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Commutativity of some Archimedean ordered algebras

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Abstract In this paper, using derivation theory, we present some results concerning the automatic order boundedness of band preserving operators on Archimedean semi-prime f-algebras. Finally, inspired by the proof of Bernau and Huijsmans (Math Proc Camb Philos Soc 107:287–308, 1990), we give necessary and sufficient conditions for Archimedean lattice-ordered algebras to be commutative.

Keywords Almost f-algebra · f-Algebra · Band preserving · Inner derivation orthogonally null

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1 Introduction

Wickstead [19] raised the problem of automatic order boundedness of all band preserving linear operators. The answer depends on the vector lattice in which the operators in question acts. There are many results that deals with this subject, see [1,4,10–13]. Abramovich et al. [1] were the first to announce an example of an order unbounded band preserving operator. Later, Bernau [4], McPolin and Wickstead [12] and De Pagter [13] proved, by using algebraic and technical tools, that if T is a band pre-

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serving linear operator on an Archimedean vector lattice A and if for every positive sequence (x_n) in A which converges to zero relatively uniformly, $\inf_n\{|T(x_n)|\} = 0$, then T is order bounded. Recently, Toumi and Toumi [18] focus their attention on the case of Dedekind σ-complete vector lattices. They proved that if A is a Dedekind σ-complete vector lattice and if $T : A \to A$ is a band preserving operator, there are some special topological circumstances that may conspire to force T to be order bounded. In fact, they proved that if the universal completion A^u of A is equipped with a Hausdorff f-compatible topology τ, then any continuous band preserving operator $T : (A, (r.u)top) \to (A^u, \tau)$ is automatically order bounded. The spirit of Sect. 3 is to give some results concerning the Wickstead Problem for the case of Archimedean semiprime f-algebras by focusing on the commutativity of the ordered algebra of all band preserving operators. More precisely, we prove that if A an Archimedean semiprime f-algebra, then the collection $B(A)$ of all band preserving operators on A are automatically order bounded if and only any derivation on $B(A)$ is null.

Vector lattices and different classes of algebras often involved in the functional analysis and their intrinsic study is related to operator theory. Many treatises have been devoted to the subject in question in the previous years. Birkhoff and Pierce [7], introduced the notion of f-algebra. Do not also forget to mention the work of Birkhoff [6] which was the precursor of the theory of almost f-algebra. Kudláček [16] introduced the notion of d-algebra. Other studies have been interested to the study of these lattice-ordered algebras. For more information about this field, we refer the reader to [5,8,9]. As Bernau and Huijsmans [5] showed that any Archimedean almost f-algebra is commutative, it is just natural to ask the following question: if A is an Archimedean lattice-ordered algebra , are there necessary and sufficient conditions for A to be commutative? Consequently and inspired by the method of Bernau and Huijsmans [5], Sect. 4 is devoted to give a positive answer to this question. In particular we prove that some behavior of inner derivations on A is proper necessary and sufficient condition. More precisely, we show that any Archimedean lattice-ordered algebra in which all inner derivations are orthogonally null, is commutative. In particular, any Archimedean almost f-algebra is commutative.

2 Preliminaries

In the next lines, we recall the definitions and some basic facts about lattice-ordered algebras. In a vector lattice A, two elements x and y are said to be disjoint (in symbols $x \perp y$) whenever $|x| \wedge |y| = 0$ holds. If D is a non-empty subset of the vector lattice A, then the disjoint complement D^d ($D^\perp$) is defined by

$$D^d = \{a \in A : a \perp b \quad \text{for all } b \in D\}.$$

We write D^{dd} for $\left(D^d\right)^d$. A lattice subspace B of A is a vector subspace of A such that the supremum and infimum of the set $\{x, y\}$ for all $x, y \in B$ is in B.

A (real) algebra A which is simultaneously a vector lattice such that the partial ordering and the multiplication on A are compatible, that is $a, b \in A^+$ implies $ab \in A$ is called a *lattice-ordered algebra* (briefly ℓ*-algebra*). In an ℓ-algebra A we denote the collection of all nilpotent elements of A by $N(A)$. An ℓ-algebra A is said to be semiprime if $N(A) = \{0\}$. An ℓ-algebra A is called an *f-algebra* if A verifies the property that $a \wedge b = 0$ and $c \geq 0$ imply $ac \wedge b = ca \wedge b = 0$. An ℓ-algebra A is called an *almost f-algebra* whenever it follows from $a \wedge b = 0$ that $ab = ba = 0$. An ℓ-algebra A is called an *d-algebra* if A verifies the property that $a \wedge b = 0$ and $c \geq 0$ imply $ac \wedge bc = ca \wedge cb = 0$.

