu)$, where ν is a vector measure defined on a δ-ring $\mathcal{R}$ and $1 \leq p < \infty$, we establish results corresponding to those of the case when the vector measure is defined on a σ-algebra.

Keywords Banach function space · Associate space · Locally σ-finite measure · δ-ring · Fatou property · Order continuous · Vector measure

Mathematics Subject Classification 46E30 · 46G10 · 46B10

1 Introduction

Let Ω be a set and $\mathcal{R}$ a δ-ring consisting of subsets of Ω. Given a vector measure $\nu : \mathcal{R} \to X$, where X is a (real or complex) Banach space, we obtain the Banach space of weakly-integrable functions $L^1_w(\nu)$, which has the space of ν-integrable functions $L^1(\nu)$ as a closed subspace. With the order given by $f \geq g$ if $f \geq g$ outside a ν-null set, we have that $L^1_w(\nu)$ is a σ-Fatou Banach lattice and $L^1(\nu)$ is an order continuous

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Banach lattice. (See Sect. 2 for definitions.) The integration theory with respect to vector measures defined on δ-rings was developed mainly by Lewis [18], Masani and Niemi [21,22] and Delgado [8]. It extends the well known theory for vector measures defined on σ-algebras [25, Ch. 3].

If E is a real or complex order continuous Banach lattice, Curbera [5, p. 22], [10, p. 246] showed that there is a set Ω, a δ-ring $\mathcal{R}$ consisting of subsets of Ω, and a vector measure $\nu : \mathcal{R} \to E$, such that E and $L^1(\nu)$ are order and isometrically isomorphic . It follows that the dual spaces $L^1(\nu)^*$ and E^*, both of which are Banach lattices, are also order and isometrically isomorphic. Hence by studying a dual space of the form $L^1(\nu)^*$ we are implicitly analyzing the dual space of the most general order continuous Banach lattice.

The study of $L^1(\nu)^*$ can be done through the associate space theory, systematically developed by Luxemburg and Zaanen [28, Ch. 15], [29, Ch. 16, Sect. 112]. This theory begins with an arbitrary positive measure space (Ω, Σ, μ) and applies to a μ-Banach function space (μ-B.f.s. for short) E which is saturated. Then also its associate space $E^\times$ is a μ-Banach function space and the operator $R : E^\times \to E^*$, defined by $R(g) := R_g$, where $R_g(f) := \int_\Omega fg d\mu$, is a linear isometry. This operator, called the canonical isometry, allows us to consider the associate space $E^\times$ as a closed subspace of the dual space E^*. When the involved measure μ is σ-finite and the saturated B.f.s. E is σ-order continuous, then the canonical isometry R is onto. Moreover, R also preserves the lattice structure and so we write $E^\times = E^*$. In the following we will maintain this notation to indicate that R is onto.

To apply the above theory for studying $L^1(\nu)^*$, in the first place we have to find a positive measure μ which is also a local control measure for ν, that is, such that μ and ν have the same null sets. In this situation $L^1(\nu)$ is a μ-B.f.s. Assume the δ-ring $\mathcal{R}$ on which the vector measure ν is defined is a σ-algebra. Then it is well known that ν has a finite local control measure μ with respect to which $L^1(\nu)$ is a saturated μ-B.f.s. [25, pp. 107–108]. It follows that in this case we have $L^1(\nu)^\times = L^1(\nu)^*$.

If the δ-ring $\mathcal{R}$ is not a σ-algebra, we cannot proceed directly as above to obtain for $L^1(\nu)$ similar results to those we have just mentioned, since in this situation ν may not have a σ-finite local control measure. This is a problem, since a key fact for the associate space theory to work is that when the measure μ is σ-finite, then the saturation of a μ-B.f.s. E implies that of $E^\times$.

However, relying on a result of Brooks and Dinculeanu [3], it has recently been pointed out by Jiménez et al. [16] the fact that any vector measure defined on a δ-ring $\mathcal{R}$, always has a local control measure μ that is also σ-finite on any set $B \in \mathcal{R}$. We will see that a measure of this kind, which we have called *Brooks–Dinculeanu measure*, is appropriate for our objectives.

In this paper we first study the associate space $E^\times$ of a saturated μ-B.f.s., for a locally σ-finite positive measure μ. Thereafter we consider a vector measure ν and apply the results to the μ-Banach function spaces of p integrable functions $L^p(\nu)$ and of weakly integrable functions $L^p_w(\nu)$, $1 \le p < \infty$, where μ is a Brooks–Dinculeanu measure for ν. The main question we discussed was that of the validity of the equality $L^1(\nu)^\times = L^1(\nu)^*$.

We divided our work in five sections, including this Introduction. In Sect. 2 we present the notation, definitions and basic results that we have needed.

The general theory for the associate space $E^\times$ of a saturated μ-B.f.s E is given in Sect. 3. As in the well known σ-finite case, it turns out that the associate space $E^\times$ always has the σ-Fatou property. When μ is a σ-Fatou property, we show that the σ-Fatou property and the Fatou properties are equivalent for E. Based on general properties of the dual space of an arbitrary Banach lattice [29, Ch. 14, Ch. 15], we establish that if $E^\times = E^*$, then E is order continuous and $E^\times$ has the Fatou property.

In Sect. 4 we restrict our considerations to a locally σ-finite measure μ. In this setting we obtain results that are well known in the σ-finite case. Particularly we show that $E^\times$ is also a saturated μ-B.f.s., that $E^{\times\times} = E$ when E has the Fatou property, and that the factorization $L^1(\mu) = EE^\times$ holds. When the saturated μ-B.f.s. is σ-order continuous, we prove that $E^\times = E^*$ if, and only if, $E^\times$ has the Fatou property. In a forthcoming work we plan to show that this last is not always the case.

In the last section we apply our development to the spaces $L^p(\nu)$ and $L^p_w(\nu)$, $1 \leq p < \infty$, obtained from a vector measure ν. These are Banach function spaces with respect to any Brooks–Dinculeanu measure for ν. Thus we show that $L^p(\nu)^{\times\times} = L^p_w(\nu)$ and $L^p(\nu)^\times = L^p_w(\nu)^\times$ for $1 \leq p < \infty$. When the vector measure ν is defined on a σ-algebra the first of these equalities was established by Curbera and Ricker [6, Prop. 2.4], [7, Prop. 1].

We give several situations where $L^1(\nu)^\times = L^1(\nu)^*$ holds, one of them being the case of a decomposable vector measure. This turns out to be important, since the vector measure ν that Calabuig et al. used to represent an order continuous Banach lattice as $L^1(\nu)$, is a decomposable vector measure [4].

Finally, we establish that $L^p(\nu)^\times = L^p(\nu)^*$ if $1 < p < \infty$ and that $L^1(\nu)^\times = L^1(\nu)^*$ when $L^1(\nu)$ is reflexive. We also verify that a reflexivity criterion proven by Fernández et al. [12, Cor. 3.10] for a vector measure defined on a σ-algebra is still valid in the δ-ring case.

To complete this introduction, we want to note that Okada was the first to obtain a description of $L^1(\nu)^*$ for a classical vector measure ν [24]. Later Galaz-Fontes gave a representation for $L^p(\nu)^*$ when $1 < p < \infty$ [14] and recently Mastylo and Sánchez Pérez have established representations of these kind for a dual Banach space in a more general context [23].

2 Notation and basic results

2.1 Banach lattices

Throughout this paper all vector spaces considered will be with respect to $\mathbb{K}$, where $\mathbb{K} = \mathbb{C}$, the field of complex numbers or $\mathbb{K} = \mathbb{R}$, the field of real numbers. Let X be a normed space. By B_X we will indicate its unit closed ball and by X^* its dual space. We will represent by $\langle \cdot, \cdot \rangle$ the duality pairing, i.e. $\langle x, x^* \rangle := x^*(x)$, $\forall\, x \in X$ and $x^* \in X^*$. If Y is other normed space, to express that $X = Y$ as sets and with equal norms, we will write $X \equiv Y$.

Let X be a real vector lattice with order $\leq$. For $A \subset X$ we will be denote by A^+ the subset of X consisting of all $f \in A$ such that $0 \leq f$. Given $f \in X$, we indicate by f^+, f^- and $|f|$ its *positive part*, *negative part* and *modulus*, respectively. Finally X is called *Dedekind σ -complete* if every non-empty countable subset which is bounded from above has a supremum.

Let J be a directed set. A net $\{f_\tau\}_{\tau\in J} \subset X$ is an *upwards directed system* if for τ_1 and τ_2 in J, there exists $\tau_3 \in J$ such that $f_{\tau_1} \leq f_{\tau_3}$ and $f_{\tau_2} \leq f_{\tau_3}$. In this case we will use the notation $f_\tau \uparrow$. If additionally there exists $f = \sup_\tau f_\tau \in X$ we write $f_\tau \uparrow f$. Similarly, if the sequence $\{f_n\} \subset X$ is such that $f_n \leq f_{n+1}, \forall\, n \in \mathbb{N}$, we will indicate $f_n \uparrow$ and if additionally $f = \sup_n f_n$ then we will write $f_n \uparrow f$. Analogously, we define a *downwards directed system* $\{f_\tau\}$ and the notations $f_\tau \downarrow$ and $f_\tau \downarrow f$.

A *real normed vector lattice* X is a real normed space that is a vector lattice and whose norm $\|\cdot\|_X$ has the *lattice property*, that is

$$\text{if } f, g \in Y \text{ satisfy}, \ |f| \leq |g|, \ \text{then } \|f\|_X \leq \|g\|_X. \tag{2.1}$$

If in addition the space is complete, we will say that X is *real Banach lattice*.

Let X be a real normed vector lattice. Then X has the *weak Fatou property* if for each upwards directed system $\{f_\tau\} \subset X^+$ with $\sup_\tau \|f_\tau\|_X < \infty$, there exists $f = \sup_\tau f_\tau \in X$; if additionally $\|f\|_X = \sup_n \|f_n\|_X$, then X has the *Fatou property*. Similarly, X has the *weak σ -Fatou property*, if given $\{f_n\} \subset X^+$ such that $f_n \uparrow$ and $\sup_n \|f_n\|_X < \infty$, then there exists $f = \sup_n f_n \in X$, and if additionally $\|f\|_X = \sup_n \|f_n\|_X$, then X is said to have the *σ-Fatou property*. We say that X is *order continuous*, if for any system $\{f_\tau\} \subset X$ satisfying $f_\tau \downarrow 0$ it follows that $\|f_\tau\|_X \downarrow 0$. Analogously, X is *σ-order continuous* if for any sequence $\{f_n\} \subset X$ satisfying $f_n \downarrow 0$ we have that $\|f_n\|_X \downarrow 0$.

Take a real Banach lattice X. Then in $Z := X + iX$, the complexification of X, the modulus is defined by $|h| := \sup\{|(\cos\theta) f + (\sin\theta) g| : 0 \leq \theta < 2\pi\}$, $\forall\, h := f + ig \in Z$ [29, Ch. 14; Thm. 91.2], the norm by $\|h\|_Z = \|\,|h|\,\|_X$, $\forall\, h \in Z$, and the order is given by $f \leq g$ in Z, if $f, g \in X$ and $f \leq g$. In this case Z is called a *complex Banach lattice*, X is its *real part* and we write $X = Z_{\mathbb{R}}$. Observe that $Z^+ = X^+$. We will say that a complex Banach lattice has one of the properties we introduced above if its real part has it. Henceforth we will say only Banach lattice (normed vector lattice) to refer to a complex or real Banach lattice (normed vector lattice).

Let X be a Banach lattice. An *ideal* Y of X is a vector subspace of X if $f \in X$ with $|f| \leq |g|$ for some $g \in Y$ implies $f \in Y$.

