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Generalized Dini theorems for nets of functions on arbitrary sets

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Abstract We characterize the uniform convergence of pointwise monotonic nets of bounded real functions defined on arbitrary sets, without any particular structure. The resulting condition trivially holds for the classical Dini theorem. Our vector-valued Dini-type theorem characterizes the uniform convergence of pointwise monotonic nets of functions with relatively compact range in Hausdorff topological ordered vector spaces. As a consequence, for such nets of continuous functions on a compact space, we get the equivalence between the pointwise and the uniform convergence. When the codomain is locally convex, we also get the equivalence between the uniform convergence and the weak-pointwise convergence; this also merges the Dini-Weston theorem on the convergence of monotonic nets from Hausdorff locally convex ordered spaces. Most of our results are free of any structural requirements on the common domain and *put compactness in the right place*: the range of the functions.

Keywords Dini theorem · Uniform convergence · Pointwise convergence · Net of functions · Topological ordered vector space · Stone-Čech compactification

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1 Introduction

Uniform convergence (denoted here by $\stackrel{\mathrm{u}}{\longrightarrow}$) of a sequence of functions is important because several properties (such as continuity and integrability), if shared by all members of the sequence, are transferred under suitable assumptions to the limit function. The pointwise convergence ($\stackrel{\mathrm{p}}{\longrightarrow}$) is easier to test, but also much weaker than the corresponding uniform convergence. In a particular setting the two convergences may coincide, as in the classical Dini theorem:

Theorem 1 (Dini) *Let $F_{\mathbb{N}} = (f_n)_{n\in\mathbb{N}}$ be a monotonic sequence of real continuous functions on a compact topological space S. Then for any continuous map $f : S \to \mathbb{R}$, we have the equivalence*

$$f_n \stackrel{u}{\longrightarrow} f \iff f_n \stackrel{p}{\longrightarrow} f.$$

There are many generalizations of the above theorem. Various authors considered: real functions with compact supports (Światkowski [15]), sequences of continuous functions satisfying generalized Alexandrov conditions (Gal [6]), topological spaces with the weak or strong Dini property (Kundu and Raha in [8]), Dini classes of upper semicontinuous real functions on compact metric spaces (Beer [2]), functions taking values in non-uniform spaces (Kupka [9] and Toma [18]), almost periodic or almost automorphic functions (Amerio [1], Bochner [3], Helmberg [7], Meisters [11], and Žikov [20]). So far such generalizations required some structure on the common domain of the functions. Nonetheless, the definitions of both convergences (pointwise and uniform) require no structure on the common domain S, and in particular no continuity of the functions.

In this paper we characterize (Theorem 4) the uniform convergence of pointwise monotonic nets (indexed by directed preordered sets $(\Delta, \preceq)$ instead of $\mathbb{N}$) of bounded real functions defined on an *arbitrary set*, without any particular structure. The resulting condition trivially holds in the setting of the classical Dini theorem.

Our vector-valued generalization (Theorem 9) characterizes the uniform convergence of pointwise monotonic nets of functions with relatively compact range in a Hausdorff topological ordered vector space. For such nets of continuous functions on a compact space, we get the equivalence between the pointwise and the uniform convergence (Corollary 15). Furthermore, when the target space is locally convex, we get (Corollary 12) the equivalence between two convergences: the *uniform* (the codomain is equipped with its original topology) and the *weak-pointwise* (pointwise convergence, when the codomain is equipped with its weak topology). This equivalence yields both Theorem 1 and the following abstract Dini theorem (see Cristescu [4], Chapter VI, Section 1.5, Prop. 2):

Theorem 2 (Dini-Weston) *If $(x_\delta)_{\delta\in\Delta}$ is a decreasing net of positive elements from a Hausdorff locally convex ordered space X, then*

$$\lim_{\delta\in\Delta} x_\delta = 0 \iff x_\delta \longrightarrow 0 \textit{ weakly}.$$

Most of our results (excepting a few corollaries) are free of any requirements on the common domain and *put compactness in the right place*: the range of the functions.

Since potential readers may not be very familiar with various notions and results on general topological ordered vector spaces,[1] whenever possible we included footnotes with details and brief explanations. For some few other needed facts on this topic, we refer the reader to [4,10,14,19].

2 "Distillation" of Dini's theorem: the scalar case

We find it interesting to present first the construction which led us in five steps of successive restatements, generalizations, and relaxations, from Dini's classical theorem to our general result. Nonetheless, the reader may jump directly to Theorem 4 and its direct proof, after understanding the notations (1) and (2), together with Definition 3.

Our next five-step discussion starts from Theorem 1; the intermediate k-th result obtained from it after the first k steps ($k \leq 4$) will be referred to as "Lemma k". Since these four lemmas are only intermediate results, we will not state them explicitly, but the reader is encouraged to do this according to the descriptions given within the corresponding steps. After finding the right setting of our general result, we will state it (Theorem 4 below) and we will prove it directly.

Step 1 (considering nets instead of sequences). Dini's theorem still holds (with almost the same proof; see also [13]) for monotonic nets of continuous functions. Hence in Theorem 1 we can replace the sequence $F_{\mathbb{N}}$ by a monotonic net $F_\Delta = (f_\delta)_{\delta\in\Delta}$ of functions from $\mathrm{C}(S, \mathbb{R})$ (we thus get Lemma 1, for monotonic nets of continuous functions on a compact space). Here $(\Delta, \preceq)$ is a directed preordered set. We can view this net as a map[2]

$$F_\Delta : S \to \mathbb{R}^\Delta, \quad F_\Delta(s) := (f_\delta(s))_{\delta\in\Delta}. \tag{1}$$

Since all components f_δ of F_Δ are continuous, we have

$$F_\Delta \in \mathrm{C}(S, \mathbb{R})^\Delta = \mathrm{C}(S, \mathbb{R}^\Delta),$$

where $\mathbb{R}^\Delta$ is equipped with the product topology. Thus, $F_\Delta(S)$ is a compact subset of $\mathbb{R}^\Delta$.