In the following lines, we recall some definitions on derivations. A*derivation* on an algebra A (or A-*derivation*) is a linear mapping D from A into A such that

$$D(ab) = D(a)b + aD(b) \quad \text{for all } a, b \in A.$$

We denote the collection of all derivations on A by $Der(A)$. Next we provide an example of derivations. Let A be an algebra and $a \in A$. The mapping D_a from A into A defined by

$$D_a(b) = [a, b] = ab - ba \quad b \in A$$

is a derivation which called an *inner derivation*. Let A be a ℓ-algebra, an inner derivation $D_a : A \longrightarrow A$ is called *orthogonally null* if it satisfies the following property:

$$|a| \wedge |b| = 0 \Rightarrow D_a(b) = 0.$$

Let A be a vector lattice and let $0 \leq a \in A$. An element $0 \leq e \in A$ is called a component of a if $e \wedge (a - e) = 0$.

Definition 1 ([18], *Definition 1*) A vector lattice A is called a Freudenthal vector lattice if A satisfies the following property: if $0 \leq x \leq e$ holds in A, then there exist positive real numbers $\alpha_1, \ldots, \alpha_n$ and components $e_1, \ldots, e_n$ of e satisfying $x = \sum_{i=1}^{n} \alpha_i e_i$.

Let A and B be vector lattices. A bilinear map Ψ from $A \times A$ is said to be orthosymmetric if for all $(a, b) \in A \times A$ such that $a \wedge b = 0$ implies $\Psi(a, b) = 0$, see [9].

We end this section with some definitions about orthomorphisms. Let A be a vector lattice. A linear operator $T : A \to A$ is called band preserving if $T(x) \perp y$ whenever $x \perp y$ in A. A linear mapping $T \in L(A, B)$ is called order bounded if T maps order bounded subsets of A onto order bounded subsets of B. An order bounded band preserving operator on A is called *an orthomorphism*. For a vector lattice A we denote the collection of all orthomorphisms by $Orth(A)$ and the collection of all band preserving operators by $B(A)$.

3 The Wickstead problem

Let A be an Archimedean semiprime f-algebra and let $B(A)$ be the collection of all band preserving operators on A furnished with pointwise addition and ordering.

Lemma 1 *Under composition, $B(A)$ is an Archimedean ordered algebra.*

If A is a universally complete semiprime f-algebra which is not locally one-dimensional, then $B(A)$ cannot be a lattice-ordered algebra, see [19]. Consequently a natural question raised; when $B(A)$ becomes an ℓ-algebra? In order to hit this mark, we need the following:

Theorem 1 *Let A be an Archimedean semiprime f-algebra. Then the following properties are equivalent:*

1. $Der(B(A)) = \{0\}$.
2. $B(A))$ *is commutative.*
3. *Any band preserving operator on A is order bounded.*

Proof *(1)* $\Rightarrow$ (2) This path is trivial.
(2) $\Rightarrow$ *(3)* Since $B(A)$ is commutative. It follows that

$$y\pi(x) = x\pi(y)$$

for all $x, y \in A$ and for all $\pi \in B(A)$.
Let $\pi \in B(A)$ and $|x| \leq a$ in A. Let $y \in A$, then

$$\begin{aligned} |y|\,|\pi(x)| &= |y\pi(x)| \\ &= |x\pi(y)| \\ &= |x|\,|\pi(y)| \\ &\leq a\,|\pi(y)|. \end{aligned}$$

But

$$\begin{aligned} a\,|\pi(y)| &= |a\pi(y)| \\ &= |y\pi(a)| \\ &= |y|\,|\pi(a)|. \end{aligned}$$

Consequently

$$|y|\,|\pi(x)| \leq |y|\,|\pi(a)|$$

for all $y \in A$. In particular, if $y = \pi(x)$, we have

$$(\pi(x))^2 \leq |\pi(x)|\,|\pi(a)|.$$

According to ([14], Lemma 12.3), we deduce that

$$|\pi(x)| \leq |\pi(a)|.$$

Hence π is order bounded.