Let $T : X \to Y$ be a linear operator between Banach lattices. Then T is said to be *positive* if for each $f \in X^+$ we have that $T(f) \in Y^+$. In this case $T(X_{\mathbb{R}}) \subset Y_{\mathbb{R}}$ and T is bounded [1, Lemma 3.22]. We will say that T is an *order isometry* if T is an isometry, T is onto and both T and T^{-1} are positive operators. This last condition is equivalent to

$$Tf \geq 0 \text{ if and only if } f \geq 0, \quad \forall f \in X.$$

In this case we have that $T(X_{\mathbb{R}}) = Y_{\mathbb{R}}$, $T(\sup\{f, g\}) = \sup\{Tf, Tg\}, \forall f, g \in X_{\mathbb{R}}$ and $T|f| = |Tf|$, $\forall f \in X$.

Let X be a real Banach lattice. Then its dual space X^* is a Banach lattice with the order given by

$$\varphi \le \psi, \text{ if } \varphi(f) \le \psi(f), \quad \forall f \in X^+, \ \varphi, \psi \in X^*. \tag{2.2}$$

In this case the supremum and infimum are uniquely determined by

$$\begin{aligned} \sup\{\varphi, \psi\}(f) &:= \sup\{\varphi(g) + \psi(h) : f = g + h, g \ge 0, \ h \ge 0\}, \\ \inf\{\varphi, \psi\}(f) &:= \inf\{\varphi(g) + \psi(h) : f = g + h, g \ge 0, \ h \ge 0\}, \end{aligned} \tag{2.3}$$

for each $\varphi, \psi \in X^*$ and $f \in X^+$ [27][Chap II, Props. 4.2, 5.5] .

Now assume that X is a complex Banach lattice. Given $\varphi \in X^*_{\mathbb{R}}$, we will indicate by $\widetilde{\varphi} : X \to \mathbb{C}$ its canonical extension, that is, $\widetilde{\varphi}(x + iy) = \varphi(x) + i\varphi(y)$. If $\Phi : X \to \mathbb{C}$ is a bounded linear functional, then Φ has the form $\Phi = \widetilde{\varphi} + i\widetilde{\psi}$, where $\widetilde{\varphi}$ and $\widetilde{\psi}$ are the canonical extensions of linear functionals $\varphi, \psi \in X^*_{\mathbb{R}}$. Identifying $X^*_{\mathbb{R}}$ with $\widetilde{X}^*_{\mathbb{R}} \subset X^*$, we have that $X^* = X^*_{\mathbb{R}} + iX^*_{\mathbb{R}}$ is a Banach lattice. As can be seen in [27, §11], in this case

$$|\Phi|(f) = \sup_{|g| \le f} |\Phi(g)|, \quad \forall f \in X^+. \tag{2.4}$$

2.2 μ-Banach function spaces

Given a measurable space (Ω, Σ) we will denote by $L^0(\Sigma)$ the space formed by the Σ-measurable functions $f : \Omega \to \mathbb{K}$. If additionally we have a positive measure μ defined on Σ, we indicate by $\mathcal{N}_0(\mu)$ the family of *μ-null* subsets, i. e., the sets $A \in \Sigma$ such that $\mu(A) = 0$. As usual a property holds *μ-almost everywhere* (briefly μ-a.e.) if it holds except on a μ-null set. We indicate by $L^0(\mu)$ the space of equivalence classes of functions in $L^0(\Sigma)$, where two functions are identified when they are equal μ-a.e.

Note that, when $\mathbb{K} = \mathbb{C}$, the space $L^0(\mu)$ is the complexification of the real space $L^0(\mu)_{\mathbb{R}} := \{f \in L^0(\mu) : f \text{ take its values in } \mathbb{R} \ \mu\text{-a.e.}\}$.

In $L^0(\mu)_{\mathbb{R}}$ we will always consider the μ-a.e. pointwise order. Let $f \in L^0(\mu)$. So $\mathrm{Re} f, \mathrm{Im} f \in L^0(\mu)_{\mathbb{R}}$ and $f = \mathrm{Re} f + i\mathrm{Im} f$. Moreover,

$$\sup_{0 \le \theta < 2\pi} |(\cos\theta)\mathrm{Re} f + (\sin\theta)\mathrm{Im} f| = \sqrt{(\mathrm{Re} f)^2 + (\mathrm{Im} f)^2} = |f|. \tag{2.5}$$

We will say that a normed space $E \subset L^0(\mu)$ is a *normed function space* related to μ (briefly μ-n.f.s.) if E is a vector subspace of $L^0(\mu)$ such that $f \in L^0(\mu)$ with $|f| \le |g|$ for some $g \in E$, implies $f \in E$ and the lattice property (2.1) holds (with E instead of X). If additionally E is complete we will call it *Banach function space* related to μ (briefly μ-B.f.s.). We must note that in the literature there appear other definitions with the same name, such as in [19, Def. 1.b.17] and in [2, Def. I.1.3].

Let E be a μ-B.f.s. Then E is Dedekind σ-complete. So, E is σ-order continuous if and only if E is order continuous [29, 103.9]. Now if E is a μ-n.f.s. with the σ-Fatou property, then E is complete [28, Ch. 15, §65, Thm. 1]. Thus, E is a μ-B.f.s. with the σ -Fatou property.

Let E be a complex μ-B.f.s. Note that $E_{\mathbb{R}} = E \cap L^0(\mu)_{\mathbb{R}}$, with the μ-a.e. pointwise order, is a real Banach lattice and $E = E_{\mathbb{R}} + iE_{\mathbb{R}}$. It follows from (2.5) that E is a complex Banach lattice. Moreover, $f \in E$ if and only if $(\mathrm{Re} f)^+, (\mathrm{Re} f)^-, (\mathrm{Im} f)^+, (\mathrm{Im} f)^- \in E^+$.

2.3 Integration with respect to measures defined on δ-rings

Let Ω be a set. A family $\mathcal{R}$ of subsets of Ω is a *δ-ring* if $\mathcal{R}$ is a ring which is closed under countable intersections. From now on in this paper $\mathcal{R}$ will be a δ-ring. We denote by $\mathcal{R}^{loc}$ the σ-algebra of all sets $A \subset \Omega$ such that $A \cap B \in \mathcal{R}, \forall\, B \in \mathcal{R}$. Given $A \in \mathcal{R}^{loc}$ we indicate by $\mathcal{R}_A$ the δ-ring $\{B \subset A : B \in \mathcal{R}\}$ and by π_A the collection of finite families of pairwise disjoint sets in $\mathcal{R}_A$. Note that if $\Omega \in \mathcal{R}$, then $\mathcal{R}$ is a σ-algebra, and in this case we have that $\mathcal{R}^{loc} = \mathcal{R}$. Moreover, for each $B \in \mathcal{R}$ it turns out that $\mathcal{R}_B$ is a σ-algebra.

A *scalar measure* (*positive measure*) is a function $\lambda : \mathcal{R} \to \mathbb{K}$ ($\lambda : \mathcal{R} \to [0, \infty]$) satisfying that if $\{B_n\} \subset \mathcal{R}$, is a family of pairwise disjoint sets such that $\bigcup_{n=1}^{\infty} B_n \in \mathcal{R}$, then $\sum_{n=1}^{\infty} \lambda(B_n) = \lambda\left(\bigcup_{n=1}^{\infty} B_n\right)$. The *variation of* λ is the countably additive measure $|\lambda| : \mathcal{R}^{loc} \to [0, \infty]$ defined by

$$|\lambda|(A) := \sup\left\{\sum_{j=1}^{n} |\lambda(A_j)| : \{A_j\} \in \pi_A\right\}.$$

A function $f \in L^0(\mathcal{R}^{loc})$ is *λ-integrable* if $f \in L^1(|\lambda|)$. We denote by $L^1(\lambda)$ the subspace of $L^0(\lambda)$ formed by the λ-integrable functions. Then $L^1(\lambda)$ with norm given by $|f|_{1,\lambda} := \int_\Omega |f| d|\lambda|$ is a σ-order continuous $|\lambda|$-B.f.s. with the σ-Fatou property. The following result is basic in the theory; when λ is a scalar measure it was established by Masani and Niemi [21, Lemma 2.30, Thm. 2.32], for the case of a positive measure we can proceed similarly.

Proposition 2.1 *If $f \in L^0(\mathcal{R}^{loc})$, then*

$$\int_A |f| d|\lambda| = \sup_{B \in \mathcal{R}_A} \int_B |f| d|\lambda|, \quad \forall\, A \in \mathcal{R}^{loc}. \tag{2.6}$$

Therefore, $f \in L^1(|\lambda|)$ if and only if $\sup_{B \in \mathcal{R}} \int_B |f| d|\lambda| < \infty$.

Let X be a Banach space. A function $\nu : \mathcal{R} \to X$ is a *vector measure* if $\sum_{n=1}^{\infty} \nu(B_n) = \nu\left(\bigcup_{n=1}^{\infty} B_n\right)$, for any collection $\{B_n\} \subset \mathcal{R}$ of pairwise disjoint sets such that $\bigcup_{n=1}^{\infty} B_n \in \mathcal{R}$. The *variation of* ν is the positive measure $|\nu|$ defined in $\mathcal{R}^{loc}$ by

$$|\nu|(A) := \sup\left\{\sum_{j} \|\nu(A_j)\|_X : \{A_j\} \in \pi_A\right\}.$$

The *semivariation of* ν is the function $\|\nu\| : \mathcal{R}^{loc} \to [0, \infty]$ given by

$$\|\nu\|(A) := \sup\{|\langle \nu, x^*\rangle|(A) : x^* \in B_{X^*}\},$$

where $|\langle \nu, x^*\rangle|$ is the variation of the scalar measure $\langle \nu, x^*\rangle : \mathcal{R} \to \mathbb{K}$, where

$$\langle \nu, x^*\rangle(B) := \langle \nu(B), x^*\rangle, \quad \forall\, B \in \mathcal{R}.$$

The semivariation of ν is finite in $\mathcal{R}$ and for any $A \in \mathcal{R}^{loc}$ satisfies $\|\nu\|(A) \le |\nu|(A)$. A set $A \in \mathcal{R}^{loc}$ is said to be ν-null if $\|\nu\|(A) = 0$. We will denote by $\mathcal{N}_0(\nu)$ the collection of ν-null sets. It turns out that $\mathcal{N}_0(\nu) = \mathcal{N}_0(|\nu|)$. Moreover $A \in \mathcal{N}_0(\nu)$ if and only if $\nu(B) = 0$, $\forall\, B \in \mathcal{R}_A$. We say that a positive measure $\lambda : \mathcal{R} \to [0, \infty]$ is a *local control measure* for ν, if $\mathcal{N}_0(|\lambda|) = \mathcal{N}_0(\nu)$ [8, p. 437]. Then, $|\nu|$ is a local control measure for ν. We define $L^0(\nu)$ as the space of equivalence classes of functions in $L^0(\mathcal{R}^{loc})$, where two functions are identified when they are equal ν-a.e. So, $L^0(\nu) = L^0(|\lambda|)$, where λ is any local control measure for ν.

A function $f \in L^0(\mathcal{R}^{loc})$ is *weakly* ν*-integrable*, if $f \in L^1(\langle \nu, x^*\rangle)$, for each $x^* \in X^*$. We will denote by $L^1_w(\nu)$ the subspace of $L^0(\nu)$ of all weakly ν-integrable functions. With the norm $\|f\|_\nu := \sup\{\int_\Omega |f| d|\langle \nu, x^*\rangle| : x^* \in B_{X^*}\}$, $L^1_w(\nu)$ is a $|\lambda|$-B.f.s. with the σ-Fatou property, where λ is a local control measure for ν.