Step 2 (monotonicity relaxation). The monotonicity condition from Lemma 1 can be weakened by using the following notion (see also Mong [12] for the case of sequences of functions):

Definition 3 (*Pointwise monotonicity*) The net F_Δ is called *pointwise monotonic*, if and only if

$$F_\Delta(s) = (f_\delta(s))_{\delta\in\Delta} \text{ is a monotonic net, for every } s \in S.$$

[1] All results of the paper hold in the particular case when the codomain of the functions is a normed lattice.

[2] The map F_Δ below and pointwise monotonicity from Definition 3 may be considered for an arbitrary set S.

Indeed, assume that F_Δ is only pointwise monotonic and $f_\delta \xrightarrow{\text{p}} f$. Hence $(|f_\delta - f|)_{\delta\in\Delta}$ is a decreasing net of functions from $\mathrm{C}(S,\mathbb{R})$. Now Lemma 1 yields the needed equivalence. We thus get the slightly more general Lemma 2 (stated in this paper as Corollary 6), for pointwise monotonic nets of continuous functions on a compact space.

Step 3 (considering $f \equiv 0$ and the vector subspace $\mathrm{c}_0(\Delta)$ of $\mathbb{R}^\Delta$). Since in Lemma 2 we may replace the pointwise monotonic net F_Δ by the translated net $(f_\delta - f)_{\delta\in\Delta}$, there is no loss of generality in restating this lemma with $f \equiv 0$. With the standard notation

$$\mathrm{c}_0(\Delta) := \left\{(r_\delta)_{\delta\in\Delta} \in \mathbb{R}^\Delta \;\middle|\; \lim_{\delta\in\Delta} r_\delta = 0\right\}, \tag{2}$$

the pointwise convergence $f_\delta \xrightarrow{\text{p}} 0$ means that $F_\Delta(s) = (f_\delta(s))_{\delta\in\Delta} \in \mathrm{c}_0(\Delta)$ for every $s \in S$, which is equivalent to the inclusion $F_\Delta(S) \subset \mathrm{c}_0(\Delta)$. We thus get from Lemma 2 the equivalent Lemma 3, for $f \equiv 0$ and with the pointwise convergence $f_\delta \xrightarrow{\text{p}} 0$ replaced by the inclusion $F_\Delta(S) \subset \mathrm{c}_0(\Delta)$.

Step 4 (compactness relaxation). So far, in Lemmas 1–3 the common domain S of the continuous functions was a compact space. Our next idea is to apply Lemma 3 to a suitable compactification. Let us consider a completely regular space S and a pointwise monotonic net $F_\Delta = (f_\delta)_{\delta\in\Delta}$ of functions from $\mathrm{C}(S,\mathbb{R})$. Then S is dense in its Stone-Čech compactification βS (see Dugundji [5], Chapter XI, Section 8).[3] For the uniform convergence, we have the obvious equivalence

$$f_\delta \xrightarrow{\text{u}} 0 \iff \lim_{\delta\in\Delta}\left(\sup_{s\in S}|f_\delta(s)|\right) = 0. \tag{3}$$

The above supremum must be finite at least starting from some $\delta_0 \in \Delta$ (for $\delta \succeq \delta_0$), since otherwise the uniform convergence is impossible. Therefore, we will assume that all functions f_δ are bounded. Consequently, $F_\Delta \in \mathrm{C}(S,\mathbb{R}^\Delta)$ extends uniquely to a map

$$\beta F_\Delta = (\beta f_\delta)_{\delta\in\Delta} \in \mathrm{C}(\beta S,\mathbb{R}^\Delta).$$

Hence βF_Δ is a pointwise monotonic net, since the original net F_Δ has this property on the dense subset S of βS. By (3) and the equality $\sup_{s\in S}|f_\delta(s)| = \max_{s\in\beta S}|\beta f_\delta(s)|$ for every $\delta \in \Delta$, we get the equivalence

$$f_\delta \xrightarrow{\text{u}} 0 \iff \beta f_\delta \xrightarrow{\text{u}} 0.$$

Since βS is compact, by applying Lemma 3 to the extended net βF_Δ, it follows that the last above uniform convergence is equivalent to the inclusion $\beta F_\Delta(\beta S) \subset \mathrm{c}_0(\Delta)$. As S is a dense subset of its compactification βS and βF_Δ is a continuous extension of F_Δ, in the product space $\mathbb{R}^\Delta$ we have

[3] Recall: for every Hausdorff compact space K, any $f \in \mathrm{C}(S,K)$ extends uniquely to $\beta f \in \mathrm{C}(\beta S,K)$.

$$\beta F_\Delta(\beta S) = \beta F_\Delta(\overline{S}) \subset \overline{\beta F_\Delta(S)} = \overline{F_\Delta(S)} \subset \beta F_\Delta(\beta S),$$

and so $\beta F_\Delta(\beta S) = \overline{F_\Delta(S)}$, where the closure is taken in $\mathbb{R}^\Delta$. Hence the uniform convergence $f_\delta \xrightarrow{\text{u}} 0$ is equivalent to the inclusion

$$\overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta) \text{ in } \mathbb{R}^\Delta. \tag{4}$$

We thus get Lemma 4, for pointwise monotonic nets of bounded real continuous functions on a completely regular space (in Lemma 3 we replace the inclusion $F_\Delta(S) \subset \mathrm{c}_0(\Delta)$ by (4)).