(3) $\Rightarrow$ *(1)* Let $D : Orth(A) \to Orth(A)$ be a derivation and let $a \in A$. Let $\bar{D}_a : A \to A$ defined by $\bar{D}_a(x) = D(\pi_a)(x) + D(\pi_x)(a)$ where $\pi_x : A \to A$ is defined by $\pi_x(a) = ax$ for all $a \in A$. Let $x, y \in A^+$ such that $x \wedge y = 0$. Since A is an f-algebra, it follows that $xy = 0$. Hence

$$
\begin{aligned}
0 &= \left[D(\pi_{axy}) \circ \pi_y\right](z) \\
&= \left[\left(D(\pi_{ax}) \circ \pi_y + \pi_{ax} \circ D(\pi_y)\right) \circ \pi_y\right](z) \\
&= \left[D(\pi_{ax}) \circ \pi_{y^2} + \pi_{axy} \circ D(\pi_y)\right](z) \\
&= \left[D(\pi_{ax}) \circ \pi_{y^2}\right](z) \\
&= D(\pi_{ax})\left(y^2 z\right) \\
&= y^2 D(\pi_{ax})(z)
\end{aligned}
$$

for all $z \in A$. Consequently,

$$
yD(\pi_{ax})(z) \in N(A)
$$

for all $z \in A$. Since A is semiprime, we deduce that

$$
yD(\pi_{ax})(z) = 0
$$

for all $z \in A$. But

$$
\begin{aligned}
D(\pi_{ax})(z) &= [\pi_a \circ D(\pi_x) + D(\pi_a) \circ \pi_x](z) \\
&= aD(\pi_x)(z) + D(\pi_a)(xz) \\
&= zD(\pi_x)(a) + zD(\pi_a)(x) \\
&= z\bar{D}_a(x).
\end{aligned}
$$

Then

$$
yz\bar{D}_a(x) = 0.
$$

for all $z \in A$. Hence

$$
y\bar{D}_a(x) \in N(A).
$$

Since A is semiprime, we deduce that

$$
y\bar{D}_a(x) = 0.
$$

Hence

$$
\left|\bar{D}_a(x)\right| \wedge y = 0.
$$

Therefore $\bar{D}_a$ is band preserving operator. Then $\bar{D}_a$ is an orthomorphism on A. It follows that

$$
\begin{aligned}
\bar{D}_a(bc) &= b\bar{D}_a(c) \\
&= c\bar{D}_a(b)
\end{aligned}
$$

for all $b, c \in A$. Then, it follows that

$$a\,[D(\pi_c)\,(b) - D(\pi_b)\,(c)] = 0$$

for all $a, b, c \in A$. Hence

$$D(\pi_c)\,(b) - D(\pi_b)\,(c) \in N(A)$$

for all $b, c \in A$. Since A is semiprime, we deduce that

$$D(\pi_c)\,(b) - D(\pi_b)\,(c) = 0$$

for all $b, c \in A$. Therefore

$$\begin{aligned} D(\pi_{bc})\,(x) &= x\bar{D}_b\,(c) \\ &= 2xD(\pi_b)\,(c) \end{aligned}$$

for all $b, c \in A$. In particular

$$\begin{aligned} D(\pi_{bcd})\,(x) &= 2xD(\pi_{bc})\,(d) \\ &= 4xdD(\pi_b)\,(c) \\ &= 2xD(\pi_b)\,(cd) \\ &= 2xdD(\pi_b)\,(c) \end{aligned}$$

for all $b, c, d \in A$. Then

$$xdD(\pi_b)\,(c) = 0$$

for all $x, b, c, d \in A$. Consequently

$$D(\pi_b)\,(c) \in N(A)$$

for all $b, c \in A$. Since A is semiprime, we deduce that

$$D(\pi_b)\,(c) = 0$$

for all $b, c \in A$. Hence

$$D(\pi_b) = 0$$

for all $b \in A$.

Now taking $\pi \in Orth(A)$ and let $x, y \in A$, then

$$\begin{aligned} D(\pi_y \circ \pi)(x) &= D(\pi_{\pi(y)})(x) \\ &= 0 \\ &= \pi_y \circ D(\pi)(x) + D(\pi_y) \circ \pi(x) \\ &= \pi_y \circ D(\pi)(x). \end{aligned}$$

Hence $yD(\pi)(x) = 0$ for all $y \in A$. Consequently $D(\pi)(x) \in N(A)$ for all $x \in A$. Since A is semiprime, we deduce that

$$D(\pi)(x) = 0$$

for all $x \in A$. Then

$$D(\pi) = 0$$

and the proof is complete. □

Corollary 1 *Let A be an Archimedean f-algebra with e as a unit element. Then the following properties are equivalent:*

1. $Der(B(A)) = \{0\}$;
2. *Any band preserving operator on A is order bounded;*
3. $B(A)$ *is commutative;*
4. *The mapping $\varphi : B(A) \mapsto A$, defined by $\varphi(\pi) = \pi(e)$ for all $\pi \in B(A)$, is injective.*

Moreover, if A is universally complete then these properties are equivalent to:

5. *A is locally one dimensional.*

Proof The equivalences (*1*) ⇔ (*2*) ⇔ (*3*) ⇔ (*4*) are due to the previous theorem. Since, for the class of universally complete vector lattices, Abramovich et al. [2] and McPolin and Wickstead [12] showed that all band preserving operators on vector lattice A are automatically bounded if and only if A is locally one-dimensional, we deduce that (*2*) ⇔ (*5*) and are done. □

In the case of Freudenthal vector lattice the situation improves considerably.

