Positivity (2015) 19:625–646
DOI 10.1007/s11117-014-0319-z

Positivity

# Gaussian lower bound for the Neumann Green function of a general parabolic operator

**Mourad Choulli · Laurent Kayser**

Received: 17 November 2014 / Accepted: 3 December 2014 / Published online: 23 December 2014


**Abstract** Based on the fact that the Neumann Green function can be constructed as a perturbation of the fundamental solution by a single-layer potential, we establish a Gaussian lower bound for the Neumann Green function for a general parabolic operator. We build our analysis on classical tools coming from the construction of a fundamental solution of a general parabolic operator by means of the so-called parametrix method. At the same time we provide a simple proof for Gaussian two-sided bounds for the fundamental solution.

**Keywords** Parabolic operator · Fundamental solution · Parametrix · Neumann Green function · Gaussian lower bound · Heat kernel

**Mathematics Subject Classification** 65M80

## Contents



M. Choulli (✉) · L. Kayser
Institut Elie Cartan de Lorraine, UMR CNRS 7502, Université de Lorraine, Boulevard des Aiguillettes BP 70239 54506 VANDOEUVRE LES NANCY Cedex, Ile du Saulcy, 57045 Metz Cedex 1, France
e-mail: mourad.choulli@univ-lorraine.fr

L. Kayser
e-mail: laurent.kayser@univ-lorraine.fr

## 1 Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ with $C^{1,1}$-smooth boundary. Let $t_0 < t_1$, we set $Q = \Omega \times (t_0, t_1)$ and we consider the second order differential operator

$$L = a_{ij}(x,t)\partial^2_{ij} + b_k(x,t)\partial_k + c(x,t) - \partial_t.$$

Here and henceforth we use the usual Einstein summation convention for repeated indices.

We make the following assumptions on the coefficients of $L$:

(i) the matrix $(a_{ij}(x,t))$ is symmetric for any $(x,t) \in \overline{Q}$,
(ii) $a_{ij} \in W^{1,\infty}(Q)$, $b_k, c \in C([t_0, t_1], C^1(\overline{\Omega}))$,
(iii) $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, $(x,t) \in \overline{Q}$, $\xi \in \mathbb{R}^n$,
(iv) $\|a_{ij}\|_{W^{1,\infty}(Q)} + \|b_k\|_{L^\infty(Q)} + \|c\|_{L^\infty(Q)} \leq A$,

where $\lambda > 0$ and $A > 0$ are two given constants.

These assumptions are surely not the best possible if one wants to construct a fundamental solution or a Green function. But they are sufficient to carry out our analysis.

Since we will use the fundamental solution in the whole space, we begin by extending the coefficients of $L$ in a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ to coefficients having the same regularity. We observe that this is possible in view of the regularity of $\Omega$. For sake of simplicity, we keep the same symbols for the extended coefficients. We may also assume that the ellipticity condition holds for the extended coefficients with the same constant $\lambda$. Pick $\psi \in C_0^\infty(\widetilde{\Omega})$ satisfying $0 \leq \psi \leq 1$ and $\psi = 1$ in a neighborhood of $\overline{\Omega}$. We set

$$\widetilde{a}_{ij} = a_{ij}\psi + \lambda\delta_{ij}(1-\psi), \quad \widetilde{b}_k = b_k\psi, \quad \widetilde{c} = c\psi$$

and

$$\widetilde{L} = \widetilde{a}_{ij}(x,t)\partial^2_{ij} + \widetilde{b}_k(x,t)\partial_k + \widetilde{c}(x,t) - \partial_t.$$

Clearly, the coefficients of $\widetilde{L}$ satisfy the same assumptions as those of $L$. So in the sequel we will use the same symbol $L$ for $L$ or its extension $\widetilde{L}$.

We recall that the function

$$\mathscr{G}(x,t) = (4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n,\ t > 0,$$

is usually called the Gaussian kernel. We set

$$\mathscr{G}_c(x,t) = c^{-1}\mathscr{G}(\sqrt{c}x, t), \quad c > 0.$$

It is important to observe that the map $c \to \mathscr{G}_c$ is non increasing.

We are interested in establishing a Gaussian lower bound for the Neumann Green function associated to the operator $L$. More specifically, denoting by $G$ the Neumann Green function for $L$, we want to prove an estimate of the form

$$\mathscr{G}_C(x-\xi, t-\tau) \le G(x,t;\xi,\tau), \quad (x,t;\xi,\tau) \in Q^2, \quad t > \tau,$$

where the constant $C$ depends only on $\Omega$, $\lambda$, $T = t_1 - t_0$ and $A$.

We succeed in proving that the above Gaussian lower bound holds true provided that $\Omega$ satisfies the chain condition. That is, there exists a constant $c > 0$ such that for any two points $x, y \in \Omega$ and for any positive integer $m$ there exists a sequence $(x_i)_{0 \le i \le m}$ of points in $\Omega$ such that $x_0 = x$, $x_m = y$ and

$$|x_{i+1} - x_i| \le \frac{c}{m}|x-y|, \quad i = 0, \ldots, m-1.$$

The sequence $(x_i)_{0 \le i \le m}$ is referred to as a chain connecting $x$ and $y$.

We see that any convex subset of $\mathbb{R}^n$ satisfies the chain condition with $c = 1$. In two dimensional case, the spherical shell $\mathcal{C} = B(0,2)\setminus\overline{B(0,1)}$ has the chain property with $c = \sqrt{2}$. This follows from the fact that any two points of $\mathcal{C}$ can be connected by a broken line consisting of two segments parallel to axes of coordinates.

We point out that a $C^{1,1}$-smooth domain does not possess necessarily the chain condition.

To our knowledge a Gaussian lower bound has never been established before for the Neumann Green function of a general parabolic operator. Moreover, even in the case of parabolic operators with time-independent coefficients, we can quote only three references: Chavel [5] when the domain is convex, Choulli et al. [10] for smooth domains and Li and Yau [22] for a compact Riemannian manifold with boundary whose Ricci curvature is bounded from above and its boundary is convex.

A Gaussian upper bound for a general parabolic operator in divergence form was proved by Daners [11]. Choi and Kim [8] obtained a Gaussian upper bound for a system of operators in divergence form under the assumption that the corresponding Neumann boundary value problem possesses a De Giorgi–Nash–Moser type estimate at the boundary. In [4], the authors established a gaussian upper bound for a Neumann Green function corresponding to a time-dependent domain.

The problem is quite different for a Dirichlet Green function since the latter vanishes on the boundary. One can prove in an obvious manner, with the help of the parabolic maximum principle, that a Dirichlet Green function is non negative and dominated pointwise by a fundamental solution and so it has a Gaussian upper bound. Aronson [2, Theorem 8, page 670] established an interior Gaussian lower bound for a Dirichlet Green function. It is worthwhile to mention that [2, Theorem 8, page 670] can be used to extend the results of [15, Section 3] to a general parabolic operator. In other words, one can obtain a proof of a continuity theorem by Nash [25] and Moser–Harnack inequality [24] for a general divergence form parabolic operator, since they rely on two-sided Gaussian bounds for the fundamental solution. Later, Cho [6], Cho et al. [7] extended this result to a global weighted Gaussian lower bound involving the distance to the boundary. A Gaussian lower bound for a Dirichlet Green function when the

Euclidean distance is changed by a geodesic distance was proved by van den Berg [30,31].

For parabolic operators with time-independent coefficients, a fundamental solution or a Green function is reduced to a heat kernel. We mention that there is a tremendous literature dealing with Gaussian bounds for heat kernels. We quote the classical books by Davies [12], Grigor'yan [17], Ouhabaz [27] Saloff-Coste [28] and Stroock [29], but of course there are many other references on the subject.

As we said in the summary, the main ingredient in our analysis relies on the classical construction of the fundamental solution by means of the so-called parametrix method. We revisit this construction in the next section and we derive from it Gaussian two-sided bounds for the fundamental solution. In Sect. 3, we prove a Gaussian lower bound for the Neumann Green function. To do so, we construct the Neumann Green function as a perturbation of the fundamental solution by a single-layer potential. The Gaussian lower bound is then derived from the smoothing effect of the single-layer potential.