Step 5 (removing the topology of S). Consider an arbitrary set S (without topological structure) and a pointwise monotonic net $F_\Delta = (f_\delta)_{\delta\in\Delta} : S \to \mathbb{R}^\Delta$. We can assume that F_Δ is an injective map. Indeed, if F_Δ is not injective, we can consider the F_Δ-equivalence class $\widehat{s} := \{t \in S \mid F_\Delta(t) = F_\Delta(s)\}$ of every $s \in S$ and the quotient set $\widehat{S} = \{\widehat{s} \mid s \in S\}$. Then the map

$$\widehat{F}_\Delta = (\widehat{f}_\delta)_{\delta\in\Delta} : \widehat{S} \to \mathbb{R}^\Delta, \quad \widehat{F}_\Delta(\widehat{s}) := F_\Delta(s),$$

is well-defined and injective. The net $\widehat{F}_\Delta$ is pointwise monotonic, since F_Δ has this property. We clearly have the equivalence

$$f_\delta \xrightarrow{\text{u}} 0 \iff \widehat{f}_\delta \xrightarrow{\text{u}} 0.$$

Therefore, we next assume that F_Δ is injective. Then the F_Δ-initial topology on S, that is,

$$\tau := \left\{F_\Delta^{-1}(D) \subset S \mid D \text{ is open in } \mathbb{R}^\Delta\right\},$$

turns F_Δ into a homeomorphism between S and the completely regular space $F_\Delta(S) \subset \mathbb{R}^\Delta$. Hence (S, τ) is completely regular. By Lemma 4, we conclude that the uniform convergence $f_\delta \xrightarrow{\text{u}} 0$ is equivalent to the inclusion (4). We finally get our first general result, stated below as Theorem 4. The direct proof will show precisely where the compactness is needed and where it comes from.

Let us note that producing compactness and continuity in this way (by using bounded real functions on an arbitrary set, as in the above Steps 5 and 4) may be done in various other settings (for instance, for proving Stone-Weierstrass-type results similar to those from Timofte [17]).

Theorem 4 (Generalized Dini theorem) *If $F_\Delta = (f_\delta)_{\delta\in\Delta}$ is a pointwise monotonic net of bounded functions $f_\delta : S \to \mathbb{R}$ on an arbitrary set S (without topological structure), then*[4]

$$f_\delta \xrightarrow{u} 0 \iff \overline{F_\Delta(S)} \subset c_0(\Delta) \textit{ in } \mathbb{R}^\Delta.$$

[4] For comments on how such equivalences lead to Dini-type results in a more general setting, see Remark 8.

Proof "⇒". Assume $f_\delta \xrightarrow{\text{u}} 0$. Let us fix $r_\Delta := (r_\delta)_{\delta\in\Delta} \in \overline{F_\Delta(S)}$. In order to show that $r_\Delta \in \mathrm{c}_0(\Delta)$, consider an arbitrary $\varepsilon > 0$. We have $r_\Delta = \lim_{\lambda\in\Lambda} F_\Delta(s_\lambda)$ in $\mathbb{R}^\Delta$ for some net $(s_\lambda)_{\lambda\in\Lambda}$ from S, where Λ is another directed preordered set. This convergence in the product space $\mathbb{R}^\Delta$ means

$$r_\delta = \lim_{\lambda\in\Lambda} f_\delta(s_\lambda) \quad \text{for every } \delta \in \Delta.$$

Since $f_\delta \xrightarrow{\text{u}} 0$, there exists $\delta_\varepsilon \in \Delta$, such that $|f_\delta(s)| \le \varepsilon$ for all $s \in S$ and $\delta \succeq \delta_\varepsilon$. Now for every fixed $\delta \succeq \delta_\varepsilon$, a passage to the limit yields

$$|r_\delta| = \left|\lim_{\lambda\in\Lambda} f_\delta(s_\lambda)\right| = \lim_{\lambda\in\Lambda} |f_\delta(s_\lambda)| \le \varepsilon.$$

Hence $\lim_{\delta\in\Delta} r_\delta = 0$, that is, $r_\Delta \in \mathrm{c}_0(\Delta)$. We thus have proved the inclusion $\overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta)$.
"⇐". Assume $\overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta)$, and hence $f_\delta \xrightarrow{\text{p}} 0$. In order to show that $f_\delta \xrightarrow{\text{u}} 0$, let us fix $\varepsilon > 0$. By Tychonoff's theorem and the obvious inclusions

$$F_\Delta(S) \subset \prod_{\delta\in\Delta} f_\delta(S) \subset \prod_{\delta\in\Delta} \overline{f_\delta(S)},$$

we get the compactness of $\overline{F_\Delta(S)}$ in $\mathbb{R}^\Delta$. For every $\delta \in \Delta$, the subset $\pi_\delta^{-1}(]-\varepsilon,\varepsilon[) \subset \mathbb{R}^\Delta$ is open, where $\pi_\delta : \mathbb{R}^\Delta \to \mathbb{R}$ denotes the standard projection on the δ-component of the product space. Since

$$\overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta) \subset \bigcup_{\delta\in\Delta} \pi_\delta^{-1}\left(]-\varepsilon,\varepsilon[\right),$$

there is a finite subset $\Delta_0 \subset \Delta$, such that

$$\overline{F_\Delta(S)} \subset \bigcup_{\delta\in\Delta_0} \pi_\delta^{-1}(]-\varepsilon,\varepsilon[). \tag{5}$$

As Δ is a directed set, its finite subset Δ_0 has an upper bound $\delta_\varepsilon \in \Delta$. We claim that

$$f_\delta(S) \subset]-\varepsilon,\varepsilon[, \text{ for every } \delta \succeq \delta_\varepsilon.$$

In order to prove this, let us fix $\delta \succeq \delta_\varepsilon$ and $s \in S$. According to (5), for some $\delta_0 \in \Delta_0$ we have $F_\Delta(s) \in \pi_{\delta_0}^{-1}(]-\varepsilon,\varepsilon[)$, that is, $f_{\delta_0}(s) \in]-\varepsilon,\varepsilon[$. Since F_Δ is pointwise monotonic and $\delta \succeq \delta_\varepsilon \succeq \delta_0$, it follows that $|f_\delta(s)| \le |f_{\delta_0}(s)| < \varepsilon$. Our claim is proved. We thus conclude that $f_\delta \xrightarrow{\text{u}} 0$. □

Remark 5 Theorem 4 requires no explicit compactness, however, the boundedness of the functions (necessary for uniform convergence) yields compactness: that of all closures $\overline{f_\delta(S)} \subset \mathbb{R}$.