Corollary 2 *Let A be a Freudenthal vector lattice. Then any band preserving operator on A is order bounded.*

Proof First of all, we note that the universally completion A^u of the vector lattice A can be seen as a unital f-algebra, see for example ([18], Lemma 1). Then its multiplication will be denoted by juxtaposition.
Let $\pi \in B(A)$ and let $\Psi : A \times A \to A^u$ defined by $\Psi(x, y) = x\pi(y)$. It is not hard to prove that Ψ is orthosymmetric and hence it is symmetric ([18], Proposition 1).

It follows that

$$\Psi(x, y) = \Psi(y, x)$$

for all $x, y \in A$. This implies that

$$x\pi(y) = y\pi(x)$$

for all $x, y \in A$ and for all $\pi \in B(A)$.
If $|x| \leq a$ in A, then

$$\begin{aligned} |y|\,|\pi(x)| &= |y\pi(x)| \\ &= |x\pi(y)| \\ &= |x|\,|\pi(y)| \\ &\leq a\,|\pi(y)|. \end{aligned}$$

But

$$\begin{aligned} a\,|\pi(y)| &= |a\pi(y)| \\ &= |y\pi(a)| \\ &= |y|\,|\pi(a)|. \end{aligned}$$

Consequently

$$|y|\,|\pi(x)| \leq |y|\,|\pi(a)|$$

for all $y \in A$. In particular, if $y = \pi(x)$, we have

$$(\pi(x))^2 \leq |\pi(x)|\,|\pi(a)|.$$

According to ([14], Lemma 12.3), we deduce that

$$|\pi(x)| \leq |\pi(a)|.$$

Hence π is order bounded. □

4 Commutativity of Archimedean lattice-ordered algebras

Birkhoff and Pierce [7] showed, that if A an f-algebra and $a, b \in A^+$, then for all $n \in \mathbb{N}^*$

$$n\,|ab - ba| \leq a^2 + b^2,$$

and

$$\begin{aligned} n\,|(ab)\,c - a\,(bc)| \leq\ & ab\,(a + b + ab) + a(a + a^2 + ba) \\ & + cb\,(c + b + cb) + c(c + c^2 + bc) \end{aligned}$$

from which the commutativity and the associativity in the Archimedean case follow.

Scheffold [15] proved that any normed almost f-algebra is commutative. Basly and Triki [3] showed that the norm condition was superfluous. Both the proof of Scheffold and the proof of Basly and Triki make use of the axiom of choice. Bernau and Huijsmans [5] gave a constructive proof. Buskes and Van Rooij [9] gave another proof. Similarly Toumi [17] showed the same result by using orthosymmetric bilinear maps.

Contrary to Archimedean almost f-algebras, Archimedean d-algebras need not to be commutative. Bernau and Huijsmans [5] found many links between different classes of lattice-ordered algebras. Notably, they proved the following result:

Theorem 2 ([5], Theorem 4.3) *Any Archimedean d-algebra in which positive disjoint elements commute is an almost f-algebra. In particular, any commutative Archimedean d-algebra is an almost f-algebra.*

Motivated by the previous theorem, we notice that any Archimedean d-algebra in which positive disjoint elements commute is commutative. Hence, a natural question is raised: What we can say about the commutativity of an Archimedean ℓ-algebra in which disjoint elements commute?

This section, by adapting the proof of Bernau and Huijsmans [5], is devoted to give a positive answer to this question by making use of derivations. But in order to make this paper self-contained, we reproduce full proofs.

In order to hit our mark, we make use of the following results.

Proposition 1 *Let A be an Archimedean lattice-ordered algebra in which all inner derivations are orthogonally null. Then we have*

$$\begin{aligned} D_a(b) &= D_{a-a\wedge b}(a\wedge b) + D_{a\wedge b}(b - a\wedge b) \\ &= -D_{a\wedge b}(a - a\wedge b) + D_{a\wedge b}(b - a\wedge b) \end{aligned}$$

for all $a, b \in A$.