## 2 The parametrix method revisited

We are concerned in this section with Gaussian two-sided bounds for the fundamental solution of $Lu = 0$. For a systematic study of fundamental solutions, we refer to the classical monographs by Friedman [16] and Ladyzhenskaja et al. [20].

In the sequel $P = \mathbb{R}^n \times (t_0, t_1)$.

We recall that a fundamental solution of $Lu = 0$ in $P$ is a function $E(x, t; \xi, \tau)$ which is $C^{2,1}$ in $P^2 \cap \{t > \tau\}$, which satisfies

$$LE(\cdot\,, \cdot; \xi, \tau) = 0 \quad \text{in } \mathbb{R}^n \times \{\tau < t \le t_1\}, \quad \text{for any } (\xi, \tau) \in \mathbb{R}^n \times [t_0, t_1[$$

and, for any $f \in C_0(\mathbb{R}^n)$,

$$\lim_{t \searrow \tau} \int_{\mathbb{R}^n} E(x, t; \xi, \tau) f(\xi) d\xi = f(x), \quad x \in \mathbb{R}^n.$$

In this definition, we can also take a larger class of functions $f$. Namely, a class of continuous functions satisfying a certain growth condition at infinity (see for instance [16, formulas (6.1) and (6.2), page 22]).

The construction of a fundamental solution by means of the so-called parametrix method was initiated by Levi [21]. Let $a = (a^{ij})$ be the inverse matrix of $(a_{ij})$, $|a|$ the determinant of $a$ and

$$Z(x, t; \xi, \tau) = [4\pi(t-\tau)]^{-n/2} \sqrt{|a(\xi, \tau)|}\, e^{-\frac{a(\xi,\tau)(x-\xi)\cdot(x-\xi)}{4(t-\tau)}}, \quad (x, t; \xi, \tau) \in P^2 \cap \{t > \tau\}.$$

This function is called the parametrix. It satisfies

$$L_0 Z(\cdot\,, \cdot\,, \xi, \tau) = 0 \quad \text{in } \mathbb{R}^n \times \{\tau < t \le t_1\} \quad \text{for any } (\xi, \tau) \in \mathbb{R}^n \times [t_0, t_1[, \tag{2.1}$$

where

$$L_0 = a_{ij}(\xi, \tau)\partial^2_{ij} - \partial_t.$$

When $(\xi, \tau)$ are fixed, $L_0$ is considered as a constant coefficients operator with respect to $(x, t)$.

In the parametrix method we seek $E$, a fundamental solution of $Lu = 0$ in $P$, of the form

$$E(x, t; \xi, \tau) = Z(x, t; \xi, \tau) + \int_\tau^t \int_{\mathbb{R}^n} Z(x, t; \eta, \sigma)\Phi(\eta, \sigma; \xi, \tau) d\eta d\sigma, \qquad (2.2)$$

where $\Phi$ is to be determined in order to satisfy $LE(\cdot\,, \cdot\,; \xi, \tau) = 0$ for any $(\xi, \tau) \in \mathbb{R}^n \times [t_0, t_1[$.

Following [16, Formulas (4.4) and (4.5), page 14], $\Phi$ is given by the series

$$\Phi = \sum_{\ell=1}^{\infty} \Phi_\ell,$$

where $\Phi_1(x, t; \xi, \tau) = LZ(x, t; \xi, \tau)$ and

$$\Phi_{\ell+1}(x, t; \xi, \tau) = \int_\tau^t \int_{\mathbb{R}^n} \Phi_1(x, t; \eta, \sigma)\Phi_\ell(\eta, \sigma; \xi, \tau) d\eta d\sigma, \quad \ell \geq 1.$$

Here, for simplicity, we write $LZ(x, t; \xi, \tau)$ instead of $[LZ(\cdot\,, \cdot\,; \xi, \tau)](x, t)$.

Let $d_i$, $1 \leq i \leq n$, given by

$$d_i = d_i(x, t; \xi, \tau) = -\frac{a^{ij}(\xi, \tau)(x_j - \xi_j)}{2(t - \tau)}, \quad (x, t; \xi, \tau) \in P^2 \cap \{t > \tau\}.$$

Then

$$\partial_i Z = d_i Z, \quad \partial^2_{ij} Z = \left[-\frac{a^{ij}(\xi, \tau)}{2(t - \tau)} + d_j d_i\right] Z.$$

Therefore, taking into account (2.1), we get

$$LZ = LZ - L_0 Z = \left\{\left(a_{ij}(x, t) - a_{ij}(\xi, \tau)\right)\left[-\frac{a^{ij}(\xi, \tau)}{2(t - \tau)} + d_j d_i\right] + b_k d_k + c\right\} Z.$$

We write $LZ = \Psi Z$, where

$$\Psi = \left(a_{ij}(x, t) - a_{ij}(\xi, \tau)\right)\left[-\frac{a^{ij}(\xi, \tau)}{2(t - \tau)} + d_j d_i\right] + b_k d_k + c.$$

Let

$$M = \max_{i,j} \|a_{ij}\|_{W^{1,\infty}(Q)}, \quad N = \max\left(\max_k \|b_k\|_{L^\infty(Q)}, \|c\|_{L^\infty(Q)}, 1\right).$$

Since

$$|d_i| \leq \frac{|x-\xi|}{2\lambda(t-\tau)},$$
$$|a_{ij}(x,t) - a_{ij}(\xi,\tau)| \leq M(|x-\xi| + t - \tau),$$

we have

$$|\Psi(x,t;\xi,\tau)| \leq N\frac{1}{\sqrt{t-\tau}}\mathscr{P}\left(\frac{|x-\xi|}{\sqrt{t-\tau}}\right). \tag{2.3}$$

Here $\mathscr{P}$ is a polynomial function of degree less than three whose coefficients depend only on $M$.

Unless otherwise stated, all the constants we use now do not depend on $N$.

In light of (2.3) we obtain

$$\begin{aligned}|LZ| &\leq CN(t-\tau)^{-(n+1)/2}P(\rho)e^{-(\lambda/4)\rho^2} \\ &= CN(t-\tau)^{-(n+1)/2}\left[P(\eta)e^{-(\lambda/8)\rho^2}\right]e^{-(\lambda/8)\rho^2},\end{aligned}$$

with

$$\rho = \frac{|x-\xi|}{\sqrt{t-\tau}}.$$

But the function $\rho \in (0,+\infty) \longrightarrow P(\rho)e^{-(\lambda/8)\rho^2}$ is bounded. Consequently,

$$|\Phi_1(x,t;\xi,\tau)| = |LZ(x,t;\xi,\tau)| \leq N\widetilde{C}(t-\tau)^{-(n+1)/2}e^{-\frac{\lambda^*|x-\xi|^2}{t-\tau}}, \tag{2.4}$$

where $\lambda^* = \lambda/8$.

The following lemma will be useful in the sequel. Its proof is given in [16, page 15].

**Lemma 2.1** *Let $c > 0$ and $-\infty < \gamma, \beta < n/2+1$. Then*

$$\begin{aligned}&\int_\tau^t \int_{\mathbb{R}^n} (t-\sigma)^{-\gamma}e^{-\frac{c|x-\eta|^2}{t-\sigma}}(\sigma-\tau)^{-\beta}e^{-\frac{c|\eta-\xi|^2}{\sigma-\tau}}d\eta d\sigma \\ &\quad = \left(\frac{4\pi}{c}\right)^{n/2} B(n/2-\gamma+1, n/2-\beta+1)(t-\tau)^{n/2+1-\gamma-\beta}e^{-\frac{c|x-\xi|^2}{t-\tau}},\end{aligned}$$

*where $B$ is the usual beta function.*

We want to show

$$|\Phi_\ell(x,t;\xi,\tau)| \le (N\widetilde{C})^\ell \widehat{C}^{\ell-1}(t-\tau)^{-(n+2-\ell)/2} \prod_{j=1}^{\ell-1} B(1/2, j/2) e^{-\frac{\lambda^*|x-\xi|^2}{t-\tau}}, \quad \ell \ge 2. \tag{2.5}$$

Here $\widetilde{C}$ is the same constant as in (2.4) and $\widehat{C} = \left(\frac{4\pi}{\lambda^*}\right)^{n/2}$.