Dini's classical convergence theorem now follows as an immediate corollary.

Corollary 6 (Dini's theorem for nets) *Let us consider a pointwise monotonic net* $(f_\delta)_{\delta\in\Delta}$ *of real continuous functions on a compact space* S. *Then for any continuous map* $f : S \to \mathbb{R}$, *we have*

$$f_\delta \xrightarrow{u} f \iff f_\delta \xrightarrow{p} f.$$

Proof Set $G_\Delta := (f_\delta - f)_{\delta\in\Delta} \in \mathrm{C}(S, \mathbb{R}^\Delta)$. As S is compact, $G_\Delta(S)$ is a closed subset of $\mathbb{R}^\Delta$. Since all functions $f_\delta - f \in \mathrm{C}(S, \mathbb{R})$ are bounded, by Theorem 4 we get the equivalences

$$f_\delta - f \xrightarrow{\mathrm{u}} 0 \iff \overline{G_\Delta(S)} \subset \mathrm{c}_0(\Delta) \iff G_\Delta(S) \subset \mathrm{c}_0(\Delta) \iff f_\delta - f \xrightarrow{\mathrm{p}} 0.$$

□

3 Dini-type results for nets of vector-valued functions

Setting 1 *Throughout this section, X is a Hausdorff topological ordered vector space*[5] *and*

$$F_\Delta = (f_\delta)_{\delta\in\Delta} : S \to X^\Delta$$

is a net of bounded[6] *functions* $f_\delta : S \to X$ *defined on an arbitrary set* S.

Pointwise monotonicity is considered as in Definition 3 (all $(f_\delta(s))_{\delta\in\Delta}$ are monotonic nets in X). With the natural notation[7]

$$\mathrm{c}_0(\Delta, X) := \left\{(x_\delta)_{\delta\in\Delta} \in X^\Delta \,\middle|\, \lim_{\delta\in\Delta} x_\delta = 0\right\},$$

the pointwise convergence $f_\delta \xrightarrow{\mathrm{p}} 0$ is equivalent to the inclusion $F_\Delta(S) \subset \mathrm{c}_0(\Delta, X)$. Even without any kind of monotonicity of the net, the uniform convergence implies a property similar to (4):

Proposition 7 *If* $f_\delta \xrightarrow{u} 0$, *then* $\overline{F_\Delta(S)} \subset c_0(\Delta, X)$, *where the closure is taken in* X^Δ.

Proof The proof is similar to that of the corresponding part of the implication "⇒" from Theorem 4. Indeed, for fixed $x_\Delta := (x_\delta)_{\delta\in\Delta} = \lim_{\lambda\in\Lambda} F_\Delta(s_\lambda) \in \overline{F_\Delta(S)}$, instead of $\varepsilon > 0$ we fix an arbitrary closed neighborhood $W \subset X$ of the origin in X. Since $f_\delta \xrightarrow{\mathrm{u}} 0$, there exists $\delta_W \in \Delta$, such that $f_\delta(S) \subset W$ for every $\delta \succeq \delta_W$. As W is closed,

[5] Topological ordered vector space: a real vector space, endowed with a linear (partial) ordering and a linear locally full topology (with a local base consisting of full neighborhoods of the origin).

[6] A subset $A \subset X$ is bounded $\overset{\mathrm{def}}{\iff}$ every neighborhood W of $0 \in X$ absorbs A ($\varepsilon A \subset W$ for some $\varepsilon > 0$).

[7] We always consider the limits in the topological sense (not as order limits).

for every fixed $\delta \succeq \delta_W$, a passage to the limit yields $x_\delta = \lim_{\lambda\in\Lambda} f_\delta(s_\lambda) \in \overline{W} = W$. We thus conclude that $x_\Delta \in \mathrm{c}_0(\Delta, X)$. □

Remark 8 Our Dini-type results for vector-valued functions will point out various settings under which the uniform convergence $f_\delta \xrightarrow{\mathrm{u}} 0$ is equivalent to the inclusion $\overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta, X)$ (or to a very similar one). If in addition $F_\Delta(S)$ is compact (or just closed in X^Δ), this inclusion simplifies to $F_\Delta(S) \subset \mathrm{c}_0(\Delta, X)$, and is equivalent to the convergence $f_\delta \xrightarrow{\mathrm{p}} 0$. In such cases, by using translated nets of the form $(f_\delta - f)_{\delta\in\Delta}$, we may get the equivalence between the convergences $f_\delta \xrightarrow{\mathrm{p}} f$ and $f_\delta \xrightarrow{\mathrm{u}} f$. *In the particular case of a net of continuous functions on a compact space S, the inclusion* $\overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta, X)$ *is equivalent to the pointwise convergence* $f_\delta \xrightarrow{\mathrm{p}} 0$.

3.1 Dini-type results for nets of functions with relatively compact range

According to Remarks 5 and 8, it is natural to consider the vector space

$$\mathrm{K}(S, X) := \{f : S \to X \mid \overline{f(S)} \text{ is compact in } X\},$$

endowed with the uniform convergence topology[8] and the pointwise ordering induced by the cone

$$\mathrm{K}(S, X)_+ := \{f \in \mathrm{K}(S, X) \mid f(S) \subset X_+\}.$$

Here X_+ denotes the positive cone[9] of the ordered vector space X.