A function $f \in L^1_w(\nu)$ is ν*-integrable*, if for each $A \in \mathcal{R}^{loc}$ there exists a vector $x_A \in X$, such that $\langle x_A, x^*\rangle = \int_A f d\langle \nu, x^*\rangle$, $\forall\, x^* \in X^*$. The subset of all ν-integrable functions is a closed subspace of $L^1_w(\nu)$ and it will be denoted by $L^1(\nu)$. We indicate by $S(\mathcal{R})$ the collection of simple functions in $L^0(\mathcal{R}^{loc})$ which have support in $\mathcal{R}$. It turns out that $L^1(\nu)$, with norm $\|\cdot\|_\nu$ is a σ-order continuous μ-B.f.s. where $S(\mathcal{R})$ is a dense subspace.

We also notice that $L^1(\nu) = L^1_w(\nu)$ if X does not contain a copy of c_0 [18, Thm. 5.1].

3 Associate space

Let (Ω, Σ, μ) be a positive measure space and let us consider a μ-B.f.s. E. We will show that several of the basic results about the associate space of E when μ is a σ-finite measure, can also be established in the case that μ is not σ-finite. We will begin by just enunciating some of these results; they can be proven as in the σ-finite case [28, Ch. 15], [2, Ch. 1].

The vector space defined by $E^\times := \{g \in L^0(\mu) : gf \in L^1(\mu),\ \forall\, f \in E\}$ is called *the associate space* of E and the function

$$\|g\|_{E^\times} := \sup\left\{\int_\Omega |gf| d\mu : f \in B_E\right\}, \quad \forall\, g \in E^\times, \tag{3.1}$$

is a seminorm. For each $f \in E$ and $g \in E^\times$ the Hölder inequality is satisfied:

$$\int_\Omega |gf| d\mu \le \|g\|_{E^\times} \|f\|_E.$$

We also have that for the function $\|\cdot\|_{E^\times}$ to be a norm, it is necessary and sufficient that E be *saturated*, that is, for each $A \in \Sigma$ with positive measure there exists $B \in \Sigma_A$ such that $\mu(B) > 0$ and $\chi_B \in E$. Next we give an equivalent condition for saturation. For this, let us first recall that a μ-B.f.s. Y is *order dense* in $L^0(\mu)$ if for any $f \in L^0(\mu)^+$ there exists an upwards directed system $\{f_\tau\} \subset Y^+$ such that $f_\tau \uparrow f$.

Lemma 3.1 *Let E a μ-B.f.s. The following statements are equivalent:*

(i) *The space E is saturated.*
(ii) *The space E is order dense in $L^0(\mu)$.*
(iii) *The seminorm $\|\cdot\|_{E^\times}$ is a norm.*

Proof The equivalence (i) $\Leftrightarrow$ (iii) is proved as in the σ-finite case [28, Ch. 15, §69, Thm. 4]. Let us prove (i) $\Leftrightarrow$ (ii).

Since E is a Banach lattice we have that E is archimedean. Then, it is enough to prove that E is saturated if and only if for each $0 \neq f \in L^0(\mu)$ there exists $g \in E$ such that $0 < |g| \leq |f|$ [20, Thm. 22.3(vi)]. Assume that E is saturated. Consider $0 \neq f \in L^0(\mu)$ and define

$$A_n := \left\{x \in \Omega : \frac{1}{n} \leq |f(x)|\right\}, \quad \forall\, n \in \mathbb{N}.$$

Let us fix $N \in \mathbb{N}$ such that $\mu(A_N) > 0$. Since E is saturated there exists $B \subset A_N$ with $\mu(B) > 0$ and $\chi_B \in E$. Then, $g := \frac{1}{N}\chi_B \in E$ and $0 < |g| \leq |f|$.

To establish the other implication, take $A \in \Sigma$ such that $\mu(A) > 0$. Hence $0 \neq \chi_A \in L^0(\mu)$. Thus there exists $g \in E$ satisfying $0 < |g| \leq \chi_A$. As $0 \neq g \in L^0(\mu)$ we can take $\varphi \in S(\Sigma)$ with $0 < \varphi \leq |g|$. Therefore $\varphi \in E$ and so, there exists $B \in \Sigma$ such that $B \subset \operatorname{supp}\varphi$, $\mu(B) > 0$ and $\chi_B \in E$. □

Henceforth we will assume that E is a saturated μ-B.f.s. Then as in the σ-finite case we have:

Proposition 3.2 *The space $E^\times$ is a μ-B.f.s. with the σ-Fatou property.*

For our next result, let us recall that a Banach lattice E is *super Dedekind complete* if every non-empty subset D of E which is bounded from above has a supremum and it contains an at most countable subset possessing the same supremum as D.

Proposition 3.3 *If there exists a σ-order continuous μ-B.f.s. F with the σ-Fatou property, such that $E^\times \subset F$, then $E^\times$ has the Fatou property.*

Proof Consider $\{g_\tau\} \subset E^\times$ such that $0 \leq g_\tau \uparrow$ and $\sup_\tau \|g_\tau\|_{E^\times} < \infty$. Then $\{g_\tau\} \subset F^+$ is an upwards directed system such that $\sup_\tau \|g_\tau\|_F < \infty$. Since F is σ-order continuous and has the σ-Fatou property, then F has the Fatou property and is super Dedekind complete [29, Thm. 113.4]. Hence $g := \sup_\tau g_\tau \in F$ and there exists a sequence $\{g_{\tau_n}\} \subset \{g_\tau\}$ such that $g_{\tau_n} \uparrow g$ [20, Thm. 23.2.(iii)]. From the σ-Fatou property in $E^\times$ we obtain that $g \in E^\times$. And since $\|\cdot\|_E^\times$ is a lattice norm, $\sup_\tau \|g_\tau\|_{E^\times} \leq \|g\|_{E^\times}$.

Now take $f \in B_E$. Using the monotone convergence theorem and the Hölder inequality we have

$$\int_{\Omega} |gf| d\mu = \sup_n \int_{\Omega} |g_{\tau_n} f| d\mu \leq \sup_n \|g_{\tau_n}\|_{E^\times} \|f\|_E \leq \sup_n \|g_\tau\|_{E^\times}.$$

Thus, $\|g\|_{E^\times} \leq \sup_\tau \|g_\tau\|_{E^\times} < \infty$. Hence, $E^\times$ has the Fatou property. □

Since $L^1(\mu)$ is σ-order continuous and has the σ-Fatou property, we obtain:

Corollary 3.4 *If $\chi_\Omega \in E$, then $E^\times$ has the Fatou property.*

Next we will show that when μ is σ-finite, it turns out that the associate space always has the Fatou property. For this it is necessary to make before a brief discussion.

Let A be in Σ. We denote by μ_A the restriction of the measure μ to the σ-algebra Σ_A formed by the measurable subsets of A. Thus (A, Σ_A, μ_A) is a measure space.

For each $f \in L^0(\Sigma_A)$ define the function $f^\Omega : \Omega \to \mathbb{K}$ by $f^\Omega(x) = f(x)$, if $x \in A$ and $f^\Omega(x) = 0$ otherwise. Then f^Ω is a Σ-measurable function which is called *canonical extension* of f. Now, the set E_A defined by

$$E_A := \left\{ f \in L^0(\mu_A) : f^\Omega \in E \right\} \tag{3.2}$$

is a vector space. If $h \in E$, then $(h_A)^\Omega = h\chi_A \in E$, where h_A is the restriction of h to A. Thus, $h_A \in E_A$. In E_A we define the norm $\|\cdot\|_A$ by

$$\|f\|_A := \|f^\Omega\|_E, \quad \forall f \in E_A. \tag{3.3}$$

Since E is a saturated μ-B.f.s., it follows that also E_A is a saturated μ_A-B.f.s. On the other hand if $A = \bigcup_{n=1}^\infty A_n$, where $A_n \in \Sigma$ and $\mu(A_n) < \infty$, $\forall n \in \mathbb{N}$, we obtain that μ_A is σ-finite. In this case it is well known that $E_A^\times := (E_A)^\times$ is saturated [28, Ch. 15, §71, Thm. 4]. Furthermore $E_A^\times = (E^\times)_A$ and

$$\|g\|_{E_A^\times} = \sup\left\{ \int_A |gf| d\mu_A : f \in B_{E_A} \right\} = \|g^\Omega\|_{E^\times}, \quad \forall g \in E_A^\times. \tag{3.4}$$

Theorem 3.5 *If the measure μ is σ-finite, then $E^\times$ has the Fatou property.*

Proof We will assume that $\mu(\Omega) > 0$. Let us take an upwards directed system $\{g_\tau\}_{\tau\in I} \subset E^\times$ such that $g_\tau \geq 0$, $\forall \tau \in I$ and $\sup_\tau \|g_\tau\|_{E^\times} < \infty$. Now since μ is σ-finite and E is a saturated μ-B-f.s., there exist $\{\Omega_n\} \subset \Sigma$ and $N \in \mathcal{N}_0(\mu)$ such that $\Omega_n \subset \Omega_{n+1}$, $0 \neq \chi_{\Omega_n} \in E$, $\forall n \in \mathbb{N}$ and $\Omega = \bigcup_{n=1}^\infty \Omega_n \cup N$ [28, Ch. 15, §67, Thm. 4].

Fix $n \in \mathbb{N}$. Let us denote by Σ_n the σ-algebra Σ_{Ω_n} and by μ_n the restriction of μ to Σ_n. Thus $(\Omega_n, \Sigma_n, \mu_n)$ is a finite measure space. Then the space $E_n := E_{\Omega_n}$ with the norm $\|\cdot\|_n := \|\cdot\|_{\Omega_n}$ is a saturated μ_n-B.f.s. such that $\chi_{\Omega_n} \in E_n$. By the above corollary we have that $E_n^\times$ has the Fatou property.

Let $g_{\tau,n}$ be the restriction of g_τ to the set Ω_n. Thus $\{g_{\tau,n}\}_{\tau\in I} \subset E_n^\times$ is an upwards directed system with $\sup_\tau \|g_{\tau,n}\|_{E_n^\times} \leq \sup_\tau \|g_\tau\|_{E^\times} < \infty$. Therefore $g_{(n)} := \sup_\tau g_{\tau,n} \in E_n^\times$ and $\|g_{(n)}\|_{E_n^\times} = \sup_\tau \|g_{\tau,n}\|_{E_n^\times}$.

Now let $g_n := (g_{(n)})^\Omega$ be the canonical extension of $g_{(n)}$. Then $\{g_n\} \subset E^\times$ is an increasing sequence. By (3.4),

$$\sup_n \|g_n\|_{E^\times} = \sup_n \|g_{(n)}\|_{E_n^\times} \leq \sup_\tau \|g_\tau\|_{E^\times}. \tag{3.5}$$

From the σ-Fatou property in $E^\times$ and (3.5), it follows that

$$g := \sup_n g_n \in E^\times \quad \text{and} \quad \|g\|_{E^\times} \leq \sup_\tau \|g_\tau\|_{E^\times}. \tag{3.6}$$

Let us prove that $g = \sup_\tau g_\tau$. Fix $\tau \in I$. Since $g_\tau \chi_{\Omega_n} \leq g_n \leq g$ for each $n \in \mathbb{N}$, we have that $g_\tau \leq g$. Suppose that $g' \in E^\times$ satisfies $g_\tau \leq g'$, $\forall\, \tau \in I$. Now fix $n \in \mathbb{N}$. Then $g_\tau \chi_{\Omega_n} \leq g' \chi_{\Omega_n}$, $\forall\, \tau$, so $g_n \leq g'$. Therefore, $g \leq g'$ and we obtain that $g = \sup_\tau g_\tau$. Finally $\|\cdot\|_E^\times$ is a lattice norm and so the conclusion follows from (3.6). □

Since we are assuming that E is a saturated μ-B.f.s., then $E^\times$ is a μ-B.f.s. and we can consider its associate space, which is called *second associate space* of E and it is denoted by $E^{\times\times}$. Thus

$$E^{\times\times} := (E^\times)^\times = \left\{h \in L^0(\mu) : hg \in L^1(\mu) \quad \forall\, g \in E^\times\right\}$$

and the seminorm $\|\cdot\|_{E^{\times\times}} : E^{\times\times} \to [0, \infty)$ is given by

$$\|h\|_{E^{\times\times}} := \sup\left\{\int_\Omega |hg|d\mu : g \in B_{E^\times}\right\}.$$

Unlike the σ-finite case, we will see in Example 4.5 that it can happen that $E^\times$ is not saturated. Nevertheless, we have that $E \subset E^{\times\times}$ and

$$\|f\|_{E^{\times\times}} \leq \|f\|_E \quad \forall\, f \in E. \tag{3.7}$$

Corollary 3.6 *Let E be a saturated μ-B.f.s. If μ is σ-finite, then*

(i) *$E^{\times\times}$ is a μ-B.f.s. with the Fatou property.*
(ii) *E has the σ-Fatou property if and only if E has the Fatou property.*

Proof Since μ is σ-finite we have that $E^\times$ is a saturated μ-B.f.s. [28, Ch. 15, §71, Thm. 4]. Thus, from Theorem 3.5 we obtain (i).