As

$$\Phi_2(x,t;\xi,\tau) = \int_\tau^t \int_{\mathbb{R}^n} \Phi_1(x,t;\eta,\sigma)\Phi_1(\eta,\sigma;\xi,\tau)d\eta d\sigma,$$

estimate (2.4) and Lemma 2.1 with $\gamma = \beta = n/2 + 1$ show that (2.5) holds true with $\ell = 2$. The general case follows by an induction argument in $\ell$. Indeed, using

$$\Phi_{\ell+1}(x,t;\xi,\tau) = \int_\tau^t \int_{\mathbb{R}^n} \Phi_1(x,t;\eta,\sigma)\Phi_\ell(\eta,\sigma;\xi,\tau)d\eta d\sigma,$$

(2.4), (2.5) for $\ell$ and Lemma 2.1 with $\gamma = n/2+1$ and $\beta = (n+2-\ell)/2$, we obtain easily that (2.5) holds true with $\ell + 1$ in place of $\ell$.

If $\Gamma$ is the usual gamma function, we recall that

$$B(1/2, j/2) = \frac{\Gamma(1/2)\Gamma(j/2)}{\Gamma((j+1)/2)}.$$

Therefore

$$\prod_{j=1}^{\ell-1} B(1/2, j/2) = \frac{\Gamma(1/2)^\ell}{\Gamma(\ell/2)} = \frac{\sqrt{\pi}^\ell}{\Gamma(\ell/2)}. \tag{2.6}$$

Hence, (2.4)–(2.6) entail

$$|\Phi(x,t;\xi,\tau)| \le \sum_{\ell\ge1} |\Phi_\ell(x,t;\xi,\tau)| \le N\widetilde{C}(1+S)(t-\tau)^{-(n+1)/2} e^{-\frac{\lambda^*|x-\xi|^2}{t-\tau}}, \tag{2.7}$$

with

$$S = \sum_{\ell\ge1} \left[CN(t-\tau)^{1/2}\right]^\ell / \Gamma((\ell+1)/2).$$

We have $\Gamma((\ell+1)/2) = \Gamma(m+1/2) \geq \Gamma(m) = (m-1)!$ if $\ell = 2m$ and $\Gamma((\ell+1)/2) = \Gamma(m+1) = m!$ if $\ell = 2m+1$. Then

$$
\begin{aligned}
S &= \frac{1}{\Gamma(3/2)}[CN(t-\tau)^{1/2}]^2 + \sum_{m\geq 2} \frac{1}{\Gamma(m+1/2)} \left[CN(t-\tau)^{1/2}\right]^{2m} \\
&\quad + \sum_{m\geq 0} \frac{1}{\Gamma(m+1)} \left[CN(t-\tau)^{1/2}\right]^{2m+1} \\
&\leq \frac{1}{\Gamma(3/2)}[CN(t-\tau)^{1/2}]^2 + \sum_{m\geq 2} \frac{1}{(m-1)!} \left[CN(t-\tau)^{1/2}\right]^{2m} \\
&\quad + \sum_{m\geq 0} \frac{1}{m!} \left[CN(t-\tau)^{1/2}\right]^{2m+1}.
\end{aligned}
$$

Whence

$$
1 + S \leq \widetilde{C} e^{\widetilde{C} N^2 (t-\tau)}.
$$

Plugging this estimate into (2.7), we obtain

$$
|\Phi(x,t;\xi,\tau)| \leq \widetilde{C} N (t-\tau)^{-(n+1)/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + \widetilde{C} N^2 (t-\tau)}. \tag{2.8}
$$

With the help of Lemma 2.1, estimate (2.8) yields

$$
\left| \int_\tau^t \int_{\mathbb{R}^n} Z(x,t;\eta,\sigma) \Phi(\eta,\sigma;\xi,\tau) d\eta d\sigma \right| \leq \widetilde{C} N (t-\tau)^{-(n-1)/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + \widetilde{C} N^2 (t-\tau)}. \tag{2.9}
$$

Noting that this inequality can be rewritten as

$$
\begin{aligned}
&\left| \int_\tau^t \int_{\mathbb{R}^n} Z(x,t;\eta,\sigma) \Phi(\eta,\sigma;\xi,\tau) d\eta d\sigma \right| \\
&\quad \leq \widetilde{C} \left[ N (t-\tau)^{1/2} e^{-\widetilde{C} N^2 (t-\tau)} \right] (t-\tau)^{-n/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + 2\widetilde{C} N^2 (t-\tau)}
\end{aligned}
$$

and, using that $\rho \to \rho e^{-\widetilde{C}\rho^2}$ is a bounded function on $[0, +\infty)$, we obtain

$$
\left| \int_\tau^t \int_{\mathbb{R}^n} Z(x,t;\eta,\sigma) \Phi(\eta,\sigma;\xi,\tau) d\eta d\sigma \right| \leq \widehat{C} (t-\tau)^{-n/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + 2\widetilde{C} N^2 (t-\tau)}. \tag{2.10}
$$

An immediate consequence of (2.10) is

$$
|E(x,t;\xi,\tau)| \leq \widetilde{C} (t-\tau)^{-n/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + \widetilde{C} N^2 (t-\tau)}. \tag{2.11}
$$

In the rest of this section, we forsake the explicit dependence on $N$. So the constants below may depend on $\Omega$, $\lambda$, $A$, and $T$.

From (2.9) we deduce in a straightforward manner that there exists $\delta > 0$ such that

$$E(x,t;\xi,\tau) \geq \widehat{C}(t-\tau)^{-n/2}, \quad (x,t;\xi,\tau) \in P^2,\ t > \tau, \quad \widetilde{C}|x-\xi|^2 < t-\tau \leq \delta. \tag{2.12}$$

By [16, Theorem 11, page 44], $E$ is positive. Moreover, $E$ satisfies the following identity, usually called the reproducing property,

$$E(x,t;\xi,\tau) = \int_{\mathbb{R}^n} E(x,t;\eta,\sigma)E(\eta,\sigma;\xi,\tau)d\eta,\ x,\xi \in \mathbb{R}^n, \quad t_0 \leq \tau < \sigma < t \leq t_1. \tag{2.13}$$

We can then paraphrase the proof of [15, Theorem 2.7, page 334] to get a Gaussian lower bound for $E$ when $0 < t-\tau \leq \delta$. To pass from $t-\tau \leq \delta$ to $t-\tau \leq T$, we use again an argument based on the reproducing property. We detail the same argument in the proof of Theorem 3.1.

We sum up our analysis in the following theorem.

**Theorem 2.1** *The fundamental solution E satisfies the Gaussian two-sided bounds:*

$$\mathcal{G}_C(x-\xi,t-\tau) \leq E(x,t;\xi,\tau) \leq \mathcal{G}_{\widetilde{C}}(x-\xi,t-\tau), \quad (x,t;\xi,\tau) \in P^2 \cap \{t > \tau\}. \tag{2.14}$$

*Remark 2.1* Let us assume that conditions (i)–(iv) above hold in all of the whole space $\mathbb{R}^n \times \mathbb{R}$ instead of $Q$ only. Taking into account the exponential term in $N^2$ in (2.11), we prove, once again with the help of the reproducing property, the following global estimate in time:

$$e^{-\kappa N^2(t-\tau)}\mathcal{G}_C(x-\xi,t-\tau) \leq E(x,t;\xi,\tau) \leq e^{\kappa N^2(t-\tau)}\mathcal{G}_{\widetilde{C}}(x-\xi,t-\tau), \tag{2.15}$$

for some constant $\kappa > 0$, where $(x,t;\xi,\tau) \in (\mathbb{R}^n \times \mathbb{R})^2$.

We point out that (2.15) does not give the two-sided Gaussian bounds by Fabes and Stroock [15] for the divergence form operator $\partial_i(a_{ij}(x,t)\partial_j\,\cdot) - \partial_t$ with ($C^\infty$-) smooth coefficients. This is not surprising since the arguments we used for proving (2.15) are not well adapted to divergence form operator. We note however that the approach developed in [15] for establishing Gaussian two-sided bounds is more involved.

Gaussian two-sided bounds were obtained by Eidel'man and Porper [13] when the coefficients of $L$ satisfy the uniform Dini condition with respect to $x$. The main tool in [13] is a parabolic Harnack inequality. We refer also to [1,14,19,26], where the reader can find various results on bounds for the fundamental solution.

