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Statistical $\mathscr{A}$-summation process and Korovkin type approximation theorem on modular spaces

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Abstract In this paper, we obtain an extension of the classical Korovkin theorem for a sequence of positive linear operators on a modular space using a statistical $\mathscr{A}$-summation process. Also, we give an example which satisfies this theorem.

Keywords Positive linear operators · Modular space · Matrix summability · Korovkin theorem

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1 Introduction

The Korovkin theorem is the object of study of many mathematicians. In the classical Korovkin theorem [1,18] the uniform convergence in $C([a,b])$, the space of all continuous real-valued functions defined on the compact interval $[a,b]$, is proved for a sequence of positive linear operators, assuming the convergence only on the test functions $1, x, x^2$. Recently some versions of Korovkin theorems were proved in the setting of modular spaces, which include as particular cases L_p, Orlicz and Musielak-Orlicz spaces [24]. Also, in [2], some versions of abstract Korovkin-type theorems in

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modular function spaces, with respect to filter convergence for linear positive operators, by considering several kinds of test functions have studied. Note that for certain function spaces, as for example L_p spaces, in general it is not possible to get the convergence in L_p of a sequence of positive linear operators for all the L_p functions, but it is necessary to consider spaces, depending on the form of the operators involved. In the present paper, we give a modular version of the Korovkin theorem for a sequence of positive linear operators using a statistical $\mathscr{A}$-summation process, is an extension of Theorem 2 in [24].

We now recall some basic definitions and notations used in the paper.

Let

$$\mathscr{B} := (b_{lk}) \quad (l, k \in \mathbb{N} := \{1, 2, 3, \ldots\})$$

be an infinite summability matrix. For a given sequence of real numbers $x := \{x_k\}$, the *$\mathscr{B}$-transform of x*, denoted by

$$\mathscr{B}x := \{(\mathscr{B}x)_l\},$$

is given by

$$(\mathscr{B}x)_l = \sum_{k=1}^{\infty} b_{lk} x_k,$$

provided the series converges for each $l \in \mathbb{N}$. We say that $\mathscr{B}$ is *regular* (see [15]) if

$$\lim \mathscr{B}x = L \text{ whenever } \lim x = L.$$

Assume that $\mathscr{B}$ is a non-negative regular summability matrix. Then the sequence $x = \{x_k\}$ is called *$\mathscr{B}$-statistically convergent* to L provided that, for every $\varepsilon > 0$,

$$\lim_l \sum_{k:\ |x_k - L| \geq \varepsilon} b_{lk} = 0. \tag{1}$$

We denote this limit as follows (*cf.* [11]; see also [7,8,17])

$$st_{\mathscr{B}} - \lim x = L.$$

Actually, this convergence method is based on the concept of $\mathscr{B}$ -density. Recall the *$\mathscr{B}$-density* of a subset $K \subset \mathbb{N}$, denoted by

$$\delta_{\mathscr{B}}\{K\},$$

is given by

$$\delta_{\mathscr{B}}\{K\} = \lim_l \sum_{k=1}^{\infty} b_{lk} \chi_K(k),$$

provided the limit exists, where χ_K is the characteristic function of K; or equivalently

$$\delta_{\mathscr{B}}\{K\} = \lim_l \sum_{k \in K} b_{lk}.$$

So, by (1), we easily see that

$$st_{\mathscr{B}} - \lim x = L \text{ iff } \delta_{\mathscr{B}}\{k : |x_k - L| \geq \varepsilon\} = 0 \text{ for every } \varepsilon > 0.$$

We should note that if we take $\mathscr{B} = C_1 := (c_{lk})$, the *Cesáro matrix* defined by

$$c_{lk} := \begin{cases} \dfrac{1}{l}, & \text{if } 1 \leq k \leq l, \\ 0, & \text{otherwise,} \end{cases}$$

then $\mathscr{B}$-statistical convergence reduces to the concept of *statistical convergence* (*cf.* [10]; see also [12–14]). In this case, we write

$$st - \lim x = L \text{ instead of } st_{C_1} - \lim x = L.$$

Further, taking $\mathscr{B} = \mathscr{I}$, the identity matrix, $\mathscr{B}$-statistical convergence coincides with the ordinary convergence, i.e.,

$$st_{\mathscr{I}} - \lim x = \lim x = L.$$

Observe that every convergent sequence (in the usual sense) is $\mathscr{B}$-statistically convergent to the same value for any non-negative regular matrix $\mathscr{B}$, but its converse is not always true. Actually, in [17], Kolk proved that $\mathscr{B}$-statistical convergence is stronger than convergence when $\mathscr{B} = (b_{lk})$ is a non-negative regular summability matrix such that