Proof Let $a, b \in A$. It follows from

$$(a - a\wedge b)\wedge(b - a\wedge b) = 0$$

that

$$\begin{aligned} D_{(a-a\wedge b)}(b - a\wedge b) &= (a - a\wedge b)(b - a\wedge b) - (b - a\wedge b)(a - a\wedge b) \\ &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} ab - ba &= a(a\wedge b) - (a\wedge b)a + (a\wedge b)b - b(a\wedge b) \\ &= D_a(a\wedge b) + D_{a\wedge b}(b) \\ &= -D_{a\wedge b}(a) + D_{a\wedge b}(b). \end{aligned}$$

But $D_{a\wedge b}(a) = D_{a\wedge b}(a - a\wedge b)$ and $D_{a\wedge b}(b) = D_{a\wedge b}(b - a\wedge b)$, the result follows. □

Lemma 2 *Let A be an Archimedean lattice-ordered algebra in which all inner derivations are orthogonally null. Then*

$$\left|D_{a\wedge b}\left(a-a\wedge b\right)-D_{a\wedge b\wedge\theta^{-1}(a-a\wedge b)}\left(a-a\wedge\left(1+\theta\right)b\right)\right|\leq\theta b^{2}.$$

for all $a, b \in A^{+}$ *and for all* $\theta > 0$.

Proof Let $a, b \in A$ and let $\theta > 0$. It follows

$$a\wedge b\leq a\wedge\left(1+\theta\right)b\leq a\wedge b+a\wedge\theta b$$

Then

$$0\leq a\wedge\left(1+\theta\right)b-a\wedge b\leq\theta b.$$

Multiplying on the right by $a \wedge b$ we obtain:

$$\begin{aligned}
&0\leq\left(a\wedge\left(1+\theta\right)b\right)\left(a\wedge b\right)-\left(a\wedge b\right)^{2}\leq\theta b\left(a\wedge b\right)\leq\theta b^{2}\\
&0\leq\left(a\wedge\left(1+\theta\right)b\right)\left(a\wedge b\right)-\left(a\wedge b\right)^{2}+a\left(a\wedge b\right)-a\left(a\wedge b\right)\leq\theta b^{2}\\
&0\leq\left(a-a\wedge b\right)\left(a\wedge b\right)-\left(a-a\wedge\left(1+\theta\right)b\right)\left(a\wedge b\right)\leq\theta b^{2}.
\end{aligned}$$

Similarly we have:

$$0\leq\left(a\wedge b\right)\left(a-a\wedge b\right)-\left(a\wedge b\right)\left(a-a\wedge\left(1+\theta\right)b\right)\leq\theta b^{2}.$$

Hence

$$\begin{aligned}
-\theta b^{2}&\leq-\left\{\left(a-a\wedge b\right)\left(a\wedge b\right)-\left(a-a\wedge\left(1+\theta\right)b\right)\left(a\wedge b\right)\right\}\\
&\leq\left\{\left(a\wedge b\right)\left(a-a\wedge b\right)-\left(a\wedge b\right)\left(a-a\wedge\left(1+\theta\right)b\right)\right\}\\
&\quad-\left\{\left(a-a\wedge b\right)\left(a\wedge b\right)-\left(a-a\wedge\left(1+\theta\right)b\right)\left(a\wedge b\right)\right\}\\
&\leq\left\{\left(a\wedge b\right)\left(a-a\wedge b\right)-\left(a\wedge b\right)\left(a-a\wedge\left(1+\theta\right)b\right)\right\}\\
&\leq\theta b^{2}.
\end{aligned}$$

Then

$$\left|D_{a\wedge b}\left(a-a\wedge b\right)-D_{a\wedge b}\left(a-a\wedge\left(1+\theta\right)b\right)\right|\leq\theta b^{2}\tag{4.1}$$

In addition

$$\begin{aligned}
0&\leq a\wedge b-a\wedge b\wedge\theta^{-1}\left(a-a\wedge b\right)\\
&=\theta^{-1}\left[\theta\left(a\wedge b\right)-\left(\theta\left(a\wedge b\right)\right)\wedge\left(a-a\wedge b\right)\right]\\
&=\theta^{-1}\left[\theta\left(a\wedge b\right)-\left(\theta\left(a\wedge b\right)+a\wedge b-a\wedge b\right)\wedge\left(a-a\wedge b\right)\right]\\
&=\theta^{-1}\left[\theta\left(a\wedge b\right)-\left(\left(\theta+1\right)\left(a\wedge b\right)-a\wedge b\right)\wedge\left(a-a\wedge b\right)\right]\\
&=\theta^{-1}\left[\theta\left(a\wedge b\right)-\left(\left(\theta+1\right)\left(a\wedge b\right)\wedge a\right)+a\wedge b\right]
\end{aligned}$$

$$= \theta^{-1}\left[(\theta+1)(a\wedge b) - (\theta+1)(a\wedge b)\wedge a\right]$$
$$= \theta^{-1}\left[(\theta+1)(a\wedge b) - a\wedge(\theta+1)\,b\right]$$
$$\leq \theta^{-1}\left[(\theta+1)\,b - a\wedge(\theta+1)\,b\right].$$