We mentioned in the Introduction that the Moser–Harnack inequality in [15] can be extended to a general divergence form parabolic operator. Let us show briefly how this Moser–Harnack inequality still holds for a general parabolic operator. First, we recall that a Dirichlet Green function was constructed in [20, formula (16.7), page 408] as a perturbation of the fundamental solution by a double-layer potential. Therefore, in light of [20, formula (16.10), page 409] and [20, estimate (16.14), page 411], we can assert that [15, Lemma 5.1] remains true for our $L$. Next, paraphrasing the proofs of [15, Lemma 5.2 and Theorem 5.4] (more detailed proofs are given in [29]), we can state the following Moser–Harnack inequality.

**Theorem 2.2** *Let $\eta, \mu, \varrho \in (0,1)$. Then there is $M > 0$, depending on $n$, $\lambda$, $A$, $\eta$, $\mu$ and $\varrho$ such that for all $(x,s) \in \mathbb{R}^n \times \mathbb{R}$, all $R > 0$ and all non negative $u \in C^{2,1}(\overline{B}(x,R) \times [s-R^2, s])$ satisfying $Lu = 0$ one has*

$$u(y,t) \le Mu(x,s) \quad \text{for all } (y,t) \in \overline{B}(x, \varrho R) \times [s-\eta R^2, s-\mu R^2].$$

## 3 Gaussian lower bound for the Neumann Green function

We recall that the derivative of $U = U(x,t)$ at $(x,t) \in \partial\Omega \times [t_0, t_1]$ in the conormal direction is given by

$$\partial_\nu U(x,t) = a_{ij}(x,t)\mathbf{n}_j(x)\partial_i U(x,t),$$

where $\mathbf{n}(x) = (\mathbf{n}_1(x), \ldots, \mathbf{n}_n(x))$ is the unit outward normal vector at $x$.

For $\tau \in [t_0, t_1[$, we set $Q_\tau = \Omega \times (\tau, t_1)$, $\Sigma_\tau = \partial\Omega \times (\tau, t_1)$ and we consider the Neumann initial-boundary value problem (abbreviated to IBVP in the sequel) for the operator $L$:

$$\begin{cases} Lu = 0 & \text{in } Q_\tau, \\ u(\cdot,\tau) = \psi & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \Sigma_\tau. \end{cases} \tag{3.1}$$

From [16, Theorem 2, page 144] and its proof, for any $\psi \in C_0^\infty(\Omega)$, the IBVP (3.1) has a unique solution $u \in C^{1,0}(\overline{Q}_\tau) \cap C^{2,1}(Q_\tau)$ given by

$$u(x,t) = \int_\tau^t \int_{\partial\Omega} E(x,t;\xi,\sigma)\varphi(\xi,\sigma)d\xi d\sigma + \int_\Omega E(x,t;\xi,\tau)\psi(\xi)d\xi. \tag{3.2}$$

Here

$$\varphi(x,t) = F_\tau(x,t) - 2\sum_{\ell \ge 1} \int_\tau^t \int_{\partial\Omega} M_\ell(x,t;\xi,\sigma)F_\tau(\xi,\sigma)d\xi d\sigma, \tag{3.3}$$

with

$$\begin{aligned} F_\tau(x,t) &= -2\int_\Omega \partial_\nu E(x,t;\xi,\tau)\psi(\xi)d\xi, \\ M_1 &= -2\partial_\nu E, \\ M_{\ell+1}(x,t;\xi,\tau) &= \int_\tau^t \int_{\partial\Omega} M_1(x,t;\eta,\sigma)M_\ell(\eta,\sigma;\xi,\tau)d\eta d\sigma. \end{aligned}$$

For $(x,t) \in \Sigma_\tau$ and $\xi \in \Omega$, let

$$\begin{aligned} \mathcal{N}(x,t;\xi,\tau) = &-2\partial_\nu E(x,t;\xi,\tau) \\ &-2\sum_{\ell \ge 1} \int_\tau^t \int_{\partial\Omega} M_\ell(x,t;\eta,\sigma)\partial_\nu E(\eta,\sigma;\xi,\tau)d\eta d\sigma. \end{aligned}$$

Assume for the moment (see the proof below) that

$$\varphi(x,t) = \int_\Omega \mathcal{N}(x,t;\xi,\tau)\psi(\xi)d\xi. \tag{3.4}$$

We set

$$G(x,t,\xi,\tau) = \int_\tau^t \int_{\partial\Omega} E(x,t;\eta,\sigma)\mathcal{N}(\eta,\sigma;\xi,\tau)d\eta d\sigma + E(x,t;\xi,\tau). \tag{3.5}$$

It follows from Fubini's theorem that

$$u(x,t) = \int_\Omega G(x,t;\xi,\tau)\psi(\xi)d\xi. \tag{3.6}$$

The function $G$ is called the Neumann Green function associated to the equation $Lu = 0$ in $Q$.

We have, for any $0 \le \psi \in C_0^\infty(\Omega)$, $u \ge 0$, according to the maximum principle (see for instance [23, Theorem 2.9, page 15] and remarks following it). Whence, $G \ge 0$.

From the uniqueness of the solution of the IBVP (3.1), we have also

$$\int_\Omega G(x,t;\xi,\tau)\psi(\xi)d\xi = \int_\Omega G(x,t;\eta,\sigma)d\eta \int_\Omega G(\eta,\sigma;\xi,\tau)\psi(\xi)d\xi$$
$$\text{for any } \psi \in C_0^\infty(\Omega),\ \tau < \sigma < t.$$

Therefore,

$$G(x,t;\xi,\tau) = \int_\Omega G(x,t;\eta,\sigma)G(\eta,\sigma;\xi,\tau)d\eta, \quad \tau < \sigma < t. \tag{3.7}$$

That is, $G$ has the reproducing property.

We note that when $c = 0$, $G$ satisfies in addition

$$\int_\Omega G(x,t;\xi,\tau)d\xi = 1.$$

The key point in the proof of our Gaussian lower bound for $G$ is the following lemma.

**Lemma 3.1** *For* $1/2 < \mu < \frac{n}{2}$, *we have*

$$|\mathcal{N}(x,t;\xi,\tau)| \le C(t-\tau)^{-\mu}|x-\xi|^{-n+2\mu}, \quad (x,t) \in \Sigma_\tau,\ \xi \in \Omega,\ x \ne \xi. \tag{3.8}$$

The lemma below appears in [16, page 137] as Lemma 1. It is needed for proving Lemma 3.1.

**Lemma 3.2** *Let* $0 < a, b < n-1$ *with* $a + b \neq n-1$. *Then*

$$\int_{\partial\Omega} |x-\eta|^{-a}|\eta-\xi|^{-b}d\eta \leq \begin{cases} \widehat{C}|x-\xi|^{n-1-(a+b)} & if\ a+b > n-1 \\ \widehat{C} & if\ a+b < n-1. \end{cases} \tag{3.9}$$

*Proof of Lemma 3.1* Since $\Omega$ is of class $C^{1,1}$, we obtain by paraphrasing the proof of [16, formula (2.12), page 137]:

$$|\partial_\nu E(x,t;\xi,\tau)| \leq C(t-\tau)^{-\mu}|x-\xi|^{-n+2\mu},$$

for any $\mu > 0$, and then

$$|M_1(x,t;\xi,\tau)| \leq C(t-\tau)^{-\mu}|x-\xi|^{-n+2\mu}. \tag{3.10}$$

We assume first that $1/2 < \mu < 1$. Since

$$|M_2(x,t;\xi,\tau)| \leq \int_\tau^t \int_{\partial\Omega} |M_1(x,t;\eta,\sigma)||M_1(\eta,\sigma;\xi,\tau)|d\eta d\sigma,$$

(3.10) leads

$$|M_2(x,t;\xi,\tau)| \leq C^2 \int_\tau^t (t-\sigma)^{-\mu}(\sigma-\tau)^{-\mu}d\sigma \int_{\partial\Omega} |x-\eta|^{-n+2\mu}|\xi-\eta|^{-n+2\mu}d\eta. \tag{3.11}$$

By Lemma 3.2,

$$\begin{aligned} &\int_{\partial\Omega} |x-\eta|^{-n+2\mu}|\xi-\eta|^{-n+2\mu}d\eta \\ &\quad\leq \begin{cases} \widehat{C}|x-\xi|^{-n+4\mu-1} & \text{if } n \geq 3 \quad \text{or} \quad n=2 \quad \text{and} \quad \frac{1}{2} < \mu < \frac{3}{4}, \\ \widehat{C} & \text{if } n=2 \quad \text{and} \quad \frac{3}{4} < \mu < 1. \end{cases} \end{aligned} \tag{3.12}$$