$$\lim_l \max_k \{b_{lk}\} = 0.$$

The concepts of *statistical limit superior* and *limit inferior* have been introduced by Fridy and Orhan [14]. $\mathscr{B}$-statistical analogs of these concepts have been examined by Connor and Kline [7], and Demirci [8] as follows. The *$\mathscr{B}$-statistical limit superior* of a number sequence $x = \{x_k\}$, denoted by

$$st_{\mathscr{B}} - \limsup x,$$

is defined by

$$st_{\mathscr{B}} - \limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \phi, \\ -\infty, & \text{if } B_x = \phi, \end{cases}$$

where $B_x := \{b \in \mathbb{R} : \delta_{\mathscr{B}}\{k : x_k > b\} \neq 0\}$ and ϕ denotes the empty set. We note that by $\delta_{\mathscr{B}}\{K\} \neq 0$ we mean either $\delta_{\mathscr{B}}\{K\} > 0$ or K fails to have $\mathscr{B}$-density. Similarly, the *$\mathscr{B}$-statistical limit inferior* of $\{x_k\}$, denoted by

$$st_{\mathscr{B}} - \liminf x,$$

is defined by

$$st_{\mathscr{B}} - \liminf x = \begin{cases} \inf C_x, & \text{if } C_x \neq \phi, \\ +\infty, & \text{if } C_x = \phi, \end{cases}$$

where $C_x := \{c \in \mathbb{R} : \delta_{\mathscr{B}}\{k : x_k < c\} \neq 0\}$. Of course, if we take $\mathscr{B} = C_1$, then the above definitions reduce to the concepts of $st - \limsup x$ and $st - \liminf x$ given in [14], respectively. As in the ordinary limit superior or inferior, it was proved that

$$st_{\mathscr{B}} - \liminf x \leq st_{\mathscr{B}} - \limsup x$$

and also that, for any sequence $x = \{x_k\}$ satisfying $\delta_{\mathscr{B}}\{k : |x_k| > M\} = 0$ for some $M > 0$,

$$st_{\mathscr{B}} - \lim x = L \text{ iff } st_{\mathscr{B}} - \liminf x = st_{\mathscr{B}} - \limsup x = L.$$

We now focus on modular spaces.

Let $I = [a, b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, by $X(I)$ we denote the space of all real-valued measurable functions on I provided with equality *a.e.* As usual, let $C(I)$ denote the space of all continuous real-valued functions, and $C^{\infty}(I)$ denote the space of all infinitely differentiable functions on I. In this case, we say that a functional $\rho : X(I) \to [0, +\infty]$ is a *modular* on $X(I)$ provided that the following conditions hold:

(i) $\rho(f) = 0$ if and only if $f = 0$ *a.e.* in I,
(ii) $\rho(-f) = \rho(f)$ for every $f \in X(I)$,
(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for every $f, g \in X(I)$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

A modular ρ is said to be *N-quasi convex* if there exists a constant $N \geq 1$ such that

$$\rho(\alpha f + \beta g) \leq N\alpha\rho(Nf) + N\beta\rho(Ng)$$

holds for every $f, g \in X(I)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. In particular, if $N = 1$, then ρ is called *convex*.

A modular ρ is said to be *N-quasi semiconvex* if there exists a constant $N \geq 1$ such that

$$\rho(af) \leq Na\rho(Nf)$$

holds for every $f \in X(I)$ and $a \in (0, 1]$.

It is clear that every N-quasi semiconvex modular is N-quasi convex. We should recall that the above two concepts were introduced and discussed in details by Bardaro et. al. [4].

We now consider some appropriate vector subspaces of $X(I)$ by means of a modular ρ as follows:

$$L^{\rho}(I) := \left\{ f \in X(I) : \lim_{\lambda \to 0^{+}} \rho(\lambda f) = 0 \right\}$$

and

$$E^{\rho}(I) := \left\{ f \in L^{\rho}(I) : \rho(\lambda f) < +\infty \quad \text{for all } \lambda > 0 \right\}.$$

Here, $L^{\rho}(I)$ is called the *modular space* generated by ρ; and $E^{\rho}(I)$ is called the space of the finite elements of $L^{\rho}(I)$. Observe that if ρ is N-quasi semiconvex, then the space

$$\{ f \in X(I) : \rho(\lambda f) < +\infty \quad \text{for some } \lambda > 0 \}$$

coincides with $L^{\rho}(I)$. The notions about modulars are introduced in [23] and widely discussed in [4] (see also [19,22]).