Our next Dini-type theorem shifts the traditional compactness requirement from the common domain to the range of the functions (for a similar shift of the compactness related to a uniform density result, see Timofte [16], Th.1, p.293).

Theorem 9 *If* F_Δ *is a decreasing net from* $K(S, X)_+$*, then*

$$f_\delta \xrightarrow{u} 0 \iff \overline{F_\Delta(S)} \subset c_0(\Delta, X) \text{ in } X^\Delta.$$

If in addition the positive cone X_+ *is closed or* X *is locally convex, the above equivalence also holds for pointwise monotonic nets* F_Δ *from* $K(S, X)$.

Proof According to Proposition 7, in all (three) cases we only need to prove the implication "⇐". Assume $\overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta, X)$, which yields in particular $f_\delta \xrightarrow{\mathrm{p}} 0$. In order to show that $f_\delta \xrightarrow{\mathrm{u}} 0$, we next consider three cases.

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8 $f_\delta \xrightarrow{\mathrm{u}} f \overset{\mathrm{def}}{\iff}$ for every neighborhood W of $0 \in X$, there is $\delta_W \in \Delta$, such that $(f_\delta - f)(S) \subset W$ for $\delta \succeq \delta_0$.

9 The positive cone $X_+ := \{x \in X \mid x \geq 0\}$ has the properties: $X_+ + X_+ \subset X_+$, $\mathbb{R}_+ \cdot X_+ \subset X_+$, $X_+ \cap (-X_+) = \{0\}$.

Case 1. Assume F_Δ is a decreasing net from $\mathrm{K}(S, X)_+$. In order to prove that $f_\delta \xrightarrow{\mathrm{u}} 0$, let us fix a full[10] neighborhood W of the origin in X. By Tychonoff's theorem, the subsets $\overline{F_\Delta(S)} \subset \prod_{\delta\in\Delta} \overline{f_\delta(S)}$ are compact in the product space X^Δ. For every $\delta \in \Delta$, the subset $\pi_\delta^{-1}(\overset{\mathrm{o}}{W}) \subset X^\Delta$ is open, where $\pi_\delta : X^\Delta \to X$ denotes the standard projection on the δ-component. It is easily seen that

$$\overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta, X) \subset \bigcup_{\delta\in\Delta} \pi_\delta^{-1}(\overset{\mathrm{o}}{W}).$$

As in the proof of the implication "$\Leftarrow$" from Theorem 4, it follows that $f_\delta \xrightarrow{\mathrm{u}} 0$. Indeed, we first get a finite subcover $\overline{F_\Delta(S)} \subset \bigcup_{\delta\in\Delta_0} \pi_\delta^{-1}(\overset{\mathrm{o}}{W})$, then we choose an upper bound $\delta_W \in \Delta$ of the finite subset $\Delta_0 \subset \Delta$, and we finally show that $f_\delta(S) \subset W$ for $\delta \succeq \delta_W$ (for fixed $s \in S$, we will use that W is a full set in this way[11]: if $f_{\delta_0}(s) \in \overset{\mathrm{o}}{W}$ and $\delta \succeq \delta_W \succeq \delta_0 \in \Delta_0$, then $f_\delta(s) \in [0, f_{\delta_0}(s)]_{\mathrm{o}} \subset W$). We thus conclude that the needed equivalence holds for decreasing nets from $\mathrm{K}(S, X)_+$.

Case 2. Assume X_+ is closed and F_Δ is a pointwise monotonic net from $\mathrm{K}(S, X)$. Let us first note that since the positive cone X_+ is closed, any decreasing (respectively, increasing) net from $\mathrm{c}_0(\Delta, X)$ is necessarily contained in X_+ (respectively, in $-X_+$). Indeed, if such a net $(x_\delta)_{\delta\in\Delta}$ is decreasing, then for every $\delta_0 \in \Delta$ we have $x_{\delta_0} = \lim_{\delta\succeq\delta_0}(x_{\delta_0} - x_\delta) \in \overline{X_+} = X_+$. Let us consider the sets

$$S_\downarrow := \{s \in S \mid F_\Delta(s) \text{ is decreasing}\}, \qquad S_\uparrow := \{s \in S \mid F_\Delta(s) \text{ is increasing}\}. \tag{6}$$

Since F_Δ is pointwise monotonic, we have $S = S_\downarrow \cup S_\uparrow$. Hence the needed uniform convergence is equivalent to that of the following two decreasing nets of functions, defined by

$$F_\Delta|_{S_\downarrow} = \left(f_\delta|_{S_\downarrow}\right)_{\delta\in\Delta}, \qquad -F_\Delta|_{S_\uparrow} = \left(-f_\delta|_{S_\uparrow}\right)_{\delta\in\Delta}. \tag{7}$$

As X_+ is closed, these are nets from $\mathrm{K}(S_\downarrow, X)_+$, and respectively $\mathrm{K}(S_\uparrow, X)_+$. Since

$$\overline{F_\Delta(S_\downarrow)} \subset \overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta, X), \qquad \overline{F_\Delta(S_\uparrow)} \subset \overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta, X),$$

the uniform convergence of both nets from (7) follows by the conclusion from the first case.

Case 3. Assume X is locally convex and F_Δ is a pointwise monotonic net from $\mathrm{K}(S, X)$. As X is Hausdorff, the closure $\overline{X_+}$ is a cone defining on X a linear ordering, which is weaker than the original. This new ordering turns X into a locally convex ordered space with a closed positive cone. The net F_Δ remains pointwise monotonic with respect to the weaker ordering. Therefore, we can assume that the original positive

10 $A \subset X$ is full $\overset{\text{def}}{\Longleftrightarrow}$ A contains all order intervals $[a, b]_{\mathrm{o}} := \{x \in X \mid a \le x \le b\}$ with endpoints $a, b \in A$.