On the other hand

$$\begin{aligned} \int_\tau^t (t-\sigma)^{-\mu}(\sigma-\tau)^{-\mu}d\sigma &= (t-\tau)^{-\mu+(1-\mu)} \int_0^1 s^{-\mu}(1-s)^{1-\mu}ds \\ &= (t-\tau)^{-\mu+(1-\mu)}B(1-\mu,1-\mu). \end{aligned} \tag{3.13}$$

We plug (3.12) and (3.13) into (3.11), and we obtain

$$|M_2(x,t;\xi,\tau)| \leq C^2\widehat{C}(t-\tau)^{-\mu+(1-\mu)}B(1-\mu,1-\mu)|x-\xi|^{-n+2\mu+(2\mu-1)},$$
$$\text{if } n \geq 3 \text{ or } n=2 \text{ and } \frac{1}{2} < \mu < \frac{3}{4}$$

and

$$|M_2(x,t;\xi,\tau)| \leq C^2\widehat{C}(t-\tau)^{-\mu+(1-\mu)}B(1-\mu,1-\mu), \quad \text{if } n=2 \text{ and } \frac{3}{4} < \mu < 1.$$

Let $\ell(n)$ be the smallest integer $\ell$ so that $n+1<2\ell$ and fix $\frac{n+1}{2\ell(n)}<\mu<1$. Then an induction argument yields

$$|M_\ell(x,t;\xi,\tau)|\le C^\ell\widehat{C}^{\ell-1}(t-\tau)^{-\mu+(\ell-1)(1-\mu)}\frac{\Gamma(1-\mu)^\ell}{\Gamma(\ell(1-\mu))},\quad \ell\ge\ell(n).$$

By Stirling's formula

$$\Gamma(\ell(1-\mu))\sim(e^{-1}(\ell(1-\mu)-1))^{\ell(1-\mu)-1}\sqrt{2\pi(\ell(1-\mu)-1))}\qquad \text{as } \ell\to+\infty,$$

implying that the series

$$S=\sum_{\ell\ge\ell(n)}C^\ell\widehat{C}^{\ell-1}T^{-\mu+(\ell-1)(1-\mu)}\frac{\Gamma(1-\mu)^\ell}{\Gamma(\ell(1-\mu))}$$

converges.

Clearly,

$$|\mathcal{N}(x,t;\xi,\tau)|\le\sum_{\ell=1}^{\ell(n)-1}|M_\ell(x,t;\xi,\tau)|+S.$$

Therefore, it is enough to prove that

$$\widetilde{\mathcal{N}}(x,t;\xi,\tau)=\sum_{\ell=1}^{\ell(n)-1}|M_\ell(x,t;\xi,\tau)|$$

satisfies (3.8). To this end, we observe that

$$\begin{aligned}|M_2(x,t;\xi,\tau)|\le&\int_\tau^{(\tau+t)/2}\int_{\partial\Omega}|M_1(x,t;\eta,\sigma)||M_1(\eta,\sigma;\xi,\tau)|d\eta d\sigma\\&+\int_{(\tau+t)/2}^{t}\int_{\partial\Omega}|M_1(x,t;\eta,\sigma)||M_1(\eta,\sigma;\xi,\tau)|d\eta d\sigma.\end{aligned}$$

Assume that $1/2<\mu<n/2$ and pick $1/2<\alpha<\min(1,(n/2-\mu)+1/2)$. From Lemma 3.2, we have

$$\begin{aligned}&\int_\tau^{(\tau+t)/2}\int_{\partial\Omega}|M_1(x,t;\eta,\sigma)||M_1(\eta,\sigma;\xi,\tau)|d\eta d\sigma\\&\quad\le C^2\int_\tau^{(\tau+t)/2}(\sigma-\tau)^{-\alpha}(t-\sigma)^{-\mu}d\sigma\\&\qquad\int_{\partial\Omega}|x-\eta|^{-n+2\mu}|\xi-\eta|^{-n+2\alpha}d\eta\end{aligned}$$

$$
\begin{aligned}
&\le C^2\left(\frac{t-\tau}{2}\right)^{-\mu}\int_\tau^{(\tau+t)/2}(\sigma-\tau)^{-\alpha}d\sigma|x-\xi|^{-n+2\mu+2\alpha-1}\\
&\le C^2\left(\frac{t-\tau}{2}\right)^{-\mu-\alpha+1}|x-\xi|^{-n+2\mu+2\alpha-1}\\
&\le C'(t-\tau)^{-\mu}|x-\xi|^{-n+2\mu}.
\end{aligned}
$$

Similarly,

$$
\int_{(\tau+t)/2}^{t}\int_{\partial\Omega}|M_1(x,t;\eta,\sigma)||M_1(\eta,\sigma;\xi,\tau)|d\eta d\sigma \le C(t-\tau)^{-\mu}|x-\xi|^{-n+2\mu}.
$$

Thus, $M_2$ satisfies (3.8). We repeat the previous argument to deduce that also $\widetilde{\mathcal{N}}$ obeys (3.8). □

*Proof of (3.4)* Let

$$
\begin{aligned}
\mathcal{N}_k(x,t;\xi,\tau) =& -2\partial_\nu E(x,t;\xi,\tau)\\
&-2\sum_{\ell\ge1}^{k}\int_\tau^t\int_{\partial\Omega}M_\ell(x,t;\eta,\sigma)\partial_\nu E(\eta,\sigma;\xi,\tau)d\eta d\sigma,\\
\varphi_k(x,t) =& -2F_\tau(x,t)-2\sum_{\ell\ge1}^{k}\int_\tau^t\int_{\partial\Omega}M_\ell(x,t;\xi,\sigma)F_\tau(\xi,\sigma)d\xi d\sigma.
\end{aligned}
$$

In light of Lemma 3.2 and with the help of Lebesgue's dominated convergence theorem, we can assert that

$$
\int_\Omega \mathcal{N}_k(x,t;\xi,\tau)\psi(\xi)d\xi \longrightarrow \int_\Omega \mathcal{N}(x,t;\xi,\tau)\psi(\xi)d\xi \quad \text{as } k\longrightarrow +\infty.
$$

According to Funini's theorem

$$
\varphi_k(x,t)=\int_\Omega \mathcal{N}_k(x,t;\xi,\tau)\psi(\xi)d\xi.
$$

But $\varphi_k(x,t)\to\varphi(x,t)$ when $k$ tends to infinity. Then the uniqueness of the limit yields

$$
\varphi(x,t)=\int_\Omega \mathcal{N}(x,t;\xi,\tau)\psi(\xi)d\xi.
$$

□

We are now ready to prove

**Theorem 3.1** *Under the assumption that* $\Omega$ *obeys the chain condition, the Neumann Green function* $G$ *satisfies the Gaussian lower bound:*

$$\mathscr{G}_C(x-\xi, t-\tau) \leq G(x,t;\xi,\tau), \quad (x,t;\xi,\tau) \in Q^2 \cap \{t > \tau\}. \tag{3.14}$$

*Proof* Let

$$G_0(x,t;\xi,\tau) = \int_\tau^t \int_{\partial\Omega} E(x,t;\eta,\sigma)\mathcal{N}(\eta,\sigma;\xi,\tau)d\eta d\sigma.$$

From the Gaussian upper bound for $E$ we obtain in a straightforward way that, for any $\beta > 0$,

$$|E(x,t;\xi,\tau)| \leq C(t-\tau)^{-\beta}|x-\xi|^{-n+2\beta}.$$

On the other hand, by Lemma 3.1,

$$|\mathcal{N}(\eta,\sigma;\xi,\tau)| \leq C(t-\tau)^{-\mu}|x-\xi|^{-n+2\mu}.$$

where $\frac{1}{2} < \mu < \frac{n}{2}$.