It follows from

$$[a - a\wedge(\theta+1)\,b]\wedge[(\theta+1)\,b - a\wedge(\theta+1)\,b] = 0$$

that

$$[a\wedge b - a\wedge b\wedge\theta^{-1}(a - a\wedge b)]\perp[a - a\wedge(\theta+1)\,b].$$

Hence

$$[a\wedge b - a\wedge b\wedge\theta^{-1}(a - a\wedge b)]\,[a - a\wedge(\theta+1)\,b]$$
$$= [a - a\wedge(\theta+1)\,b]\,[a\wedge b - a\wedge b\wedge\theta^{-1}(a - a\wedge b)].$$

This gives

$$D_{a\wedge b}(a - a\wedge(1+\theta)\,b) = D_{a\wedge b\wedge\theta^{-1}(a-a\wedge b)}(a - a\wedge(1+\theta)\,b). \tag{4.2}$$

By replacing (4.2) in (4.1), the desired result is obtained. □

To be more clear, we introduce the following notation.

Definition 2 ([5], *Definition 2.5*) Let A be an Archimedean lattice- ordered algebra. For $a, b \in A^+$ and $\theta > 0$

Define: $\begin{cases} f_0(a,b,\theta) = a\wedge b\wedge\theta^{-1}(a - a\wedge b) \\ \text{and } f_1(a,b,\theta) = a - a\wedge(1+\theta)\,b \end{cases}$

Note that

$$0 \leq f_0(a,b,\theta) + f_1(a,b,\theta) \leq a \tag{4.3}$$

With this notation, the inequality of Lemma 2 read as follows:

Corollary 3 *Let A be an Archimedean lattice-ordered algebra in which all inner derivations are orthogonally null. If $a, b \in A^+$ and $\theta > 0$, then*

$$\left|D_{a\wedge b}(a - a\wedge b) - D_{f_0(a,b,\theta)}(f_1(a,b,\theta))\right| \leq \theta b^2.$$

Lemma 3 *Let A be an Archimedean lattice-ordered algebra in which all inner derivations are orthogonally null. If $a, b \in A^+$ and $\theta > 0$, then*

$$\left|D_a(b) + D_{f_0(a,b,\theta)}(f_1(a,b,\theta)) - D_{f_0(b,a,\theta)}(f_1(b,a,\theta))\right| \leq \theta(a^2 + b^2).$$

Proof Since

$$D_a(b) = -D_{a\wedge b}(a - a\wedge b) + D_{a\wedge b}(b - a\wedge b),$$

and by using the previous corollary it follows that

$$\begin{aligned}
&\left|D_a(b) + D_{f_0(a,b,\theta)}(f_1(a,b,\theta)) - D_{f_0(b,a,\theta)}(f_1(b,a,\theta))\right| \\
&\quad \le \left|D_{a\wedge b}(a - a\wedge b) - D_{f_0(a,b,\theta)}(f_1(a,b,\theta))\right| \\
&\quad\quad + \left|D_{a\wedge b}(b - a\wedge b) - D_{f_0(b,a,\theta)}(f_1(b,a,\theta))\right| \\
&\quad \le \theta(a^2 + b^2).
\end{aligned}$$

□

We can now give the approximating terms.

Definition 3 ([5], *Definition 2.9*) Let A be an Archimedean lattice-ordered algebra. For $a, b \in A^+$ and $\eta = (\eta_1, \eta_2, \eta_3 \ldots)$ a sequence of positive real numbers. For $k \in \mathbb{N}^*$, we define

$$B_k = \{(\epsilon_1, \epsilon_2, \ldots \epsilon_k) \text{ such that } \epsilon_i \in \{0,1\}\, \forall 1 \le i \le k\}.$$

For all $\epsilon \in B_k$, we defined by induction elements $a(\epsilon) = a(\eta, a, b; \epsilon)$ of A^+ by $a(0) = a$, $a(1) = b$ and for $\epsilon = (\epsilon_1, \epsilon_2, \ldots \epsilon_k) \in B_k$ and $i = 0, 1$.

$$\begin{aligned}
a((\epsilon, i)) &= a((\epsilon_1, \epsilon_2, \ldots \epsilon_k, i)) \\
&\qquad = f_i(a((\epsilon_1, \ldots \epsilon_k)), a((\epsilon_1, \ldots \epsilon_{k-1}, 1 - \epsilon_k)), \eta_k).
\end{aligned}$$

For example

$$a((1,0)) = f_0(a(1), a(0), \eta_1) = a \wedge b \wedge \eta_1^{-1}(b - a\wedge b)$$

In the following lemma we collect some properties of these elements.