11 This is what we get if we follow closely the proof of Theorem 4, with $\overset{\mathrm{o}}{W}$ and δ_W instead of $]-\varepsilon, \varepsilon[$ and δ_ε.

cone X_+ is closed. Now the uniform convergence $f_\delta \xrightarrow{\text{u}} 0$ follows by the conclusion from the second case. □

If X is a Hausdorff locally convex ordered space,[12] considerations similar to the above apply to its weak topology $\sigma = \sigma(X, X^*)$. The space endowed with the weak topology will be denoted by X_σ. We have the obvious inclusions

$$\mathrm{K}(S, X) \subset \mathrm{K}(S, X_\sigma), \qquad \mathrm{c}_0(\Delta, X) \subset \mathrm{c}_0(\Delta, X_\sigma). \tag{8}$$

Furthermore, let us note that with the above notation, the Dini-Weston theorem may be restated as: "*Every monotonic net from* $\mathrm{c}_0(\Delta, X_\sigma)$ *belongs to* $\mathrm{c}_0(\Delta, X)$".

Theorem 10 *If X is a Hausdorff locally convex ordered space and F_Δ is a pointwise monotonic net from $K(S, X)$, then*

$$f_\delta \xrightarrow{u} 0 \iff \overline{F_\Delta(S)} \subset c_0(\Delta, X_\sigma) \text{ in } X^\Delta$$

(where the closure $\overline{F_\Delta(S)}$ is considered in X^Δ, and not in $(X_\sigma)^\Delta$).

Proof "⇒". If $f_\delta \xrightarrow{\text{u}} 0$, then by Proposition 7 and the second inclusion from (8) we get

$$\overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta, X) \subset \mathrm{c}_0(\Delta, X_\sigma).$$

"⇐". Assume $\overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta, X_\sigma)$. As in the proof of Theorem 9 (Case 3), we can assume that X has a closed positive cone (otherwise, we replace the linear ordering of X by the weaker defined by the cone $\overline{X_+}$). For the sets $S_\downarrow$ and $S_\uparrow$ defined as in (6), we have $S = S_\downarrow \cup S_\uparrow$, and so

$$\overline{F_\Delta(S_\downarrow)} \cup \overline{F_\Delta(S_\uparrow)} = \overline{F_\Delta(S_\downarrow) \cup F_\Delta(S_\uparrow)} = \overline{F_\Delta(S_\downarrow \cup S_\uparrow)} = \overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta, X_\sigma).$$

We claim that $\overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta, X)$. Let us fix $x_\Delta := (x_\delta)_{\delta\in\Delta} \in \overline{F_\Delta(S_\downarrow)}$. Then $x_\Delta = \lim_{\lambda\in\Lambda} F_\Delta(s_\lambda)$ in X^Δ, for some net $(s_\lambda)_{\lambda\in\Lambda}$ from $S_\downarrow$. Hence x_Δ is a decreasing net in X, since every $F_\Delta(s_\lambda)$ is decreasing and the positive cone X_+ is closed. Indeed, since a passage to the limit preserves non-strict inequalities in X (because X_+ is closed), for arbitrary $\delta \preceq \delta'$ in Δ, we have

$$x_\delta = \lim_{\lambda\in\Lambda} f_\delta(s_\lambda) \geq \lim_{\lambda\in\Lambda} f_{\delta'}(s_\lambda) = x_{\delta'}.$$

As $x_\Delta \in \mathrm{c}_0(\Delta, X_\sigma)$ is monotonic, by the Dini-Weston theorem it follows that $x_\Delta \in \mathrm{c}_0(\Delta, X)$. We thus get the inclusion $\overline{F_\Delta(S_\downarrow)} \subset \mathrm{c}_0(\Delta, X)$. In the same way we deduce that $\overline{F_\Delta(S_\uparrow)} \subset \mathrm{c}_0(\Delta, X)$, and hence that $\overline{F_\Delta(S)} \subset \mathrm{c}_0(\Delta, X)$. By Theorem 9, we conclude that $f_\delta \xrightarrow{\text{u}} 0$. □

[12] That is, X has a locally convex and locally full topology.

Remark 11 On any locally convex ordered space X we may consider either its original topology, or its weak topology. For both possible choices, we may consider two convergences (of a net F_Δ as in Setting 1): the uniform and the pointwise. We thus get four convergences:

(a) the *uniform* and the *pointwise* (when X is equipped with its original topology),
(b) the *weak-uniform* and the *weak-pointwise* (when X is equipped with its weak topology); we denote these convergences by $\xrightarrow{\text{wu}}$ and respectively $\xrightarrow{\text{wp}}$.

Among these convergences, the strongest is the uniform and the weakest is the weak-pointwise. All four convergences coincide for nets as in the following corollary:

Corollary 12 *Let us consider a Hausdorff locally convex ordered space X, a compact space S, and a pointwise monotonic net $(f_\delta)_{\delta\in\Delta}$ from $C(S, X)$. Then for every map $f \in C(S, X)$, we have the equivalence*

$$f_\delta \xrightarrow{u} f \iff f_\delta \xrightarrow{wp} f.$$

Proof Set $G_\Delta := (f_\delta - f)_{\delta\in\Delta} \in \mathrm{C}(S, X^\Delta)$. As S is compact, $G_\Delta(S)$ is a closed subset of X^Δ. Since $\mathrm{C}(S, X) \subset \mathrm{K}(S, X)$, by Theorem 10 we get the equivalences

$$\begin{aligned} f_\delta - f \xrightarrow{\text{u}} 0 &\iff \overline{G_\Delta(S)} \subset \mathrm{c}_0(\Delta, X_\sigma) \iff G_\Delta(S) \subset \mathrm{c}_0(\Delta, X_\sigma) \\ &\iff f_\delta - f \xrightarrow{\text{wp}} 0. \end{aligned}$$

□

Let us note that the above result merges both classical convergence theorems of Dini ($X = \mathbb{R}$) and Dini-Weston ($S = \{s_0\}$), together with Corollary 6 ($X = \mathbb{R}$).

