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Positivity

A note on "Higher-order optimality conditions in set-valued optimization using Studniarski derivatives and applications to duality" [Positivity. 18, 449–473(2014)]

Xu Yihong[1] · **Li Min**[1] · **Peng Zhenhua**[1]

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Abstract The note points out that the sufficiency of proposition 2.1 in Anh (Positivity 18:449–473, 2014) is erroneous and we provide an example to illustrate it. Also the proof of proposition 2.2 in Anh (Positivity 18:449–473, 2014) is incorrect and we give a new proof.

Keywords Generalized subconvexlike · Convex cone · Set-valued map

Mathematics Subject Classification 32F17 · 46G05 · 90C29 · 90C46

1 Introduction

The concept of generalized convexity plays an important role in operations research and applied mathematics. Yang et al. [1,2] introduced the concepts of generalized cone subconvexlike set-valued map and nearly cone-subconvexlike set-valued map. Sach [3] introduced a new convexity notion for set-valued maps, called ic-cone-convexlikeness. Xu and Song [4] obtained the following results: (a) when the ordering cone has non-empty interior, ic-cone-convexness is equivalent to near cone-subconvexlikeness; (b) when the ordering cone has empty interior, ic-cone-convexness implies near cone-subconvexlikeness, a counter example is given to show that the converse implication is not true.

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✉ Xu Yihong
xuyihong@ncu.edu.cn

1 Department of Mathematics, Nanchang University, Nanchang 330031, Jiangxi, China

Anh [5] gave an equivalent characterization of generalized cone subconvexlikeness in Proposition 2.1 and applied it to obtain Proposition 2.2, under the assumption of generalized cone subconvexlikeness, some higher-order optimality conditions were established.

In this paper, we will point out that the sufficiency of [5, Proposition 2.1] is invalid and the proof of [5, Proposition 2.2] is incorrect.

2 Preliminaries

Throughout the paper, suppose X, Y are two normed spaces, 0_X and 0_Y denote the original points of X and Y, respectively. $C \subset Y$ is a convex cone with nonempty interior $\text{int}C$ such that $0_Y \in C$.

Definition 2.1 (See [1,5]) Suppose S is a nonempty set in X and $F : S \to 2^Y$ is a set-valued map. F is said to be generalized $C-$subconvexlike on S if $\exists v \in \text{int}C$, $\forall x_1, x_2 \in S$, $\forall \lambda \in (0, 1)$, $\forall \epsilon > 0$, $\exists x_3 \in S$ and $\exists r > 0$ such that

$$\epsilon v + \lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq rF(x_3) + C.$$

3 Main result

By virtue of generalized $C-$subconvexlike of set-valued maps, Anh obtained an equivalent characterization for generalized $C-$subconvexlikeness, as is shown in the following proposition.

Proposition 3.1 *(See [5, Proposition 2.1]) The map $F : S \to 2^Y$ is generalized $C-$subconvexlike on S if and only if* $\text{cone}_+(F(S)) + \text{int}C$ *is convex, where* $\text{cone}_+(F(S)) := \{ry : r > 0, y \in F(S)\}$.

Remark 3.1 The sufficiency of [5, Proposition 2.1] is incorrect, to illustrate the point, we need the following Lemma.

Lemma 3.1 *The following statements are equivalent for the set-valued map F:*

(i) $\forall \hat{u} \in \text{int}C$, $\forall x_1, x_2 \in S$, $\forall \alpha \in [0, 1]$, $\exists x_3 \in S$ *and* $\exists \rho > 0$, *such that*

$$\hat{u} + \alpha F(x_1) + (1 - \alpha)F(x_2) \subseteq \rho F(x_3) + C;$$

(ii) *F is generalized $C-$subconvexlike on S;*

(iii) $\forall x_1, x_2 \in S$, $\forall \alpha \in [0, 1]$, $\exists u = u(x_1, x_2, \alpha) \in \text{int}C$, *and* $\forall \varepsilon > 0$, $\exists x_3 = x_3(u, \varepsilon) \in S$ *and* $\exists \rho = \rho(u, \varepsilon) > 0$ *such that*

$$\varepsilon u + \alpha F(x_1) + (1 - \alpha)F(x_2) \subseteq \rho F(x_3) + C.$$

Proof By Theorem 2.1 of [1], (i) implies (ii) and (ii) implies (iii).