We fix $0 < \epsilon < \frac{1}{2}$ and $0 < \alpha < \frac{1}{2}$. In the preceding inequalities, we take $\mu = \frac{n}{2} - \epsilon$ and $\beta = 1 + \epsilon - \alpha$. In light of the fact that $-n + 2\mu + 2\beta - 1 = 1 - 2\alpha > 0$, we get from Lemma 3.2

$$|G_0(x,t;\xi,\tau)| \leq C(t-\tau)^{-n/2+\alpha}.$$

But, we know from (2.12) that

$$E(x,t;\xi,\tau) \geq C(t-\tau)^{-n/2}, \quad (x,t;\xi,\tau) \in P^2,\ t > \tau, \quad \widehat{C}|x-\xi|^2 < t-\tau.$$

Hence,

$$\begin{aligned} G(x,t;\xi,\tau) &\geq E(x,t;\xi,\tau) - |G_0(x,t;\xi,\tau)| \\ &\geq C(t-\tau)^{-n/2}(1 - \widetilde{C}(t-\tau)^{\alpha}),\ t > \tau, \quad \widehat{C}|x-\xi|^2 < t-\tau. \end{aligned}$$

Consequently, we find $\delta > 0$ so that

$$\begin{aligned} G(x,t;\xi,\tau) \geq C(t-\tau)^{-n/2}, \quad (x,t;\xi,\tau) \in Q^2,\ 0 < t-\tau \leq \delta, \\ \widetilde{C}|x-\xi|^2 < t-\tau. \end{aligned}$$

Or equivalently

$$\begin{aligned} G(x,t;\xi,\tau) \geq C(t-\tau)^{-n/2}, \quad (x,t;\xi,\tau) \in Q^2,\ 0 < t-\tau \leq \delta, \\ |x-\xi| < \widehat{C}(t-\tau)^{1/2}. \end{aligned} \tag{3.15}$$

As $\Omega$ has the chain condition, there exists a constant $c > 0$, independent on $x$ and $\xi$, such that for any positive integer $k$ there exists a sequence $(x_i)_{0\le i\le k}$ of points in $\Omega$ so that $x_0 = x$, $x_k = \xi$ and

$$|x_{i+1} - x_i| \le \frac{c}{k}|x-\xi|, \quad 0 \le i \le k-1. \tag{3.16}$$

When $2c|x-\xi| \le \widehat{C}(t-\tau)^{1/2}$ (implying $|x-\xi| \le \widehat{C}(t-\tau)^{1/2}$), Eq. (3.14) follows immediately from (3.15). Therefore we may assume that $2c|x-\xi| > \widehat{C}(t-\tau)^{1/2}$. We choose $m \ge 2$ to be the smallest integer satisfying

$$2c\frac{|x-y|}{m^{1/2}} \le \widehat{C}(t-\tau)^{1/2}.$$

Let $(x_i)_{0\le i\le m}$ be the sequence given by (3.16) when $k = m$ and

$$r = \frac{1}{4}\widehat{C}\left(\frac{t-\tau}{m}\right)^{1/2}.$$

In light of the reproducing property and the positivity of $G$, we obtain

$$\begin{aligned} G(x,t;\xi,\tau) &= \int_\Omega \dots \int_\Omega G\left(x,t;\xi_1,\frac{(m-1)t+\tau}{m}\right) \\ &\quad \dots G\left(\xi_{m-1},\frac{t+(m-1)\tau}{m};\xi,\tau\right) d\xi_1 \dots d\xi_{m-1} \\ &\ge \int_{B(x_1,r)\cap\Omega} \dots \int_{B(x_{m-1},r)\cap\Omega} G\left(x,t;\xi_1,\frac{(m-1)t+\tau}{m}\right) \\ &\quad \dots G\left(\xi_{m-1},\frac{t+(m-1)\tau}{m};\xi,\tau\right) d\xi_1 \dots d\xi_{m-1}. \end{aligned} \tag{3.17}$$

Using that $\Omega$ is $C^{1,1}$-smooth, we obtain from the result in Appendix A: there exist two positive constants $d$ and $r_0$ such that, for any $z \in \overline{\Omega}$ and $0 < \rho \le r_0$,

$$d\rho^n \le |B(z,\rho)\cap\Omega|. \tag{3.18}$$

We mention that Choi and Kim [8] observed that this condition is necessary for domains having a De Giorgi–Nash–Moser type estimate at the boundary.

In the sequel, replacing $\widehat{C}$ by a smaller constant, we may assume that $r \le r_0$.

Let $\xi_0 = x$, $\xi_i \in B(x_i, r)$ and $\xi_m = \xi$. Then we have

$$\begin{aligned} |\xi_{i+1} - \xi_i| &\le |x_{i+1} - x_i| + 2r \le c\frac{|x-\xi|}{m} + 2r \\ &\le c\frac{|x-\xi|}{m^{1/2}} + 2r \le 4r, \quad 0 \le i \le m-1. \end{aligned}$$

Whence,

$$|\xi_{i+1}-\xi_i| \le \widehat{C}\left(\frac{t-\tau}{m}\right)^{1/2}, \quad 0 \le i \le m-1.$$

It follows from (3.15) and (3.18) that

$$\begin{aligned} G(x,t;\xi,\tau) &\ge \int_{B(x_1,r)\cap\Omega}\dots\int_{B(x_{m-1},r)\cap\Omega} C^m\left(\frac{t-\tau}{m}\right)^{-nm/2} d\xi_1\dots d\xi_{m-1} \\ &\ge (dr^n)^{m-1}C^m\left(\frac{t-\tau}{m}\right)^{-nm/2} \\ &\ge d^{m-1}\left[\frac{\widehat{C}^2}{16}\left(\frac{t-\tau}{m}\right)\right]^{n(m-1)/2} C^m\left(\frac{t-\tau}{m}\right)^{-nm/2} \\ &\ge \widetilde{C}C^m(t-\tau)^{-n/2}. \end{aligned}$$

Hence

$$G(x,t;\xi,\tau) \ge \widetilde{C}e^{-Cm}(t-\tau)^{-n/2}. \tag{3.19}$$

From the definition of $m$, we have

$$m-1 \le \left(\frac{2c}{\widehat{C}}\right)^2 \frac{|x-y|^2}{t-\tau}. \tag{3.20}$$

Finally, a combination of (3.19) and (3.20) leads to (3.14) when $t-\tau \le \delta$.

We complete the proof by showing that we can remove the assumption $t-\tau \le \delta$ in (3.15). Let then $0 < t-\tau \le T$, such that $t-\tau > \delta$, and let $m \ge 2$ be the smallest integer such that $\delta^{-1}(t-\tau) \le m$. We set

$$r = r_0 T^{-1/2} m^{-1/2}(t-\tau)^{1/2} \tag{3.21}$$

and we denote by $p$ the smallest integer satisfying

$$\frac{2cD}{rm} \le p, \quad \text{with } D = \operatorname{diam}(\Omega).$$

If we choose $k = pm$ in (3.16), we obtain

$$|x_{i+1}-x_i| \le \frac{c|x-\xi|}{pm} \le \frac{cD}{pm} \le r/2.$$

Let us denote by (3.17*) the inequality (3.17) in which we take $r/2$ in place of $r$, with $r$ given as in (3.21), and $m$ changed by $pm$.

Taking into account that

$$p < 1 + 2cDr_0^{-1}T^{1/2}\delta^{-1/2} = p^*,$$

we get, for $\xi_i \in B(x_i, r)$, $1 \le i \le m-1$.

$$\frac{pm|\xi_{i+1}-\xi_i|^2}{t-\tau} \le \frac{pmr^2}{t-\tau} < p^* r_0^2 T^{-1}. \tag{3.22}$$

As $(pm)^{-1}(t-\tau) \le \delta$, (3.14) holds true. Therefore, in light of (3.22), we obtain from (3.17*)

$$\begin{aligned} G(x,t;\xi,\tau) &\ge \left(d\left[2^{-1}r_0T^{-1/2}m^{-1/2}(t-\tau)^{1/2}\right]^n\right)^{pm-1} C^{pm}\left[(pm)^{-1}(t-\tau)\right]^{-pnm/2} \\ &\ge \left(d\left[2^{-1}r_0T^{-1/2}\right]^n\right)^{pm-1} C^{pm}m^{n/2}p^{pnm/2}(t-\tau)^{-n/2} \\ &\ge \left(d\left[2^{-1}r_0T^{-1/2}\right]^n\right)^{pm-1} (t-\tau)^{-n/2} \\ &\ge \widetilde{C}\widehat{C}^{pm}(t-\tau)^{-n/2} \\ &\ge \widetilde{C}e^{-pm|\ln\widehat{C}|}(t-\tau)^{-n/2} \\ &\ge \widetilde{C}e^{-p^*m^*|\ln\widehat{C}|}(t-\tau)^{-n/2}, \quad \text{with } m^* = \delta^{-1}T+1. \end{aligned}$$