3.2 Dini-type results for nets of bounded functions

So far the statements of our Dini-type results for vector-valued functions involved compactness in some way. Our next two theorems are "compactness-free".

Notation 1 For every function $p : X \to \mathbb{R}$ and for the net F_Δ as in Setting 1, we may consider the net $pF_\Delta := (p \circ f_\delta)_{\delta\in\Delta}$ and the associated map

$$pF_\Delta : S \to \mathbb{R}^\Delta, \qquad pF_\Delta(s) = \big((p \circ f_\delta)(s)\big)_{\delta\in\Delta}.$$

Theorem 13 *Assume X is a Hausdorff locally convex ordered space and F_Δ is pointwise monotonic. Then for any set $\mathcal{P}$ of seminorms defining the topology of X, we have the equivalence*

$$f_\delta \xrightarrow{u} 0 \iff \overline{pF_\Delta(S)} \subset c_0(\Delta) \textit{ in } \mathbb{R}^\Delta, \textit{for every } p \in \mathcal{P}.$$

Proof "⇒". Assume $f_\delta \xrightarrow{\text{u}} 0$. For every $p \in \mathcal{P}$ we have $p \circ f_\delta \xrightarrow{\text{u}} 0$, which yields $\overline{pF_\Delta(S)} \subset \mathrm{c}_0(\Delta)$, by Proposition 7.

"⇐". As in the proof of Theorem 9 (Case 3), we can assume that X has a closed positive cone (otherwise, we replace the linear ordering of X by the weaker defined by the cone $\overline{X_+}$). Since X is a locally convex ordered space, there is a set $\mathcal{Q}$ consisting of monotonic seminorms[13] defining its topology. We have the obvious equivalence

$$f_\delta \xrightarrow{\mathrm{u}} 0 \iff q \circ f_\delta \xrightarrow{\mathrm{u}} 0, \text{ for every } q \in \mathcal{Q}. \tag{9}$$

In order to show that $f_\delta \xrightarrow{\mathrm{u}} 0$, let us fix $q \in \mathcal{Q}$. Since $pF_\Delta(S) \subset \mathrm{c}_0(\Delta)$ for every $p \in \mathcal{P}$, we have $F_\Delta(S) \subset \mathrm{c}_0(\Delta, X)$, and so $f_\delta \xrightarrow{\mathrm{p}} 0$. Hence the net $(f_\delta(s))_{\delta\in\Delta}$ converges monotonically to 0, for every $s \in S$. As q is a monotonic seminorm, $qF_\Delta = (q \circ f_\delta)_{\delta\in\Delta}$ is a decreasing net of bounded real functions. According to Theorem 4, we have the equivalence

$$q \circ f_\delta \xrightarrow{\mathrm{u}} 0 \iff \overline{qF_\Delta(S)} \subset \mathrm{c}_0(\Delta) \text{ in } \mathbb{R}^\Delta. \tag{10}$$

In order to prove the above convergence, let us fix $r_\Delta \in \overline{qF_\Delta(S)}$. We have $r_\Delta = \lim_{\lambda\in\Lambda} qF_\Delta(s_\lambda)$ in $\mathbb{R}^\Delta$, for some net $(s_\lambda)_{\lambda\in\Lambda}$ from S (where Λ is a directed preordered set). As $\mathcal{Q}$ and $\mathcal{P}$ define on X the same topology, for the seminorm $q \in \mathcal{Q}$ we have a domination

$$q \le p' := \alpha \sum_{i=1}^{k} p_i \qquad (\alpha \in \mathbb{R}_+,\ k \in \mathbb{N}^*,\ \{p_1, \ldots, p_k\} \subset \mathcal{P}).$$

For every $i \in \{1, \ldots, k\}$, the net $(p_i F_\Delta(s_\lambda))_{\lambda\in\Lambda}$ is contained in the compact $\overline{p_i F_\Delta(S)} \subset \mathbb{R}^\Delta$. By passing repeatedly (k times) to convergent subnets, we find a subnet $(s_{\lambda_\omega})_{\omega\in\Omega}$ of $(s_\lambda)_{\lambda\in\Lambda}$, such that each $(p_i F_\Delta(s_{\lambda_\omega}))_{\omega\in\Omega}$ is convergent. Set

$$r_\Delta^i := \lim_{\omega\in\Omega} p_i F_\Delta(s_{\lambda_\omega}) \in \overline{p_i F_\Delta(S)} \subset \mathrm{c}_0(\Delta) \quad (1 \le i \le k),$$

$$r'_\Delta := \lim_{\omega\in\Omega} p' F_\Delta(s_{\lambda_\omega}) = \alpha \sum_{i=1}^{k} r_\Delta^i \in \mathrm{c}_0(\Delta).$$

As $q \le p'$, in the ordered vector space $\mathbb{R}^\Delta$ (with componentwise ordering) we have $qF_\Delta(s) \le p'F_\Delta(s)$ for every $s \in S$, and so

$$0 \le r_\Delta = \lim_{\omega\in\Omega} qF_\Delta(s_{\lambda_\omega}) \le \lim_{\omega\in\Omega} p'F_\Delta(s_{\lambda_\omega}) = r'_\Delta \in \mathrm{c}_0(\Delta).$$

This forces $r_\Delta \in \mathrm{c}_0(\Delta)$. We thus have proved the inclusion $\overline{qF_\Delta(S)} \subset \mathrm{c}_0(\Delta)$, and hence the convergence $q \circ f_\delta \xrightarrow{\mathrm{u}} 0$, by (10). As $q \in \mathcal{Q}$ was arbitrarily fixed, by (9) we conclude that $f_\delta \xrightarrow{\mathrm{u}} 0$. □

A version of the above theorem for metrizable spaces is:

[13] The seminorm $q : X \to \mathbb{R}_+$ is monotonic $\overset{\text{def}}{\iff}$ $q(x) \le q(y)$ whenever $0 \le x \le y$.