In what follows, we show that (iii) implies (i). Let

$$\hat{u} \in \text{int}C, x_1, x_2 \in S, \alpha \in [0, 1].$$

Then, from (iii), $\exists u = u(x_1, x_2, \alpha) \in \text{int}C$, and $\forall \varepsilon > 0$, $\exists \bar{x}_3 = \bar{x}_3(u, \varepsilon) \in S$ and $\exists \bar{\rho} = \bar{\rho}(u, \varepsilon) > 0$ such that

$$\varepsilon u + \alpha F(x_1) + (1-\alpha)F(x_2) \subseteq \bar{\rho}F(\bar{x}_3) + C.$$

Since $\hat{u} \in \text{int}C$, one can find $\varepsilon_0 > 0$ and $u_0 \in \text{int}C$ such that

$$\hat{u} - \varepsilon_0 u = u_0.$$

From (iii), $\exists x_3 = x_3(u, \varepsilon_0) \in S$ and $\rho = \rho(u, \varepsilon_0) > 0$ such that

$$\varepsilon_0 u + \alpha F(x_1) + (1-\alpha)F(x_2) \subseteq \rho F(x_3) + C.$$

Hence

$$\begin{aligned} \hat{u} + \alpha F(x_1) + (1-\alpha)F(x_2) &= [\varepsilon_0 u + \alpha F(x_1) + (1-\alpha)F(x_2)] + u_0 \\ &\subseteq \rho F(x_3) + C + u_0 \\ &\subseteq \rho F(x_3) + \text{int}C \\ &\subseteq \rho F(x_3) + C. \end{aligned}$$

Thus we complete the proof. □

Example 3.1 Let us set

$$S = \left\{(x_1, x_2) \in R^2 : x_1 + x_2 = 1\right\}, \quad C = R^2_+ = \left\{(x_1, x_2) \in R^2 : x_1 \geq 0, x_2 \geq 0\right\},$$
$$F(x_1, x_2) = \{(x_1, x_2), (1/2, 1/2)\}, \forall (x_1, x_2) \in S.$$

A direct calculation gives

$$\text{cone}_+(F(S)) + \text{int}C = \left\{(y_1, y_2) \in R^2 : y_1 + y_2 > 0\right\}.$$

Then $\text{cone}_+(F(S)) + \text{int}C$ is convex.

However, F is not generalized C−subconvexlike on S, as is illustrated in the following.

Let $z_1 = (-1, 2)$, $z_2 = (2, -1)$, $\lambda_0 = 1/2$, $\hat{u} = (1/8, 1/8)$. Then

$$\hat{u} + \lambda_0 F(z_1) + (1-\lambda_0)F(z_2) = \{(-1/8, 11/8), (11/8, -1/8), (5/8, 5/8)\}.$$

In what follows, we show that

$$\hat{u} + \lambda_0 F(z_1) + (1-\lambda_0)F(z_2) \nsubseteq \rho F(z) + C, \forall z \in S, \forall \rho > 0.$$

In fact, $\forall z = (x_1, x_2) \in \{(x_1, x_2) : x_1 + x_2 = 1\}, \forall \rho > 0$,

(1) If $x_1 \geq 0, x_2 < 0$, then $(-1/8, 11/8) \notin \rho F(z) + C$;
(2) If $x_1 \geq 0, x_2 \geq 0$, then $(-1/8, 11/8) \notin \rho F(z) + C$;
(3) If $x_1 < 0, x_2 > 0$, then $(11/8, -1/8) \notin \rho F(z) + C$.

From above discussions, we deduce that $\exists \hat{u} \in \mathrm{int}C, \exists z_1, z_2 \in S, \exists \lambda_0 = 1/2$ such that $\forall z \in S, \forall \rho > 0$,

$$\hat{u} + \lambda_0 F(z_1) + (1 - \lambda_0) F(z_2) \nsubseteq \rho F(z) + C.$$

From Lemma 3.1, it follows that F is not generalized $C-$subconvexlike on S.

Remark 3.2 Since the sufficient condition of Proposition 2.1 in Ref. [5]. was applied to the proof of Proposition 2.2 in Ref. [5]., the proof is erroneous. However, Proposition 2.2 in Ref. [5]. is true. In the following, we give the proposition and new proof.

Proposition 3.2 *(See [5, Proposition 2.2]) Suppose that the map $F : S \to 2^Y$ is generalized $C-$subconvexlike on S. Then F is also generalized $K-$subconvexlike on S, where K is a convex cone satisfying $C \subseteq K$.*

Proof Since F is generalized $C-$subconvexlike on S, there exists $v \in \mathrm{int}C$, for any $x_1, x_2 \in S, \forall \lambda \in (0, 1), \forall \epsilon > 0, \exists x_3 \in S$ and $\exists r > 0$ such that

$$\epsilon v + \lambda F(x_1) + (1 - \lambda) F(x_2) \subseteq r F(x_3) + C.$$

From $C \subseteq K$, it follows that

$$\epsilon v + \lambda F(x_1) + (1 - \lambda) F(x_2) \subseteq r F(x_3) + C \subseteq r F(x_3) + K.$$

Then F is generalized $K-$subconvexlike on S. □

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