Now from the σ-Fatou property in E it follows that $E \equiv E^{\times\times}$ [28, Ch. 15, §71, Thm. 1]. Therefore from (i) we have (ii). □

Let us fix $g \in E^{\times}$. Then the function $\varphi_g : E \to \mathbb{K}$ defined by

$$\varphi_g(f) := \int_{\Omega} gf\, d\mu, \tag{3.8}$$

is a linear and bounded functional such that $\|\varphi_g\| = \|g\|_{E^{\times}}$. Thus we consider the operator

$$R : E^{\times} \to E^{*} \text{ defined by } R(g) := \varphi_g. \tag{3.9}$$

Clearly R is a linear isometry, called *canonical isometry*. Accordingly, the associate space $E^{\times}$ can be identified with a certain closed subspace of E^{*}. The canonical isometry also preserves the order in the sense that

$$g \geq 0 \text{ if, and only if, } \varphi_g \geq 0.$$

In the case $\mathbb{K} = \mathbb{C}$ we also have that g is real if, and only if, φ_g is real. Therefore if R is onto, then R is an order isometry. Hence in what follows we will write $E^{\times} = E^{*}$ to mean that the canonical isometry R is onto.

Next we distinguish two necessary conditions for $E^{\times} = E^{*}$ to hold. We will need the following result, which is obtained from [29, Thm. 102.3, p. 415].

Lemma 3.7 *If E is a Banach lattice, then E^{*} has the Fatou property.*

We also need to recall that a functional $\varphi \in E^{*}$ is *σ-order continuous* whenever $f_n \downarrow 0$ implies $\varphi(f_n) \to 0$.

Proposition 3.8 *Let E be a saturated μ-B.f.s. If $E^{*} = E^{\times}$, then E is order continuous and $E^{\times}$ has the Fatou property.*

Proof Since E is a Dedekind σ-complete Banach lattice, we only have to show that E is σ-order continuous. And so, by [29, Lemma 84.1, Thm. 102.7] it is enough to establish that φ is σ-order continuous for any $\varphi \in (E^{*})^{+}$. So, take $\varphi \in E^{*}$ such that $\varphi \geq 0$ and consider $\{f_n\} \subset E$ satisfying that $f_n \downarrow 0$. Since φ is positive, there exists $g \in (E^{\times})^{+}$ such that

$$\varphi(f) = \int_{\Omega} gf\, d\mu, \quad \forall\, f \in E.$$

Then $\{gf_n\} \subset L^1(\mu)$ is a decreasing sequence such that $0 \leq gf_n$. Since the space $L^1(\mu)$ is Dedekind σ-complete, there exists $0 \leq h = \inf_n gf_n$. Let $A := \operatorname{supp} g$. It is clear that $h\chi_{\Omega\setminus A} = 0$. Taking $\frac{0}{0} := 0$, we have that

$$\frac{h\chi_A}{g} \leq f_n\chi_A \leq f_n, \quad \forall\, n \in \mathbb{N}.$$

It follows that $\frac{h\chi_A}{g} \in E$ and from $f_n \downarrow 0$ we have that $\frac{h\chi_A}{g} \leq 0$. Hence $h\chi_A = 0$ and then $gf_n \downarrow 0$. Therefore $\varphi(f_n) = \int_{\Omega} gf_n d\mu \downarrow 0$. So φ is σ-order continuous.

The other affirmation follows from the above lemma. □

4 Locally σ-finite measure defined on a δ-ring

Let us assume now that the σ-algebra Σ that we have been considering is given as $\Sigma = \mathcal{R}^{loc}$, where $\mathcal{R}$ is a δ-ring and $\mu = |\lambda|$, where $\lambda : \mathcal{R} \to [0, \infty]$ is a measure on $\mathcal{R}$. Noting that for any $A \in \mathcal{R}^{loc}$ with $|\lambda|(A) > 0$ we can find $B \in \mathcal{R}_A$ with $\lambda(B) > 0$, next we give a simple sufficient condition for E to be saturated.

Lemma 4.1 *Let E be a $|\lambda|$-B.f.s. If $S(\mathcal{R}) \subset E$, then E is saturated.*

Remark 4.2 When $S(\mathcal{R}) \subset E$, the space E is a B.f.s. with respect to $(\Omega, \mathcal{R}, \lambda)$ in the sense introduced by Delgado in [9, Def. 3.1]. Thus these class of spaces are always saturated.

Let $\nu : \mathcal{R} \to X$ be a vector measure having $\lambda : \mathcal{R} \to [0, \infty]$ as a local control measure. For $1 \leq p < \infty$, the spaces $L^p_w(\nu)$ and $L^p(\nu)$ are defined by

$$L^p_w(\nu) := \left\{ f \in L^0(\nu) : |f|^p \in L^1_w(\nu) \right\} \quad \text{and} \quad L^p(\nu) := \left\{ f \in L^0(\nu) : |f|^p \in L^1(\nu) \right\}.$$

Each function in $L^p_w(\nu)$ is called *weakly p-integrable* with respect to ν and each function in $L^p(\nu)$ is called *p-integrable* with respect to ν. Note that $L^p(\nu) \subset L^p_w(\nu)$. Moreover, $L^p_w(\nu)$ and $L^p(\nu)$ are $|\lambda|$-B.f.s. with norm

$$\|f\|_{p,\nu} := \||f|^p\|_\nu^{\frac{1}{p}} = \sup_{x^* \in B_{X^*}} \left(\int_\Omega |f|^p d|\langle \nu, x^* \rangle| \right)^{\frac{1}{p}}, \quad \forall f \in L^p_w(\nu).$$

Also $S(\mathcal{R})$ is a dense subspace of $L^p(\nu)$, the space $L^p(\nu)$ is σ-order continuous and $L^p_w(\nu)$ has the σ-Fatou property [17, p. 37].

From the above lemma and Theorem 3.5 we obtain:

Proposition 4.3 *Let $1 \leq p < \infty$. Then $L^p(\nu)$ is saturated. Thus, $L^p(\nu)^\times$, with norm $\|\cdot\|_{\nu^\times} := \|\cdot\|_{L^p(\nu)^\times}$, is a $|\lambda|$-B.f.s. with the σ-Fatou property. If in addition the measure λ is σ-finite, then $L^p(\nu)^\times$ has the Fatou property.*

Remark 4.4 Since $L^1(\nu)$ is always saturated with respect to any local control measure for ν, by Lemma 3.1 we have that $L^1(\nu)$ is order dense in $L^0(|\lambda|)$. Using other methods, this result was established by Calabuig et al. [4, 4.2].

Example 4.5 Given a vector measure ν, let us consider its variation $|\nu|$ as a local control measure. Then $L^1(\nu)^\times$ is a $|\nu|$-B.f.s. with the σ-Fatou property. It may happen that the range of $|\nu|$ is $\{0, \infty\}$. For instance, if Σ is the Lebesgue σ-algebra on $[0, 1]$, then the function $\nu : \Sigma \to L^2([0, 1])$ defined by $\nu(A) := \chi_A$ is a vector measure whose range is $\{0, \infty\}$ [5, p. 57]. In this case $L^1(|\nu|) = \{0\}$ and so $L^1(\nu)^\times = \{g \in L^0(|\nu|) : gf = 0, \ \forall f \in L^1(\nu)\} = \{0\}$. Thus clearly the space $L^1(\nu)^\times$ is not saturated. Then in this situation the study of the associate space will not give interesting information.

As we have just seen, when the measure involved is not σ-finite the associate space is not necessarily saturated. This motivates to look for a class of measures for which this problem does not occur. In this direction, let us recall the following definition, introduced by Brooks and Dinculeanu [3, p. 162].

Definition 4.6 A measure $\lambda : \mathcal{R} \to [0, \infty]$ is *locally σ-finite*, if for each $B \in \mathcal{R}$, there exists $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{R}$ such that $B = \bigcup_{n=1}^{\infty} B_n$ and $\lambda(B_n) < \infty, \forall\, n \in \mathbb{N}$.

Clearly any positive σ-finite measure on a σ-algebra is locally σ-finite.

Example 4.7 Let us consider an uncountable set Γ. Let $\mathcal{R} := \{B \subset \Gamma : B \text{ is finite}\}$ and $\lambda : \mathcal{R} \to [0, \infty]$ be the counting measure. Then $\mathcal{R}$ is a δ-ring and λ is a locally σ-finite measure which is not σ-finite.

Remark 4.8 Consider a locally σ-finite measure $\lambda : \mathcal{R} \to [0, \infty]$ and a $|\lambda|$-B.f.s. E. Let us take $A \in \mathcal{R}^{loc}$ and assume that

$$A := \bigcup_{n=1}^{\infty} B_n \cup N \text{ with } \{B_n\} \subset \mathcal{R},\ \lambda(B_n) < \infty, \quad \forall\, n \in \mathbb{N} \text{ and } N \in \mathcal{N}_0(|\lambda|). \quad (4.1)$$

Hence $(A, (\mathcal{R}_A)^{loc}, |\lambda_A|)$ is a σ-finite measure space, where $(\mathcal{R}_A)^{loc}$ is the σ-algebra related to $\mathcal{R}_A$ and λ_A is the restriction of λ to $\mathcal{R}_A$. It follows that the space E_A defined in (3.2), with norm $\|\cdot\|_A$, is a saturated $|\lambda_A|$-B.f.s. As in this case $|\lambda_A|$ is σ-finite, we have that $E_A^{\times}$, with norm $\|\cdot\|_{E_A^{\times}}$, is a saturated $|\lambda_A|$-B.f.s.

Note that if $B \in \mathcal{R}$, then B has the form (4.1) and in this case $\mathcal{R}_B = (\mathcal{R}_B)^{loc}$. Hence $(B, \mathcal{R}_B, \lambda_B)$ is a σ-finite measure space.

We now show that the problem of having nonsaturated associate spaces does not appear when we work with a locally σ-finite measure.