This estimate completes the proof. □

Theorem 3.1 can be easily extended to a Robin Green function. Indeed, if we replace the Neumann boundary condition by the following Robin boundary condition:

$$\partial_\nu u + q(x,t)u = 0 \quad \text{in } \Sigma_\tau,$$

where $q \in C(\Sigma_\tau)$, then $\mathcal{N}$ has to be changed by

$$\begin{aligned} \mathcal{N}_q(x,t;\xi,\tau) =& -2[\partial_\nu E(x,t;\xi,\tau) + q(x,t)E(x,t;\xi,\tau)] \\ & -2\sum_{\ell\ge 1}\int_\tau^t\int_{\partial\Omega} M_\ell(x,t;\eta,\sigma)[\partial_\nu E(\eta,\sigma;\xi,\tau) \\ & + q(\eta,\sigma)E(\eta,\sigma;\xi,\tau)]d\eta d\sigma. \end{aligned}$$

Here

$$\begin{aligned} M_1(x,t;\xi,\tau) &= -2[\partial_\nu E(x,t;\xi,\tau) + q(x,t)E(x,t;\xi,\tau)] \\ M_{\ell+1}(x,t;\xi,\tau) &= \int_\tau^t\int_{\partial\Omega} M_1(x,t;\eta,\sigma)M_\ell(\eta,\sigma;\xi,\tau)d\eta d\sigma, \quad \ell \ge 1. \end{aligned}$$

Apart the positivity of the Green function, which can be obtained by an adaptation of [9, Proposition 3.2], one can see without any difficulty, that the rest of our analysis holds true when $\mathcal{N}$ is replaced by $\mathcal{N}_q$.

We already mentioned that, for parabolic operators with time-independent coefficients, a Neumann Green function is nothing else but a Neumann heat kernel. Let us then consider a parabolic operator of the form

$$\mathcal{L} = \partial_j(a_{ij}(x)\partial_i \cdot) + b_k(x)\partial_k + c(x) - \partial_t, \tag{3.23}$$

so that the following assumptions are satisfied:

(i′) the matrix $(a_{ij}(x))$ is symmetric for any $x \in \overline{\Omega}$,
(ii′) $a_{ij} \in W^{1,\infty}(\Omega)$, $\partial_k a_{ik}$, $b_k$, $c \in C^1(\overline{\Omega})$,
(iii′) $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, $(x,t) \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$,
(iv′) $\|a_{ij}\|_{W^{1,\infty}(\Omega)} + \|\partial_k a_{ik} + b_k\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \leq A$,

where $\lambda > 0$ and $A > 0$ are two given constants.

Here again, the assumptions on the coefficients of $\mathcal{L}$ are not necessarily the best possible.

Let $\mathfrak{a}$ be the unbounded bilinear form defined by $D(\mathfrak{a}) = H^1(\Omega)$ and

$$\mathfrak{a}(u,v) = \int_\Omega a_{ij}\partial_i u \partial_j v dx + \int_\Omega b_k \partial_k u v dx + \int_\Omega c u v dx, \quad u, v \in D(\mathfrak{a}).$$

We associate to $\mathfrak{a}$ the unbounded operator $\mathcal{A}$ given by

$$D(\mathcal{A}) = \{u \in L^2(\Omega);\ \exists v \in L^2(\Omega) : \mathfrak{a}(u,\varphi) = (v,\varphi)_2,\ \varphi \in H^1(\Omega)\}, \quad \mathcal{A}u := v.$$

Here $(\cdot,\cdot)_2$ denotes the usual scalar product of $L^2(\Omega)$.

We have

$$\int_\Omega b_k \partial_k u u \leq \frac{2}{\lambda}\int_\Omega |\nabla u|^2 + \left(\frac{8 \sup_k \|b_k\|_{L^\infty(\Omega)}}{\lambda}\right)\int_\Omega u^2, \quad u \in H^1(\Omega).$$

This and (iii′) entail that $\mathfrak{a} + \kappa$ is accretive for a sufficiently large $\kappa > 0$. Since $\mathfrak{a}$ is clearly densely defined and continuous on $H^1(\Omega)$, we derive from [27, Theorem 1.52, page 29] that $-\mathcal{A}$ is the generator of an holomorphic semigroup $e^{-t\mathcal{A}}$.

Let $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$, $\psi \in C_0^\infty(\Omega)$ and $u \in C^{1,0}(\overline{Q}) \cap C^{2,1}(Q)$ ([16, Theorem2, page 144]) be the unique solution of the IBVP

$$\begin{cases} \mathcal{L}u = 0 & \text{in } Q, \\ u(\cdot,0) = \psi & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \Sigma. \end{cases} \tag{3.24}$$

By [3, Theorem 10.9, page 341], $u \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$, $u' \in L^2(0,T;[H^1(\Omega)]')$ and it is the unique solution of

$$\begin{cases} \langle u'(t), v\rangle + \mathfrak{a}(u(t),v) = 0 \quad \text{a.e. } t \in [0,T],\ v \in H^1(\Omega), \\ u(0) = \varphi \end{cases} \tag{3.25}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $H^1(\Omega)$ and its dual space $[H^1(\Omega)]'$.

We set $\widetilde{u}(t) = e^{-t\mathcal{A}}\psi$, $t \geq 0$. By using $\widetilde{u}'(t) = \mathcal{A}u(t)$, $t \in [0, T]$, we obtain in a straightforward manner

$$\langle \widetilde{u}'(t), v\rangle + \mathfrak{a}(\widetilde{u}(t), v) = 0, \quad t \in [0, T],\ v \in H^1(\Omega),$$

Using that $u(0) = \widetilde{u}(0)$, we get from the uniqueness of the solution of problem (3.25) that $u = \widetilde{u}$. Hence,

$$e^{-t\mathcal{A}}\psi(x) = \int_\Omega G(x, t; \xi, 0)\psi(\xi)d\xi, \quad 0 < t \leq T.$$

We rewrite this equality as follows:

$$e^{-t\mathcal{A}}\psi(x) = \int_\Omega K(x, \xi, t)\psi(\xi)d\xi, \quad 0 < t \leq T.$$

The function

$$K(x, \xi, t) = G(x, t; \xi, 0)$$

is usually called the heat kernel of the semigroup $e^{-t\mathcal{A}}$.

We have as an immediate consequence of Theorem 3.1:

**Corollary 3.1** *When $\Omega$ possesses the chain condition, the Neumann heat kernel $K$ satisfies the Gaussian lower bound:*

$$\mathscr{G}_C(x - \xi, t) \leq K(x, \xi, t), \quad (x, \xi) \in \Omega^2, \quad 0 < t \leq T. \tag{3.26}$$

A Gaussian lower bound for the Neumann heat kernel was proved in [10] when $L$ is the Laplace operator. The key point is the Hölder continuity of $x \longrightarrow K(x, \xi, t)$ which relies on the fact that $\mu - \mathcal{A}$ is an isomorphism from $H^s(\Omega)$ into $H^{s-2}(\Omega)$, for large $\mu$ and $s > n/2 + 1$. We note that a quick examination of the proof in [10] shows that this result can be extended to a divergence form operator with $C^\infty$-smooth coefficients.