Theorem 14 *Assume X is a metrizable topological ordered vector space. Let us consider a translation-invariant distance d defining the topology of X, such that the function $q := d(\cdot, 0)$ is increasing*[14] *on X_+. If F_Δ is pointwise monotonic, then*

$$f_\delta \xrightarrow{u} 0 \iff \overline{qF_\Delta(S)} \subset c_0(\Delta) \text{ in } \mathbb{R}^\Delta. \tag{11}$$

Proof Both conditions from (11) yield $f_\delta \xrightarrow{\mathrm{p}} 0$. Therefore, we assume this pointwise convergence to hold. As the distance d defines the topology of X, we have the equivalence

$$f_\delta \xrightarrow{\mathrm{u}} 0 \iff q \circ f_\delta \xrightarrow{\mathrm{u}} 0.$$

We claim that if $(x_\delta)_{\delta\in\Delta}$ is a decreasing net from X, with the property that $\lim_{\delta\in\Delta} x_\delta = 0$, then $(q(x_\delta))_{\delta\in\Delta}$ is a decreasing net. Indeed, for fixed $\delta_2 \succeq \delta_1$ in Δ and for arbitrary $\delta \succeq \delta_2$, we have $x_{\delta_1} - x_\delta \geq x_{\delta_2} - x_\delta \geq 0$. As q is continuous and increasing on X_+, we have

$$q(x_{\delta_1}) = \lim_{\delta\succeq\delta_2} q(x_{\delta_1} - x_\delta) \geq \lim_{\delta\succeq\delta_2} q(x_{\delta_2} - x_\delta) = q(x_{\delta_2}).$$

Our claim is proved. Since F_Δ is pointwise monotonic, with $f_\delta \xrightarrow{\mathrm{p}} 0$, and $q(-x) = q(x)$ for every $x \in X$ (because d is translation-invariant), it follows that $qF_\Delta = (q \circ f_\delta)_{\delta\in\Delta}$ is a decreasing net of bounded real functions. According to Theorem 4, we have

$$q \circ f_\delta \xrightarrow{\mathrm{u}} 0 \iff \overline{qF_\Delta(S)} \subset \mathrm{c}_0(\Delta).$$

We thus have proved the claimed equivalence (11). □

The following consequence is an analog of Corollary 6 in a much more general setting. This result generalizes Dini's theorem in three ways, by considering nets, pointwise monotonicity, and Hausdorff topological vector spaces (as codomain), instead of sequences, monotonicity, and respectively $\mathbb{R}$.

Corollary 15 *Assume X is a Hausdorff topological ordered vector space and S is a compact space. Let us consider a decreasing net $(f_\delta)_{\delta\in\Delta}$ from $C(S, X)$. Then for every lower bound $f \in C(S, X)$ of the net ($f \leq f_\delta$ pointwise, for every $\delta \in \Delta$), we have the equivalence*

$$f_\delta \xrightarrow{u} f \iff f_\delta \xrightarrow{p} f.$$

If the positive cone X_+ is closed or if X is locally convex or metrizable, the above equivalence also holds for pointwise monotonic nets $(f_\delta)_{\delta\in\Delta}$ from $C(S, X)$ and for arbitrary $f \in C(S, X)$.

[14] Such a distance always exists for a metrizable topological ordered vector space.

Proof In all cases we only need to prove the implication "$\Leftarrow$". Therefore, assume $f_\delta \xrightarrow{\text{p}} f$. Consider the pointwise monotonic net $G_\Delta = (g_\delta)_{\delta\in\Delta} := (f_\delta - f)_{\delta\in\Delta}$ of functions from $\mathrm{C}(S, X) \subset \mathrm{K}(S, X)$. Since S is compact and $G_\Delta \in \mathrm{C}(S, X^\Delta)$, by $g_\delta \xrightarrow{\text{p}} 0$ it follows that

$$\overline{G_\Delta(S)} = G_\Delta(S) \subset \mathrm{c}_0(\Delta, X). \tag{12}$$

In order to show that $g_\delta \xrightarrow{\text{u}} 0$, we next consider three cases.
Case 1. Assume $(f_\delta)_{\delta\in\Delta}$ is decreasing and f is a lower bound of this net. Then G_Δ is a decreasing net from $\mathrm{K}(S, X)_+$. According to Theorem 9, by (12) it follows that $g_\delta \xrightarrow{\text{u}} 0$.
Case 2. Assume the positive cone X_+ is closed or X is locally convex. Again, by (12) and Theorem 9 (the last part, for pointwise monotonic nets), we deduce that $g_\delta \xrightarrow{\text{u}} 0$.
Case 3. Assume X is metrizable. In this case, there is a distance d (and the associated $q := d(\cdot, 0)$) with the properties from Theorem 14. We have

$$g_\delta \xrightarrow{\text{p}} 0 \iff q \circ g_\delta \xrightarrow{\text{p}} 0 \iff qG_\Delta(S) \subset \mathrm{c}_0(\Delta).$$

Since S is compact and $qG_\Delta \in \mathrm{C}(S, \mathbb{R}^\Delta)$, it follows that $\overline{qG_\Delta(S)} = qG_\Delta(S) \subset \mathrm{c}_0(\Delta)$. According to Theorem 14, this yields $g_\delta \xrightarrow{\text{u}} 0$. □

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