Theorem 4.9 *Let E be a saturated $|\lambda|$-B.f.s. If the measure λ is locally σ-finite, then $E^{\times}$ is a saturated $|\lambda|$-B.f.s.*

Proof Let $A \in \mathcal{R}^{loc}$ be such that $|\lambda|(A) > 0$ and consider $B \in \mathcal{R}_A$ satisfying $\lambda(B) > 0$. Since E_B, defined in (3.2), is a saturated λ_B-B.f.s. and $\lambda_B(B) > 0$ there exists $C \in \mathcal{R}_B$ with $0 < \lambda_B(C) = \lambda(C)$ and $\chi_C \in E_B^{\times}$. Now take $f \in E$. Then $f_B \in E_B$ and $\int_{\Omega} |f|\chi_C d|\lambda| = \int_B |f_B|\chi_C d|\lambda_B| < \infty$. Hence $\chi_C \in E^{\times}$. □

Hereafter, to the condition that E be a saturated $|\lambda|$-B.f.s. we will add that of $\lambda : \mathcal{R} \to [0, \infty]$ being always a locally σ-finite measure and sometimes we will omit it explicitly. In the σ-finite case it is well known that $E \equiv E^{\times\times}$ when E has the σ-Fatou property. By using this fact we will establish the corresponding result in the more general context we are discussing.

Theorem 4.10 *If E has the Fatou property, then $E \equiv E^{\times\times}$.*

Proof As we have that $E \subset E^{\times\times}$ and $\|f\|_{E^{\times\times}} \le \|f\|_E, \forall\, f \in E$, it only rests to prove the another contention and the other norm inequality. For this, it is enough to establish the conclusion only for non-negative functions.

Let $0 \le f \in E^{\times\times}$. Fix $B \in \mathcal{R}$. Since E has the Fatou property we have that E_B is also a λ_B-B.f.s. with the Fatou property. Moreover, as established in Remark 4.8, λ_B is a σ-finite measure. Then we obtain $E_B \equiv E_B^{\times\times}$ [28, Ch. 15, §71, Thm. 1]. Denote by f_B the restriction of f to B. Hence $f_B \in E_B$ and $\|f_B\|_B = \|f_B\|_{E_B^{\times\times}}$. Noting that $(f_B)^\Omega = f\chi_B$, we have $f\chi_B \in E$ and $\|f\chi_B\|_E = \|f\chi_B\|_{E^{\times\times}}$.

On the other hand, as $\mathcal{R}$ is a directed set with the order given by $B \le C$ if, $B \subset C$, $\forall B, C \in \mathcal{R}$, we can consider the net $\{f\chi_B\}_{B\in\mathcal{R}} \subset E$. Then $\{f\chi_B\}_{B\in\mathcal{R}}$ is an upwards directed system and $\|f\chi_B\|_E = \|f\chi_B\|_{E^{\times\times}} \le \|f\|_{E^{\times\times}}, \forall\, B \in \mathcal{R}$. Since E has the Fatou property, there exists $h \in E \subset E^{\times\times}$ with $h = \sup_{B\in\mathcal{R}} f\chi_B$ and

$$\|h\|_E = \sup_{B\in\mathcal{R}} \|f\chi_B\|_E = \sup_{B\in\mathcal{R}} \|f\chi_B\|_{E^{\times\times}} \le \|f\|_{E^{\times\times}}. \tag{4.2}$$

Assume that there exists $A \in \mathcal{R}^{loc}$ such that $h\chi_A < f\chi_A$ and $|\lambda|(A) > 0$. Then for some $B \in \mathcal{R}_A$ with positive measure we have that $h\chi_B < f\chi_B$, which is a contradiction and it follows that $f \in E$. Moreover, as $f\chi_B \le f, \forall\, B \in \mathcal{R}$ we have that $h \le f$, λ-a.e. Therefore $h = f$, $|\lambda|$-a.e. The remaining inequality between the norms follows from (4.2). □

It is well known that if μ is a σ-finite measure and E is a μ-B.f.s., then we can write $L^1(\mu) = \{fg : f \in E, g \in E^\times\}$. Next we will show that this result remains valid when we consider a locally σ-finite measure.

Proposition 4.11 *Let λ be a locally σ-finite measure and E be a saturated $|\lambda|$-B.f.s. If $h \in L^1(\lambda)$:*

(i) *then for each $\varepsilon > 0$ there exist $f \in E$ and $g \in E^\times$ such that*

$$h = fg \quad \textit{and} \quad \|f\|_E\|g\|_{E^\times} \le (1+\varepsilon)\int_\Omega |h|d|\lambda|.$$

(ii) *if in addition E has the σ-Fatou property, then there exist $f \in E$ and $g \in E^\times$ such that*

$$h = fg \quad \textit{and} \quad \|f\|_E\|g\|_{E^\times} = \int_\Omega |h|d|\lambda|.$$

Proof If $h = 0$, the conclusion is clear. Assume that $h \ne 0$. Since $h \in L^1(\lambda)$, we have that $A := \text{supp} h = \bigcup_{n=1}^\infty B_n \cup N$, where $\{B_n\} \subset \mathcal{R}$ and $N \in \mathcal{N}_0(\lambda)$. As λ is a locally σ-finite measure we can assume that $\lambda(B_n) < \infty, \forall\, n \in \mathbb{N}$. Then λ_A is σ-finite and $h_A \in L^1(\lambda_A)$.

(i) Given $\varepsilon > 0$, [15, Thm. 1, (ii)] there exist $\tilde{f} \in E_A$ and $\tilde{g} \in E_A^\times$ such that

$$h_A = \tilde{f}\tilde{g} \quad \text{and} \quad \|\tilde{f}\|_{E_A}\|\tilde{g}\|_{E_A^\times} \le (1+\varepsilon)\int_A |h_A|d|\lambda_A|.$$

(ii) Since E has the σ-Fatou property it follows that E_A also has it. From [15, Thm. 1i)] we get $\tilde{f} \in E_A$ and $\tilde{g} \in E_A^\times$ such that

$$h_A = \tilde{f}\tilde{g} \quad \text{and} \quad \|\tilde{f}\|_{E_A}\|\tilde{g}\|_{E_A^\times} = \int_A |h_A|d|\lambda_A|.$$

As $h = h_A\chi_A$, by taking $f := \tilde{f}^{\Omega} \in E$ and $g := \tilde{g}^{\Omega} \in E^{\times}$, the conclusion follows. □

In the σ-finite case we know that $E^{\times} = E^{*}$ if, and only if, E is σ-order continuous [28, Ch. 15, §72, Thm. 5]. We proved in Proposition 3.8 that if $E^{\times} = E^{*}$, then E is σ-order continuous and $E^{\times}$ has the Fatou property. Now we will show that the converse also is valid in our context. For this let us recall that an ideal Y of a Banach lattice X is a *band* whenever, for every subset D of Y possessing a supremum in X, this supremum is already in Y.

Theorem 4.12 *If E is σ-order continuous, then the following properties are equivalent:*

(i) $E^{\times} = E^{*}$.
(ii) $E^{\times}$ *is a band of* E^{*}.
(iii) $E^{\times}$ *has the Fatou property.*
(iv) $E^{\times}$ *has the weak Fatou property.*

Proof The implications (i) ⇒ (ii) and (iii) ⇒ (iv) are clear. (ii) ⇒ (iii) Let $\{g_\tau\} \subset E^{\times}$ be an upwards directed system such that $\sup_\tau \|g_\tau\|_{E^{\times}} < \infty$. Let us take $\varphi_\tau := R(g_\tau)$. Then $\{\varphi_\tau\} \subset E^{*}$ is an upwards directed system such that $\sup_\tau \|\varphi_\tau\| < \infty$. From Lemma 3.7 we have that E^{*} has the Fatou property. Thus there exists $\varphi \in E^{*}$ such that $\varphi_\tau \uparrow \varphi$ and $\|\varphi\| = \sup_\tau \|\varphi_\tau\|$. Now since $\{\varphi_\tau\} \subset R(E^{\times})$ and $R(E^{\times})$ is a band of E^{*} we have that $\varphi \in R(E^{\times})$. Let $g \in E^{\times}$ be such that $\varphi = R(g)$. Since R is an order isometry $g_\tau \uparrow g$ and $\|g\|_{E^{\times}} = \sup_\tau \|g_\tau\|_{E^{\times}}$.

(iv) ⇒ (i) Now let us assume that $E^{\times}$ has the weak Fatou property. First note that since E^{*} is a Banach lattice and R is a linear operator it is enough to represent only the positive functionals. Take $0 \leq \varphi \in E^{*}$. Consider the δ-ring $\mathcal{R}(E) := \{B \in \mathcal{R} : \chi_B \in E\}$ and define $m : \mathcal{R}(E) \to [0, \infty)$ by $m(B) := \varphi(\chi_B)$, $\forall\, B \in \mathcal{R}(E)$. Since E is σ-order continuous and φ is a positive linear functional we have that m is a positive measure.

Fix $B \in \mathcal{R}(E)$. Then $\mathcal{R}_B = \mathcal{R}(E)_B$. Let us denote by m_B the restriction of m to $\mathcal{R}_B$.Then m_B is bounded. On the other hand as λ is a locally σ-finite measure, its restriction λ_B to $\mathcal{R}_B$ is a positive σ-finite measure.

Now let $A \in \mathcal{R}_B$ with $\lambda_B(A) = |\lambda|(A) = 0$, then $\chi_A = 0$, λ-a.e. Hence $m_B(A) = 0$. By the Radon–Nikodym Theorem [26, p. 121] there exists a unique $h_B \in L^0(\mathcal{R}_B)$ such that

$$\varphi(\chi_A) = m_B(A) = \int_A h_B d\lambda_B = \int_B h_B \chi_A d\lambda_B, \quad \forall\, A \in \mathcal{R}_B.$$

Using a standard procedure it follows now that

$$\varphi(f) = \int_B h_B f d\lambda_B, \quad \forall\, f \in E_B. \tag{4.3}$$

Let us denote by H_B the canonical extension of h_B, then $H_B \in E^{\times}$, $\varphi_B(f) := \varphi(f\chi_B) = \int_\Omega f H_B d|\lambda|$, $\forall\, f \in E$ and, by the uniqueness of h_B it turns out that $H_B\chi_C = H_{B\cap C}$, $\forall\, C \in \mathcal{R}$. Noting that $\varphi_B \leq \varphi$ we have that

$$\|H_B\|_{E^\times} = \|\varphi_B\| \le \|\varphi\| < \infty.$$

This shows that $\sup_{B\in\mathcal{R}(E)} \|H_B\|_{E^\times} < \infty$. It turns out that $\{H_B\}_{B\in\mathcal{R}(E)} \subset E^\times$ is an upwards directed system. Now since $E^\times$ has the weak Fatou property, there exists $h = \sup_{B\in\mathcal{R}(E)} H_B \in E^\times$.

Let $f \in E^+$. To prove that $\varphi(f) = \int_\Omega fhd|\lambda|$ first we will show that $H_B = h\chi_B$, $\forall\ B \in \mathcal{R}(E)$. Fix $B \in \mathcal{R}(E)$. It is clear that $H_B \le h\chi_B$. Let us assume that $H_B < h\chi_B$, so we can take $C \in \mathcal{R}_B$ with positive measure such that $H_B(t) < h\chi_B(f)$, $\forall\, t \in C$. Define $k = h\chi_{\Omega\setminus C} + H_B\chi_C$. Now let $D \in \mathcal{R}(E)$, then

$$H_D = H_D\chi_{\Omega\setminus C} + H_{D\cap C} \le h\chi_{\Omega\setminus C} + H_B\chi_C = k,$$

that is, k is an upper bound of $\{H_B\}_{B\in\mathcal{R}(E)}$ which contradicts that h is the supremum. We conclude that $H_B = h\chi_B$.