We end this section by showing that we can obtain a strong maximum from Theorem 3.1. Let $\psi \in C(\overline{\Omega})$, $f \in C(\overline{Q}_\tau)$, $g \in C(\overline{\Sigma}_\tau)$ and consider the IBVP

$$\begin{cases} Lu = f & \text{in } Q_\tau, \\ u(\cdot, \tau) = \psi & \text{in } \Omega, \\ \partial_\nu u = g & \text{on } \Sigma_\tau. \end{cases} \tag{3.27}$$

**Corollary 3.2** *We assume that $\psi \geq 0$, $f \geq 0$, $g \geq 0$ and at least one of the functions $\psi$, $f$ and $g$ is non identically equal to zero. If $u \in C^{0,1}(\overline{Q}_\tau) \cap C^{2,1}(Q_\tau)$ is the solution of the IBVP (3.27), then*

$$u > 0 \quad \textit{in } \Omega \times ]\tau, t_1].$$

*Proof* Follows from Theorem 3.1 since (see for instance [16, formula (3.5), page 144])

$$u(x,t)=\int_\Omega G(x,t;\xi,\tau)\psi(\xi)d\xi+\int_\tau^t\int_\Omega G(x,t;\xi,s)f(\xi,s)d\xi ds$$
$$+\int_\tau^t\int_{\partial\Omega} G(x,t;\xi,s)g(\xi,s)dS_\xi ds.$$

□

## 4 Appendix A

In this appendix we prove (3.18). Henceforth, $\Omega$ is a bounded domain of $\mathbb{R}^n$ with boundary $\Gamma$.

Following [18, Definition 2.4.1, page 50], we introduce the notation

$$\mathscr{C}(y,\xi,\epsilon)=\{z\in\mathbb{R}^n;\ (z-y)\cdot\xi\geq(\cos\epsilon)|z-y|,\ 0<|y-z|<\epsilon\},$$

where $y\in\mathbb{R}^n$, $\xi\in\mathbb{S}^{n-1}$ and $0<\epsilon$. That is, $C(y,\xi,\epsilon)$ is the cone, of dimension $\epsilon$, with vertex $y$, aperture $\epsilon$ and directed by $\xi$.

We say that $\Omega$ has the $\epsilon$-cone property if

$$\text{for any } x\in\Gamma,\ \text{there exists } \xi_x\in\mathbb{S}^{n-1} \text{ so that, for all } y\in\overline{\Omega}\cap B(x,\epsilon),\ \mathscr{C}(y,\xi_x,\epsilon)\subset\Omega. \tag{4.1}$$

Assume that $\Omega$ has the $\epsilon$-cone property, for some $0<\epsilon$. By using the compactness of $\Gamma$, we find a finite number of points of $\Gamma$, $x_1,\dots x_p$, so that $\Gamma=\bigcup_k[\Gamma\cap B(x_k,\epsilon/2)]$ and (4.1) is satisfied for each $x_i$, $i=1,\dots,p$. Let $K=\overline{\Omega}\setminus\bigcup_k B(x_k,\epsilon/2)$. Then, $0<\varrho=\text{dist}(K,\Gamma)(<\epsilon)$ and therefore, for each $x\in K$, we have $B(x,\varrho)\subset\Omega$. We deduce from this observation that, for each $x\in\overline{\Omega}$ and $0<r<\varrho$, $\Omega\cap B(x,r)$ contains a cone of dimension $r$ and aperture $\epsilon$. It is then straightforward to get the following inequality:

$$|\Omega\cap B(x,r)|\geq cr^n,\quad 0<r<\varrho,$$

for some constant $c=c(n,\varrho)$.

We complete the proof of (3.18) by using the following theorem.

**Theorem 4.1** *$\Omega$ has the $\epsilon$-cone property, for some $0<\epsilon$, if and only if its boundary $\Gamma$ is Lipschitz.*

We refer to [18, Theorem 2.4.7, page 53] for a detailed proof of this theorem.

## References

1. Aronson, D.G.: Bounds for the fundamental solution of a parabolic equation. Bull. Am. Math. Soc. **73**, 890–896 (1967)
2. Aronson, D.G.: Non-negative solutions of linear parabolic equations. Ann. Scuola Norm. Sup. Pisa **22**(3), 607–694 (1968)
3. Brézis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York (2011)

4. Burdzy, K., Chen, Z.-Q., Sylvester, J.: The heat equation and reflected Brownian motion in time-dependent domains. Ann. Probab. **32**, 775–804 (2004)
5. Chavel, I.: Heat diffusion in insulated convex domains. J. Lond. Math. Soc. **34**(3), 473–478 (1986)
6. Cho, S.: Two-sided global estimates of Green's function of parabolic equations. Potential Anal. **25**, 387–398 (2006)
7. Cho, S., Kim, S., Park, H.: Two-sided estimates on Dirichlet heat kernels for time-dependent parabolic operators with singular drifts. J. Differ. Equ. **252**, 1101–1145 (2012)
8. Choi, J., Kim, S.: Green's function for second order parabolic systems with Neumann boundary condition. J. Differ. Equ. **252**, 2834–2860 (2013)
9. Choulli, M.: On the determination of an unknown boundary function in a parabolic equation. Inverse Probl. **15**, 659–667 (1999)
10. Choulli, M., Ouhabaz, E.M., Yamamoto, M.: Stable determination of a semilinear term in a parabolic equation. Commun. Pure Appl. Anal. **5**(3), 447–462 (2006)
11. Daners, D.: Heat kernel estimates for operators with boundary conditions. Math. Nachr. **217**, 13–41 (2000)
12. Davies, E.B.: Heat kernels and spectral theory. In: Cambridge Tracts in Math. vol. 92. Cambridge University Press, London (1989)
13. Eidel'man, S.D., Porper, F.O.: Two-sided estimates for fundamental solutions of second-order parabolic equations, and some applications. Uspekhi. Mat. Nauk. **39**(3), 107–156 (1984); Russ. Math. Surv. **39**(3), 119–178 (1984)
14. Fabes, E.: Gaussian upper bounds on fundamental solutions of parabolic equations; the method of Nash, Dirichlet forms (Varenna, 1992). Lecture Notes in Math., vol. 1563, pp. 1–20. Springer, Berlin (1993)
15. Fabes, E., Stroock, D.W.: A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash. Arch. Ration. Mech. Anal. **96**, 327–338 (1986)
16. Friedman, A.: Partial Differential Equations of Parabolic Type. Prentice-Hall, Englewood Cliffs (1964)
17. Grigor'yan, A.: Heat kernel and analysis on manifolds. In: AMS/IP Studies in Advanced Mathematics, vol. 47. American Mathematical Society, Providence; International Press, Boston (2009)
18. Henrot, A., Pierre, M.: Variation et optimisation de formes, Mathématiques et Applications, vol. 48. Springer, Berlin (2005)
19. Itô, S.: Diffusion equations. In: Transaction of Mathematical Monographs, vol. 114. American Mathematical Society, Providence (1991)
20. Ladyzhenskaja, O.A., Solonnikov, V.A., Ural'tzeva, N.N.: Linear and quasilinear equations of parabolic type, Nauka, Moscow (1967) in Russian; English translation: American Math. Soc., Providence (1968)
21. Levi, E.E.: Sulle equazioni lineari totalmente alle derivate parziali. Rend. Circ. Mat. Palermo **24**, 275–317 (1907)
22. Li, P., Yau, S.T.: On the parabolic kernel of the Schrödinger operator. Acta Math. **156**, 153–201 (1986)
23. Lieberman, G.: Second Order Parabolic Differential Equations. World Scientifique Publishing, Singapore (1996)
24. Moser, J.: Harnack inequality for parabolic differential equations. Commun. Pure Appl. Math. **17**, 101–134 (1964); Correction to "Harnack inequality for parabolic differential equations". Commun. Pure Appl. Math. **20**, 231–236 (1967)
25. Nash, J.: Continuity of solutions of parabolic and elliptic equations. Am. J. Math. **80**, 931–954 (1958)
26. Norris, J.R., Stroock, D.W.: Estimates on the fundamental solution to heat flows with uniformly elliptic coefficients. Proc. Lond. Math. Soc. **62**(3), 373–402 (1991)
27. Ouhabaz, E.M.: Analysis of heat equations on domains. In: London Math. Soc. Monographs, vol. 31. Princeton University Press, Princeton (2004)
28. Saloff-Coste, L.: Aspects of Sobolev-type inequalities. In: London Mathematical Society. Lecture Note Series, vol. 289. Cambridge University Press, Cambridge (2002)
29. Stroock, D.W.: Partial differential equations for probabilists. In: Cambridge Studies in Advanced Mathematics, vol. 112. Cambridge University Press, Cambridge (2008)
30. van den Berg, M.: Gaussian bounds for the Dirichlet heat kernel. J. Funct. Anal. **88**, 267–278 (1990)
31. van den Berg, M.: A Gaussian lower bound for the Dirichlet heat kernel. Bull. Lond. Math. Soc. **24**, 475–477 (1992)