Lemma 4 ([5], Lemma 2.10) *Let A be an Archimedean lattice-ordered algebra.*

1. *If $\epsilon = (\epsilon_1, \epsilon_2, \ldots \epsilon_k) \in B_k$ and $1 \le r \le k$, then $a((\epsilon_1, \ldots \epsilon_k)) \le a((\epsilon_1, \ldots \epsilon_r))$.*
2. *If $\epsilon, \epsilon' \in B_k$, $\epsilon \ne \epsilon'$ and $i, j \in \{0, 1\}$, then $a((\epsilon, i)) \wedge a\left((\epsilon', j)\right) = 0$.*
3. *If $\epsilon \in B_k$, then $a((\epsilon, 0)) \le \frac{1}{k+1}(a(0) + a(1))$ and $a((\epsilon, 1)) \le a(0) + a(1)$.*

Definition 4 ([5], *Definition 2.13*) Let A be an Archimedean lattice- ordered algebra and let $a, b \in A^+$. For $k = 1, 2, \ldots$, let

$$C_k = \sum_{\epsilon \in B_k} (-1)^{|\epsilon|+k} D_{a((\epsilon,0))}(a((\epsilon,1))).$$

where $|\epsilon| = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_k$ for all $\epsilon = (\epsilon_1, \epsilon_2, \ldots \epsilon_k) \in B_k$.
For example,

$$\begin{aligned} C_1 &= \sum_{\epsilon \in B_1} (-1)^{|\epsilon|+1} D_{a((\epsilon,0))} (a((\epsilon, 1))) \\ &= -D_{a((0,0))} (a((0, 1))) + D_{a((1,0))} (a((1, 1))) \\ &= -D_{f_0(a,b,\eta_1)} (f_1(a, b, \eta_1)) + D_{f_0(b,a,\eta_1)} (f_1(b, a, \eta_1)) \end{aligned}$$

So from Lemma 3,

$$|D_a(b) - C_1| \leq \eta_1(a^2 + b^2).$$

Put $C_0 = D_a(b)$, then $|C_0 - C_1| \leq \eta_1(a^2 + b^2)$.

In the following lemma, we extend this result.

Lemma 5 *Let A be an Archimedean lattice-ordered algebra in which all inner derivations are orthogonally null. If* $a, b \in A^+$ *, then*

$$|C_k - C_{k+1}| \leq \eta_{k+1}(a^2 + b^2) \quad \text{for all } k \in \mathbb{N}.$$

Proof Note that the property is true for $k = 0$. Then put $k \geq 1$.
We have

$$\begin{aligned} C_{k+1} &= \sum_{\tilde{\epsilon} \in B_{k+1}} (-1)^{|\tilde{\epsilon}|+k+1} D_{a((\tilde{\epsilon},0))} (a((\tilde{\epsilon}, 1))) \\ &= -\sum_{\epsilon \in B_k} (-1)^{|\epsilon|+k} D_{a((\epsilon,0,0))} (a((\epsilon, 0, 1))) \\ &\quad + \sum_{\epsilon \in B_k} (-1)^{|\epsilon|+k} D_{a((\epsilon,1,0))} (a((\epsilon, 1, 1))). \end{aligned}$$

Then

$$\begin{aligned} C_k - C_{k+1} = \sum_{\epsilon \in B_k} (-1)^{|\epsilon|+k} \{ & D_{a((\epsilon,0))} (a((\epsilon, 1))) \\ & + D_{a((\epsilon,0,0))} (a((\epsilon, 0, 1))) - D_{a((\epsilon,1,0))} (a((\epsilon, 1, 1)))\}. \end{aligned}$$

From Lemma 3,

$$\begin{aligned} |C_k - C_{k+1}| &\leq \sum_{\epsilon \in B_k} \left| \begin{matrix} D_{a((\epsilon,0))} (a((\epsilon, 1))) + D_{a((\epsilon,0,0))} (a((\epsilon, 0, 1))) \\ -D_{a((\epsilon,1,0))} (a((\epsilon, 1, 1))) \end{matrix} \right| \\ &\leq \eta_{k+1} \sum_{\epsilon \in B_k} \{a((\epsilon, 0))^2 + a((\epsilon, 1))^2\} \end{aligned}$$

$$\leq \eta_{k+1}\sum_{\epsilon\in B_k}\{a((\epsilon,0))+a((\epsilon,1))\}^2 \leq \eta_{k+1}\sum_{\epsilon\in B_k} a(\epsilon)^2$$

But from the inequality 4.3

$$\sum_{\epsilon\in B_k} a(\epsilon)^2 = \sum_{\lambda\in B_{k-1}}\{a((\lambda,0))+a((\lambda,1))\}^2 \leq \sum_{\lambda\in B_{k-1}} a(\lambda)^2 .$$

By repeating this argument, we find

$$\sum_{\epsilon\in B_k} a(\epsilon)^2 \leq \sum_{\lambda\in B_1} a(\lambda)^2 = a(0)^2 + a(1)^2 .$$

Hence

$$|C_k - C_{k+1}| \leq \eta_{k+1}(a(0)^2 + a(1)^2) = \eta_{k+1}(a^2+b^2)$$

and the proof is complete. □

All the preparations have been made for the principal result in the section.