Since $fh \in L^1(|\lambda|)$, it follows that $A := \operatorname{supp} fh = \bigcup_{n=1}^\infty B_n \cup N$ where $\{B_n\}$ is a disjoint family of subsets of $\mathcal{R}$ and N is a $|\lambda|$-null set. Observe that since λ_B is σ-finite and E_B is saturated, $\forall\ B \in \mathcal{R}$, by [28, Ch. 15, §67, Thm. 4] we can take $B_n \in \mathcal{R}(E)$. Since E is order continuous, it follows that $f\chi_A = \sum_{n=1}^\infty f\chi_{B_n}$ in E. Assume that $\varphi(f\chi_{\Omega\setminus A}) > 0$. Then $f\chi_{\Omega\setminus A} > 0$, so we can choose $B \in \mathcal{R}(E)_{\Omega\setminus A}$ such that $|\lambda|(B) > 0$ and $f\chi_B > 0$. Take its corresponding $H_B \in E^\times$. So, $0 < \varphi(f\chi_B) = \int_\Omega fH_Bd|\lambda| = \int_\Omega fh\chi_Bd|\lambda|$, but $B \subset \Omega\setminus\operatorname{supp} fh$, thus $\int_\Omega fh\chi_Bd|\lambda| = 0$, which is a contradiction. Therefore $\varphi(f\chi_{\Omega\setminus A}) = 0$ and so

$$\begin{aligned}\varphi(f) &= \sum_{n=1}^\infty \varphi(f\chi_{B_n}) = \sum_{n=1}^\infty \int_\Omega fH_Bd|\lambda| \\ &= \sum_{n=1}^\infty \int_\Omega fh\chi_Bd|\lambda| = \int_\Omega \sum_{n=1}^\infty f\chi_Bhd|\lambda| = \int_\Omega fhd|\lambda|.\end{aligned}$$

This show that $\varphi = \varphi_h \in R(L^1(\nu)^\times)$ and hence the conclusion follows. □

Although the above theorem characterizes when $E^\times = E^*$, up to now we do not know if $E^\times$ always has the Fatou property. Next we present a situation where this holds. The proof follows from Theorem 4.12 and Proposition 3.3.

Corollary 4.13 *Let $\lambda : \mathcal{R} \to [0, \infty]$ a locally σ-finite measure and E be a saturated λ-B.f.s. If E and $E^\times$ are σ-order continuous, then $E^\times = E^*$.*

Using Theorem 4.12 we can give a characterization of reflexivity as follows.

Theorem 4.14 *Let λ be a locally σ-finite measure and E a saturated λ-B.f.s. If E is order continuous then the space E is reflexive if, and only if, $E = E^{\times\times}$ and $E^\times$ is σ-order continuous.*

Proof Let us denote by $R_1 : E^\times \to E^*$ and $R_2 : E^{\times\times} \to E^{\times^*}$ the corresponding canonical isometries. Then for the adjoint operator of R_1 we have $R_1^* : E^{**} \to E^{\times^*}$. First assume that $E^\times$ is σ-order continuous and $E = E^{\times\times}$. Then by Corollary 4.13

R_1 is onto and by hypothesis $E^{\times\times} = E$. So $E^{\times\times}$ is σ-order continuous and has the σ-Fatou property. It follows that $E^{\times\times}$ has the Fatou property and so we can apply Theorem 4.12 to obtain that R_2 is onto. Let us see that $R_1^* j = R_2$ where $j : E \to E^{**}$ is the canonical injection. Take $f \in E$ and $g \in E^{\times}$, then

$$\langle g, R_1^* j(f)\rangle = \langle R_1(g), j(f)\rangle = \langle f, R_1(g)\rangle = \int_\Omega fg d|\lambda| = \langle g, R_2(f)\rangle.$$

Therefore j is onto, that is, E is reflexive.

Now assume that E is reflexive. To establish that $E = E^{\times\times}$ it only rests to prove that $E^{\times\times} \subset E$. Take $h \in E^{\times\times}$. Since R_1 is an injective linear operator with closed range, it follows that R_1^* is onto [11, Thm. VI.6.2]. Hence there exists $\varphi \in E^{**}$ such that $R_1^*(\varphi) = R_2(h)$. Let $f \in E$ satisfy $j(f) = \varphi$. Thus for $g \in E^{\times}$ we have

$$\langle g, R_1^*(\varphi)\rangle = \langle R_1(g), \varphi\rangle = \langle R_1(g), j(f)\rangle = \langle f, R_1(g)\rangle = \int_\Omega fg d|\lambda|.$$

Hence, $\int_\Omega fg d|\lambda| = \langle g, R_2(h)\rangle = \int_\Omega hg d|\lambda|$. Then $h = f$, $|\lambda|$-c.t.p. and $h \in E$.

We will now prove that R_2 is onto, then by Proposition 3.8 we will obtain that $E^{\times}$ is order continuous. Consider $\varphi \in L^1(\nu)^{**}$, then $\varphi \circ R_1^{-1} : R_1(E^{\times}) \to \mathbb{K}$ is linear and bounded. By the Hahn–Banach Theorem there exists $\widetilde{\varphi} \in E^{**}$ such that $\langle \psi, \widetilde{\varphi}\rangle = \langle \psi, \varphi \circ R_1^{-1}\rangle$, $\forall\, \varphi \in R_1(E^{\times})$. Let $f \in E = E^{\times\times}$ be satisfy $j(f) = \widetilde{\varphi}$. Then for each $g \in E^{\times}$ we have

$$\begin{aligned}\langle g, R_2(f)\rangle &= \int_\Omega fg d|\lambda| = \langle f, R_1(g)\rangle = \langle R_1(g), j(f)\rangle \\ &= \langle R_1(g), \widetilde{\varphi}\rangle = \left\langle R_1(g), \varphi \circ R_1^{-1}\right\rangle = \langle g, \varphi\rangle\end{aligned}$$

It follows that $\varphi = R_2(f)$ and we conclude that R_2 is onto. □

Proposition 4.15 *Let λ be a locally σ-finite measure and E a saturated $|\lambda|$-B.f.s. If $E^{\times}$ has the weak Fatou property, then $E^{\times}$ is a band of E^*.*

Proof To prove that $R(E^{\times})$ is an ideal of E^* we can proceed as in implication (iv) ⇒ (i) of Theorem 4.12 only observing that if $0 \le \varphi \le \varphi_g \in R(E^{\times})$, then the set function $m_\varphi : \mathcal{R}(E) \to [0, \infty)$, defined by $m_\varphi(B) = \varphi(\chi_B)$ is a positive measure. Now let $A \subset E^{\times}$ be a non empty set such that there exists $\varphi := \sup_{g\in A} \varphi_g \in E^*$. We have to prove that $\varphi \in R(E^{\times})$.

Let us note that $\mathcal{F} := \{F \subset A : F \text{ is finite}\}$ is a directed set with the order given by $F_1 \le F_2$ if $F_1 \subset F_2$. For each $F \in \mathcal{F}$ define $\varphi_F := \max_{g\in F} \varphi_g$. Let us fix $F_0 \in \mathcal{F}$ and take $\mathcal{F}_0 := \{F \in \mathcal{F} : F_0 \subset F\}$. It turns out that $\sup_{F\in\mathcal{F}_0} \varphi_F = \varphi$. Then $0 \le \varphi_F - \varphi_{F_0} \le \varphi - \varphi_{F_0}$, $\forall\, F \in \mathcal{F}_0$. Thus $\{\varphi_F - \varphi_{F_0}\}_{\mathcal{F}_0}$ is an upwards directed system such that $\sup_{F\in\mathcal{F}_0} \|\varphi_F - \varphi_{F_0}\| < \infty$. Since $\{\varphi_F - \varphi_{F_0}\}_{\mathcal{F}_0} \subset R(E^{\times})$ and R is an order isometry we obtain an upwards directed system $\{g_F\}_{\mathcal{F}_0} \subset E^{\times+}$ with $\sup_{F\in\mathcal{F}_0} \|g_F\|_{E^{\times}} < \infty$. By the weak Fatou property in $E^{\times}$, there exists $g \in E^{\times}$ such that $g_F \uparrow g$. Using again that R is an order isometry we have that $\varphi_F - \varphi_{F_0} \uparrow \varphi_g$. Since $\sup_{F \in \mathcal{F}_0} \varphi_F = \varphi$ we have that $\varphi - \varphi_{F_0} = \varphi_g$. Therefore $\varphi \in R(E^\times)$. □

The following result was established in [29, p. 418]. We obtain it as consequence of the above proposition and Theorem 3.5.

Corollary 4.16 *Let E be a saturated $|\lambda|$-B.f.s. If $|\lambda|$ is σ-finite, then $E^\times$ is a band of E^*.*

5 Brooks–Dinculeanu measure

Let $\nu : \mathcal{R} \to X$ be a vector measure defined on a δ-ring. Since we are interested in providing a representation of the dual space of $L^1(\nu)$ as its associate space, it is important to know if ν has a local control measure which is locally σ-finite. Then, by Theorem 4.9, the associate space of $L^1(\nu)$, with respect to this local control measure, will be saturated. Let us distinguish this kind of measures.

Definition 5.1 A measure $\lambda : \mathcal{R} \to [0, \infty]$ is a *Brooks–Dinculeanu measure for* ν, if λ is a local control measure for ν which is locally σ-finite.

Example 5.2 1. Let $\nu : \Sigma \to X$ be a vector measure defined on a σ-algebra. If $\mu : \Sigma \to [0, \infty)$ is a Rybakov control measure for ν, then μ is a Brooks–Dinculeanu measure for ν.
2. Let $\nu : \mathcal{R} \to X$ be a σ-finite vector measure. Then ν has a bounded local control measure $\lambda : \mathcal{R} \to [0, \infty)$ [8, Thm. 3.3]. Hence, λ is a Brooks–Dinculeanu for ν.

Fortunately, it turns out that each vector measure defined in a δ-ring has a Brooks–Dinculeanu measure. This result was established by Jiménez Fernández et al. in [16, p. 3]. Given its importance, we will state it below.

Theorem 5.3 *If $\nu : \mathcal{R} \to X$ is a vector measure, then ν has a Brooks–Dinculeanu measure.*

Let us define

$$\widehat{\mathcal{R}} := \{B \in \mathcal{R} : \lambda(B) < \infty\}.$$

It is clear that $\widehat{\mathcal{R}}$ is a δ-ring satisfying that $\widehat{\mathcal{R}} \subset \mathcal{R}$. Moreover, it turns out that $\mathcal{R}^{loc} = \widehat{\mathcal{R}}^{loc}$. Now let us show that

$$|\langle \nu, x^* \rangle| = |\langle \widehat{\nu}, x^* \rangle|, \quad \forall\, x^* \in X^*, \tag{5.1}$$

where $\widehat{\nu}$ is the restriction of ν to $\widehat{\mathcal{R}}$. By definition we obtain that $|\langle \widehat{\nu}, x^* \rangle| \le |\langle \nu, x^* \rangle|$. To establish the other inequality let us fix $x^* \in X^*$ and consider $A \in \mathcal{R}^{loc}$. Take $B \in \mathcal{R}_A$. Since λ is locally σ-finite, there exists an increasing sequence, $\{B_n\} \subset \widehat{\mathcal{R}}$ such that $B = \bigcup_{n=1}^\infty B_n$. So,

$$\begin{aligned} |\langle \nu, x^* \rangle|(B) &= \sup_n |\langle \nu, x^* \rangle|(B_n) = \sup_n |\langle \widehat{\nu}, x^* \rangle|(B_n) \\ &\le \sup_{C \in \widehat{\mathcal{R}}_A} |\langle \widehat{\nu}, x^* \rangle|(C) = |\langle \widehat{\nu}, x^* \rangle|(A). \end{aligned}$$

It follows that $|\langle \nu, x^* \rangle|(A) \leq |\langle \widehat{\nu}, x^* \rangle|(A)$. Hence we have established (5.1).

From (5.1) we have that

$$\|f\|_\nu = \|f\|_{\widehat{\nu}}, \quad \forall\, f \in L^0(\mathcal{R}^{loc}). \tag{5.2}$$

Thus $L^1_w(\nu) \equiv L^1_w(\widehat{\nu})$. Since (5.2) is valid, from the density of $S(\mathcal{R})$ in $L^1(\nu)$ and of $S(\widehat{\mathcal{R}})$ in $L^1(\widehat{\nu})$, to prove that $L^1(\nu) \equiv L^1(\widehat{\nu})$, it is sufficient to check that $S(\mathcal{R}) \subset L^1(\widehat{\nu})$ and that $S(\widehat{\mathcal{R}}) \subset L^1(\nu)$. Noting that $\widehat{\mathcal{R}} \subset \mathcal{R}$ the second contention is clear. Now consider $B \in \mathcal{R}$ and take $\{B_n\} \subset \widehat{\mathcal{R}}$ satisfying that $B_n \subset B_{n+1}$ and $B = \bigcup_{n=1}^\infty B_n$. Then $\chi_{B_n} \to \chi_B$, moreover

$$\int_A \chi_{B_n} d\widehat{\nu} = \widehat{\nu}(B_n \cap A) = \nu(B_n \cap A) = \int_A \chi_{B_n} d\nu \to \int_A \chi_B d\nu, \quad \forall\, A \in \mathcal{R}^{loc}.$$

From [8, Prop. 2.3] we have that $\chi_B \in L^1(\widehat{\nu})$. It follows that $S(\mathcal{R}) \subset L^1(\widehat{\nu})$ and $I_\nu(s) = I_{\widehat{\nu}}(s)$, $\forall\, s \in S(\mathcal{R})$. By the continuity of the integration operators we have $I_\nu = I_{\widehat{\nu}}$. Therefore, we have proven the following result and so, whenever we find it convenient we can work on the δ-ring $\widehat{\mathcal{R}}$ instead of $\mathcal{R}$.

Lemma 5.4 *If $\lambda : \mathcal{R} \to [0, \infty]$ is a Brooks–Dinculeanu measure for a given vector measure ν, then*

(i) *for each $x^* \in X^*$ we have that $|\langle \nu, x^* \rangle| = |\langle \widehat{\nu}, x^* \rangle|$,*
(ii) *$L^1_w(\nu) \equiv L^1_w(\widehat{\nu})$, $L^1(\nu) \equiv L^1(\widehat{\nu})$ and $\int_\Omega f d\nu = \int_\Omega f d\widehat{\nu}$, $\forall\, f \in L^1(\nu)$.*

Curbera and Ricker established that $L^p(\nu)^{\times\times} \equiv L^p_w(\nu)$ when a vector measure defined on a σ-algebra and a Rybakov control measure are considered [7, Prop. 2]. We will show that this equality remains true if we consider instead a vector measure defined on a δ-ring and a Brooks–Dinculeanu measure. Before it is necessary to establish a useful characterization for the functions in $L^1_w(\nu)$.

Lemma 5.5 *Let $f \in L^0(\mathcal{R}^{loc})$. Then $f \in L^1_w(\nu)$ if and only if for each $B \in \mathcal{R}$, $f\chi_B \in L^1_w(\nu)$ and $\sup_{B\in\mathcal{R}} \|f\chi_B\|_\nu < \infty$. In this case $\|f\|_\nu = \sup_{B\in\mathcal{R}} \|f\chi_B\|_\nu$.*

Proof First let us assume that $f \in L^1_w(\nu)$. Since $L^1_w(\nu)$ is a Banach lattice we have that $f\chi_B \in L^1_w(\nu)$ and $\|f\chi_B\|_\nu \leq \|f\|_\nu, \forall\, B \in \mathcal{R}$. Then, $\sup_{B\in\mathcal{R}} \|f\chi_B\|_\nu \leq \|f\|_\nu < \infty$.

Now assume that $f\chi_B \in L^1_w(\nu)$, $\forall\, B \in \mathcal{R}$ and $M := \sup_{B\in\mathcal{R}} \|f\chi_B\|_\nu < \infty$. Let $x^* \in B_{X^*}$, then

$$\sup_{B\in\mathcal{R}} \int_B |f| d|\langle x^*, \nu \rangle| \leq M.$$

From Proposition 2.1 we obtain that $f \in L^1(|\langle x^*, \nu \rangle|)$, $\forall\, x^* \in B_{X^*}$. Thus $f \in L^1_w(\nu)$ and $\|f\|_\nu \leq M$. □

Although most of the time we will not state it explicitly, in what follows $\lambda : \mathcal{R} \to [0, \infty]$ will be a Brooks–Dinculeanu measure for a given vector measure ν and we will consider $L^p_w(\nu)$ and $L^p(\nu)$ as Banach function spaces with respect to $|\lambda|$.

Theorem 5.6 *Let* $1 \leq p < \infty$. *Then* $L^p(\nu)^{\times\times} \equiv L_w^p(\nu)$.

Proof First we prove that $L_w^p(\nu) \subset L^p(\nu)^{\times\times}$ and $\|f\|_{p,\nu^{\times\times}} \leq \|f\|_{p,\nu}, \forall f \in L_w^p(\nu)$. Let $\varphi \in S(\mathcal{R}^{loc})$ be such that $\varphi \in L_w^p(\nu)$ and $B \in \widehat{\mathcal{R}}$. Then $\varphi\chi_B \in S(\mathcal{R}) \subset L^p(\nu)$. By the Hölder inequality, for each $g \in L^p(\nu)^\times$

$$\int_B |g\varphi d|\lambda| \leq \|g\|_{p,\nu^\times}\|\varphi\chi_B\|_{p,\nu} \leq \|g\|_{p,\nu^\times}\|\varphi\|_{p,\nu}.$$

From Proposition 2.1 we have

$$\int_\Omega |g\varphi|d|\lambda| = \sup_{B\in\widehat{\mathcal{R}}} \int_B |g\varphi|d|\lambda| \leq \|g\|_{p,\nu^\times}\|\varphi\|_{p,\nu}.$$

It follows that $\varphi \in L^p(\nu)^{\times\times}$ and

$$\|\varphi\|_{p,\nu^{\times\times}} \leq \|\varphi\|_{p,\nu}. \tag{5.3}$$

Now consider $f \in L_w^p(\nu)$ and take $\{\varphi_n\} \subset S(\mathcal{R}^{loc})$ with $0 \leq \varphi_n \uparrow |f|$. Then, $\{\varphi_n\} \subset L_w^p(\nu)$. By (5.3), $\|\varphi_n\|_{p,\nu^{\times\times}} \leq \|\varphi_n\|_{p,\nu} \leq \|f\|_{p,\nu}, \forall n \in \mathbb{N}$. Since $L^p(\nu)^{\times\times}$ has the σ-Fatou property it turns out that $f \in L^p(\nu)^{\times\times}$ and $\|f\|_{p,\nu^{\times\times}} \leq \|f\|_{p,\nu}$.

For the other contention let us fix $B \in \mathcal{R}$ and let ν_B be the restriction of ν to the σ-algebra $\mathcal{R}_B$. As $L^1(\nu_B) \equiv L^1(\nu)_B$, it follows that $L^p(\nu_B) \equiv L^p(\nu)_B$. Hence we obtain that $L^p(\nu_B)^{\times\times} \equiv L_w^p(\nu_B)$ [7, Prop. 2].

Take $f \in L^p(\nu)^{\times\times}$ and let us denote by f_B its restriction to B, then $f_B \in L_w^p(\nu_B)$ and $\|f_B\|_{p,\nu_B} = \|f_B\|_{p,\nu_B^{\times\times}} \leq \|f\|_{p,\nu^{\times\times}}$. And so, for each $x^* \in B_{X^*}$

$$\int_B |f|^p d|\langle\nu, x^*\rangle| \leq \|f\|^p_{p,\nu^{\times\times}}.$$

From the above proposition we have $|f|^p \in L_w^1(\nu)$ and $\||f|^p\|_\nu \leq \|f\|^p_{p,\nu^{\times\times}}$. Hence $f \in L_w^p(\nu)$ and $\|f\|_{p,\nu} \leq \|f\|_{p,\nu^{\times\times}}$. ◻

The following result was established in [4, p. 77] by other methods, we obtain it as consequence of the above proposition and Corollary 3.6.

Corollary 5.7 *Let* $1 \leq p < \infty$. *If* $\nu : \mathcal{R} \to X$ *is a* σ*-finite vector measure, then* $L_w^p(\nu)$ *has the Fatou property.*

From Theorem 5.6 and Proposition 3.3 we obtain:

Corollary 5.8 *If* $L_w^p(\nu) \subset L^1(\lambda)$, $1 \leq p < \infty$, *then* $L_w^p(\nu)$ *has the Fatou property.*

The sufficiency in the next result was proven in [4, Prop. 5.4]. Since $L^1(\nu)$ is σ-order continuous we obtain it from Theorems 4.10 and 5.6.

Corollary 5.9 $L^1(\nu)$ *has the Fatou property if, and only if,* $L^1(\nu) = L_w^1(\nu)$.

As in the σ-finite case, we have the next result.

Lemma 5.10 *If E and F are μ-B.f.s such that $E \subset F$ and there exists $a > 0$ with $\|f\|_F \le a\|f\|_E$, $\forall f \in E$, then $F^\times \subset E^\times$ and*

$$\|g\|_{E^\times} \le a\|g\|_{F^\times}, \quad \forall g \in F^\times.$$

Corollary 5.11 $L^p(\nu)^\times \equiv L^p_w(\nu)^\times$ *and* $L^p_w(\nu)^{\times\times} \equiv L^p_w(\nu)$, $1 \le p < \infty$.

Proof From the above lemma, we have $L^p_w(\nu)^\times \subset L^p(\nu)^\times$ and $\|g\|_{p,\nu^\times} \le \|g\|_{L^p_w(\nu)^\times}$, $\forall g \in L^p_w(\nu)^\times$. Now consider $f \in L^p_w(\nu)$ and $g \in L^p(\nu)^\times$, from the Hölder inequality and Theorem 5.6 we have that

$$\int_\Omega |gf d|\lambda| \le \|g\|_{p,\nu^\times}\|f\|_{p,\nu^{\times\times}} = \|g\|_{p,\nu^\times}\|f\|_{p,\nu}.$$

Hence $g \in L^p_w(\nu)^\times$ and $\|g\|_{L^p_w(\nu)^\times} \le \|g\|_{p,\nu^\times}$.

The second equality follows from Theorem 5.6. □

In Proposition 3.5 we have seen that the associate space of a μ-B.f.s. has the Fatou property when μ is σ-finite. We will show that this result remains true for certain Brooks–Dinculeanu measures, introduced by Calabuig et al. [4, p. 77].

Definition 5.12 A vector measure ν is *$\mathcal{R}$-decomposable* if we can write $\Omega = \bigcup_{\alpha\in\Delta} \Omega_\alpha \cup N$, where $N \in \mathcal{N}_0(\nu)$ and $\{\Omega_\alpha\}_{\alpha\in\Delta} \subset \mathcal{R}$ is a family of pairwise disjoint sets satisfying that

(a) if $A_\alpha \in \mathcal{R}_{\Omega_\alpha}$, $\forall \alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta} A_\alpha \in \mathcal{R}^{loc}$, and
(b) if $x^* \in X^*$ and $N_\alpha \in \mathcal{N}_0(\langle\nu, x^*\rangle)$, $\forall \alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta} N_\alpha \in \mathcal{N}_0(\langle\nu, x^*\rangle)$.

Note that if ν is an $\mathcal{R}$-decomposable vector measure and $A \in \mathcal{R}^{loc}$ is such that $A \cap \Omega_\alpha \in \mathcal{N}_0(\nu)$, $\forall \alpha \in \Delta$, then $A \cap \Omega_\alpha$ is $\langle\nu, x^*\rangle$-null, $\forall \alpha \in \Delta$ and $\forall x^* \in B_{X^*}$. From b) in the above definition it follows that A is ν-null.