Theorem 3 *Let A be an Archimedean ℓ-algebra. Then the following properties are equivalent:*

1. *A is commutative.*
2. *For all $a \in A$, $\ker D_a$ is a lattice subspace of A.*
3. *Any inner derivation is orthogonally null.*

Proof The implication (*1*) ⇒ (*2*) is trivial.
(*2*) ⇒ (*3*) Let $a \in A$. Then $a \in \ker D_a$. Since $\ker D_a$ a lattice subspace of A, then

$$a^+ \in \ker D_a.$$

Then

$$a^+a = aa^+$$

It follows from

$$a^{+^2} - a^+a^- = a^{+^2} - a^-a^+$$

that

$$a^+a^- = a^-a^+$$

for all $a \in A$. Then if $a \wedge b = 0$, we find

$$\begin{aligned} ab &= (a-b)^{+}(a-b)^{-} \\ &= (a-b)^{-}(a-b)^{+} \\ &= ba. \end{aligned}$$

(3) $\Rightarrow$ *(1)* Let $a, b \in A^{+}, \theta > 0$ and let $\eta = \{\eta_k\}_{k\geq 1}$ a sequence of positive real numbers such that $\sum_{k\geq 1}\eta_k < \theta$.

Define $a(\epsilon)$ and C_k for $\epsilon \in B_k$ $(k = 1, 2 \ldots)$ as above and taking $C_0 = D_a(b)$. From Lemma 5,

$$\begin{aligned} |D_a(b) - C_k| &= \left|\sum_{r=1}^{k}(C_{r-1} - C_r)\right| && (4.4) \\ &\leq \sum_{r=1}^{k}|C_{r-1} - C_r| && (4.5) \\ &\leq \sum_{r=1}^{k}\eta_r(a^2 + b^2) \\ &\leq \theta(a^2 + b^2). && (4.6) \end{aligned}$$

It follows from

$$C_k = \sum_{\epsilon\in B_k} \pm D_{a((\epsilon,0))}(a((\epsilon, 1)))$$

that

$$\begin{aligned} |C_k| &\leq \sum_{\epsilon\in B_k}\left|D_{a((\epsilon,0))}(a((\epsilon,1)))\right| \\ &\leq \sum_{\epsilon\in B_k}\{a((\epsilon,0))\,a((\epsilon,1)) + a((\epsilon,1))\,a((\epsilon,0))\} \end{aligned}$$

According to Lemma 4(iii),

$$|C_k| \leq \frac{1}{k+1}\sum_{\epsilon\in B_k}\{(a(0) + a(1))\,a((\epsilon,1)) + a((\epsilon,1))\,(a(0) + a(1))\}.$$

According to Lemma 4(ii) and (iii),

$$\begin{aligned} \sum_{\epsilon\in B_k} a((\epsilon,1)) &= \bigvee_{\epsilon\in B_k} a((\epsilon,1)) \\ &\leq a(0) + a(1). \end{aligned}$$

Thus

$$
\begin{aligned}
|C_k| &\leq \frac{1}{k+1}\left\{((a(0)+a(1)))\sum_{\epsilon\in B_k} a((\epsilon,1)) + \sum_{\epsilon\in B_k} a((\epsilon,1))(a(0)+a(1))\right\} \\
&\leq \frac{2}{k+1}(a+b)^2 . \qquad (4.7)
\end{aligned}
$$

From (4.4) and (4.7) we have

$$
\begin{aligned}
|D_a(b)| &= |D_a(b) - C_k + C_k| \\
&\leq |D_a(b) - C_k| + |C_k| \\
&\leq \theta(a^2+b^2) + \frac{2}{k+1}(a+b)^2 \quad \text{for } k = 1, 2\ldots
\end{aligned}
$$

Since A is Archimedean, we obtain

$$
|D_a(b)| \leq \theta(a^2+b^2).
$$

But this holds for all $\theta > 0$, then as well since A is Archimedean, we have $|D_a(b)| = 0$, i.e., $ab = ba$. This holds for all $a, b \in A^+$ and hence for all $a, b \in A$ and the commutativity is proved. □

Corollary 4 ([5], Theorem 2.15) *Any Archimedean almost f-algebra is commutative.*

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