Some examples of $\mathcal{R}$-decomposable measures are the σ-finite vector measures and the discrete vector measures [4, Lemma 4.6, p. 77]. However there are $\mathcal{R}$-decomposable measures which are neither σ-finite nor discrete [4, p. 85].

Proposition 5.13 *Let $\nu : \mathcal{R} \to X$ be a vector measure, $\lambda : \mathcal{R} \to [0, \infty]$ be a Brooks–Dinculeanu measure for ν and E be $|\lambda|$-B.f.s. If $S(\mathcal{R}) \subset E$ and ν is $\mathcal{R}$-decomposable, then $E^\times$ has the Fatou property.*

Proof Since ν is $\mathcal{R}$-decomposable, $\Omega = \bigcup_{\alpha\in\Delta} \Omega_\alpha \cup N$, where $N \in \mathcal{N}_0(\nu)$ and $\{\Omega_\alpha\}_{\alpha\in\Delta} \subset \mathcal{R}$ is a family of pairwise disjoint sets satisfying (a) and (b) in Definition 5.12. Moreover, since λ is locally σ-finite we can consider that $\lambda(\Omega_\alpha) < \infty$, $\forall\alpha \in \Delta$. Let us note that by Lemma 4.1, E is a saturated $|\lambda|$-e.f.B.

Let $I \subset \Delta$ be a countable set and take $\Omega_I := \bigcup_{\alpha\in I} \Omega_\alpha$, $\mathcal{R}_I := \mathcal{R}_{\Omega_I}$ and λ_I the restriction of λ to the δ-ring $\mathcal{R}_I$. Then $(\Omega_I, (\mathcal{R}_I)^{loc}, |\lambda_I|)$ is a σ-finite measure space and $E_I := E_{\Omega_I}$ is a $|\lambda_I|$-B.f.s. Then, from Theorem 3.5, we obtain that $E_I^\times$ is a $|\lambda_I|$-B.f.s. whit the Fatou property.

Now let us consider an upwards directed system $\{g_\tau\}_{\tau\in K} \subset E^\times$ such that $g_\tau \geq 0$, $\forall\ \tau \in K$ and $M := \sup_\tau \|g_\tau\|_{E^\times} < \infty$. Denoting by $g_{\tau,I}$ to the restriction of g_τ to Ω_I, we have that $\{g_{\tau,I}\}_{\tau\in K} \subset E_I^\times$ is an upwards directed system, and from (3.4), $\sup_\tau \|g_{\tau,I}\|_{E_I^\times} \leq M < \infty$. Since $E_I^\times$ has the Fatou property, it turns out that $g_I := \sup_\tau g_{\tau,I} \in E_I^\times$ and

$$\|g_I\|_{E_I^\times} = \sup_\tau \|g_{\tau,I}\|_{E_I^\times} \leq M. \tag{5.4}$$

In particular, for each $\alpha \in \Delta$ exists $g_{\{\alpha\}} \in E_{\{\alpha\}}^\times$ such that $g_{\{\alpha\}} = \sup_\tau g_{\tau,\{\alpha\}}$ and

$$\|g_{\{\alpha\}}\|_{E_{\{\alpha\}}^\times} = \sup_\tau \|g_{\tau,\{\alpha\}}\|_{E_{\{\alpha\}}^\times}.$$

Let us denote by g^α the canonical extension of $g_{\{\alpha\}}$ and define $g := \sum_{\alpha\in\Delta} g^\alpha$. As $\mathcal{N}_0(\nu) = \mathcal{N}_0(|\lambda|)$ and from (a) in Definition 5.12 we have that $g \in L^0(|\lambda|)$.

Let us prove that $g = \sup_\tau g_\tau$. Consider $\alpha \in \Delta$ and $\tau \in K$. Since $g_\tau \chi_{\Omega_\alpha} \leq g^\alpha \leq g$, it follows that $g_\tau \leq g$. Let us assume that $g' \in L^0(|\lambda|)$ is such that $g_\tau \leq g'$. Then $g_\tau \chi_{\Omega_\alpha} \leq g' \chi_{\Omega_\alpha}$. So, $g^\alpha \leq g'$. Hence $g \leq g'$ and $g = \sup_\tau g_\tau$.

Finally we will establish that $g \in E^\times$. Let us fix $f \in B_E$ and let $I \subset \Delta$ be a countable set. Note that the canonical extension of $g_I = \sup_\tau g_{\tau,I}$ is given by $g^I = \sum_{\alpha\in I} g^\alpha$ and $f_I \in B_{E_I}$, where f_I is the restriction of f to Ω_I. Moreover, $\int_\Omega |g^I f| d|\lambda| = \int_{\Omega_I} |g_I f_I| d|\lambda_I|$. By using the monotone convergence theorem

$$\sum_{\alpha\in I} \int_\Omega |g^\alpha f| d|\lambda| = \int_\Omega |g^I f| d|\lambda| = \int_{\Omega_I} |g_I f_I| d|\lambda_I| \leq \|g_I\|_{E_I^\times}.$$

From this and (5.4) we obtain that $\sum_{\alpha\in I} \int_\Omega |g^\alpha f| d\lambda| \leq M$, for each finite subset I of Δ. Then, there exists a countable set $J \subset \Delta$ such that $\int_\Omega |g^\alpha f| d|\lambda| = 0, \forall\, \alpha \in \Delta \setminus J$. This implies that

$$\int_\Omega |gf| d|\lambda| = \sum_{\alpha\in J} \int_\Omega |g^\alpha f| d|\lambda| = \int_\Omega |g^J f| d|\lambda| \leq M. \tag{5.5}$$

We conclude that $g \in E^\times$; moreover, from lattice property of the norm in $E^\times$ and from (5.5), we have that $\|g\|_{E^\times} = \sup_\tau \|g_\tau\|_{E^\times}$. □

Let us consider the canonical isometry R between $L^1(\nu)^\times$ and $L^1(\nu)^*$. When ν is a σ-finite vector measure, then ν has a bounded local control measure $\lambda : \mathcal{R} \to [0, \infty)$ [8, Thm. 3.3]. Since $L^1(\nu)$ is a σ-order continuous $|\lambda|$-B.f.s., then we have $L^1(\nu)^* = L^1(\nu)^\times$ [28, Ch. 15, §72, Thm. 5]. In what follows we will present other situations where this holds.

Since $L^1(\nu)$ is an order continuous $|\lambda|$-B.f.s. from Theorem 4.12 we obtain the following result.

Corollary 5.14 *The following properties are equivalent:*

(i) $L^1(\nu)^\times = L^1(\nu)^*$.
(ii) $L^1(\nu)^\times$ *is a band of* $L^1(\nu)^*$.
(iii) $L^1(\nu)^\times$ *has the Fatou property.*
(iv) $L^1(\nu)^\times$ *has the weak Fatou property.*

The next result is a consequence of Proposition 5.13 and the previous result.

Corollary 5.15 *If* ν *is* $\mathcal{R}$*-decomposable, then* $L^1(\nu)^\times = L^1(\nu)^*$.

If E is a real σ-order continuous Banach lattice it is well known that there exists an $\mathcal{R}$-decomposable vector measure $\nu : \mathcal{R} \to E$ such that E is order isometric to the space $L^1(\nu)$ [10, Thm. 5]. Then from the above corollary we obtain the following result.

Corollary 5.16 *If* E *is a* σ*-order continuous Banach lattice, then there exist an* $\mathcal{R}$*-decomposable vector measure* $\nu : \mathcal{R} \to E$ *and a order isometry from* $L^1(\nu)^\times$ *onto* E^**. More precisely, if* T *is a lattice isometry from* E *onto* $L^1(\nu)$*,* $\lambda : \mathcal{R} \to [0, \infty]$ *is a Brooks–Dinculeanu measure for* ν *and* $\varphi \in E^*$*, then there exists* $g \in L^1(\nu)^\times$ *such that*

$$\varphi(f) = \int_\Omega (Tf) g d|\lambda|, \quad \forall\, f \in E.$$

Proof Note that it only rests to verify that the result mentioned before remains valid in the complex case. Since E is a σ-order continuous Banach lattice, then $E_{\mathbb{R}}$ is also a σ-order continuous Banach lattice. Thus, there exist an $\mathcal{R}$-decomposable vector measure $\tilde{\nu} : \mathcal{R} \to E_{\mathbb{R}}$ and an onto lattice isometry $S : L^1(\nu) \to E_{\mathbb{R}}$. Now let us define $\nu : \mathcal{R} \to E$, by $\nu(B) = \tilde{\nu}(B)$, $\forall\, B \in \mathcal{R}$. It turns out that ν is an $\mathcal{R}$-decomposable vector measure and $L^1(\nu)_{\mathbb{R}} = L^1(\tilde{\nu})$. Let $T : L^1(\nu) \to E$ be the canonical extension of S, then T is an onto lattice isometry [25, Lemma 3.8]. □

As a consequence of Theorem 4.14 we obtain the following result.

Corollary 5.17 $L^1(\nu)$ *is reflexive if, and only if,* $L^1(\nu) = L^1_w(\nu)$ *and* $L^1(\nu)^\times$ *is* σ*-order continuous.*

Now from the previous result and Corollary 4.13 we have:

Corollary 5.18 *If* $L^1(\nu)$ *is reflexive, then* $L^1(\nu)^\times = L^1(\nu)^*$.

If $1 < p < \infty$, Ferrando and Rodríguez established that $L^p(\nu)^*$ is order continuous when ν is defined on a σ-algebra [13, Thm 3.1]. Using the same arguments, it follows that in our context we also have that $L^p(\nu)^\times$ is order continuous. Then from Corollary 4.13 we have the following result.

Corollary 5.19 $L^p(\nu)^\times = L^p(\nu)^*$, $1 < p < \infty$.

Since $L^p(\nu)^\times$ is order continuous and $L^p(\nu) = L^p_w(\nu)$ if, and only if, $L^1(\nu) = L^1_w(\nu)$ [17, Prop. 3.1.6], the next result follows from Theorems 4.14 and 5.6. It was proven when ν is defined in a σ-algebra by Fernández et al. [12, Cor. 3.10].

Corollary 5.20 *Let* $1 < p < \infty$. *Then* $L^p(\nu)$ *is reflexive if, and only if,* $L^1(\nu) = L^1_w(\nu)$.

Let us fix $1 < p < \infty$. Then Corollary 5.19 implies that each functional in $L^p(\nu)^*$ has the form φ_g, $g \in L^p(\nu)^\times$. So we can define $S : L^p(\nu)^{\times\times} \to L^p(\nu)^{**}$ by

$$\langle \varphi_g, S(h) \rangle := \int_\Omega ghd|\lambda|.$$

It turns out that S is a linear isometry and we will write $L^p(\nu)^{\times\times} = L^p(\nu)^{**}$ to indicate that is onto. Let $R_1 : L^p(\nu)^\times \to L^p(\nu)^*$ and $R_2 : L^p(\nu)^{\times\times} \to L^p(\nu)^{\times*}$ be the corresponding canonical isometries, then $S = (R_1^*)^{-1} \circ R_2$. Thus S is onto if, and only if, R_2 is it. So from Theorems 5.6 and 4.12 we have:

Corollary 5.21 *Let* $1 < p < \infty$. *Then* $L^p_w(\nu)$ *has the Fatou property if, and only if,* $L^p_w(\nu) = L^p(\nu)^{**}$.

Remark 5.22 Calabuig, Delgado, Juan and Sánchez-Pérez asked if in general $L^1_w(\nu)$ always has the Fatou property [4, pp. 77–78]. With respect to this question we have the following. Let $1 < p < \infty$ and notice that $L^1_w(\nu)$ has the Fatou property if, and only if, $L^p_w(\nu)$ has it. Then from the previous result we obtain that

$$L^1_w(\nu) \text{ has the Fatou property if, and only if, } L^p_w(\nu) = L^p(\nu)^{**}.$$

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