With the help of the notions of modular convergence and strong convergence, some approximation theorems have recently been introduced by Bardaro and Mantellini [5].

Now we recall the convergence methods in modular spaces.

• Let $\{f_n\}$ be a function sequence whose terms belong to $L^{\rho}(I)$. Then, $\{f_n\}$ is *modularly convergent* to a function $f \in L^{\rho}(I)$ iff

$$\lim_{n} \rho(\lambda_0 (f_n - f)) = 0 \quad \text{for some } \lambda_0 > 0. \tag{2}$$

• Also, $\{f_n\}$ is *F-norm convergent* (or, *strongly convergent*) to f iff

$$\lim_{n} \rho(\lambda (f_n - f)) = 0 \quad \text{for every } \lambda > 0. \tag{3}$$

It is known from [22] that (2) and (3) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\rho(2f) \leq M\rho(f)$ for every $f \in X(I)$.

In this paper, we will need the following assumptions on a modular ρ:

- if $\rho(f) \leq \rho(g)$ for $|f| \leq |g|$, then ρ is *monotone*,
- ρ is *finite* if $\chi_A \in L^{\rho}(I)$ whenever A is measurable subset of I such that $\mu(A) < \infty$,
- if ρ is finite and, for every $\varepsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ such that $\rho(\lambda \chi_B) < \varepsilon$ for any measurable subset $B \subset I$ with $\mu(B) < \delta$, then ρ is *absolutely finite*,
- if $\chi_I \in E^{\rho}(I)$, then ρ is *strongly finite*
- ρ is *absolutely continuous* provided that there exists an $\alpha > 0$ such that, for every $f \in X(I)$ with $\rho(f) < +\infty$, the following condition holds: for every $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(\alpha f \chi_B) < \varepsilon$ whenever B is any measurable subset of I with $\mu(B) < \delta$.

Observe now that (see [5]) if a modular ρ is monotone and finite, then we have $C(I) \subset L^{\rho}(I)$. In a similar manner, if ρ is monotone and strongly finite, then $C(I) \subset E^{\rho}(I)$. Also, if ρ is monotone, absolutely finite and absolutely continuous, then $\overline{C^{\infty}(I)} = L^{\rho}(I)$. Some important relations between the above properties may be found in [3,4,21,23].

2 Korovkin type theorems

Let $\mathscr{A} := \{A^n\}_{n\geq 1}$, $A^n = \left(a_{kj}^{(n)}\right)_{k,j\in}$ be a sequence of infinite non-negative real matrices. For a sequence of real numbers, $x = \left\{x_j\right\}_{j\in}$, the double sequence

$$\mathscr{A}x := \left\{(Ax)_k^n : k, n \in\right\}$$

defined by $(Ax)_k^n := \sum_{j=1}^{\infty} a_{kj}^{(n)} x_j$ is called the $\mathscr{A}$-transform of x whenever the series converges for all k and n. A sequence x is said to be $\mathscr{A}$-summable to L if

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} x_j = L$$

uniformly in n ([6,26]).

If $A^n = \mathscr{B}$ for some matrix $\mathscr{B}$, then $\mathscr{A}$-summability is the ordinary matrix summability by $\mathscr{B}$. If, $a_{kj}^{(n)} = \frac{1}{k+1}$, for $n \leq j \leq k+n$, $(n = 1, 2, \ldots)$, and $a_{kj}^{(n)} = 0$ otherwise, then $\mathscr{A}$-summability reduces to almost convergence [20].

Let ρ be a monotone and finite modular on $X(I)$, and let $\mathscr{B} = (b_{lk})$ be a non-negative regular summability matrix. Assume that D is a set satisfying $C^{\infty}(I) \subset D \subset L^{\rho}(I)$. We can construct such a subset D when ρ is monotone and finite (see [5]). Here, D is domain of the operator $\mathbb{T}$. We will assume that $\mathbb{T} := \{T_j\}$ is a sequence of positive linear operators from D into $X(I)$ and for all $k, n \in$, $f \in D$ the series

$$A_{k,n}^{\mathbb{T}}(f) := \sum_{j=1}^{\infty} a_{kj}^{(n)} T_j f,$$

is absolutely convergent almost everywhere with respect to Lebesgue measure. Also, assume that there exists a subset $X_{\mathbb{T}} \subset D$ with $C^{\infty}(I) \subset X_{\mathbb{T}}$ and a constant $P > 0$ such that,

$$st_{\mathscr{B}} - \limsup_k \rho\left(\lambda\left(A_{k,n}^{\mathbb{T}}(f)\right)\right) \leq P\rho(\lambda f), \text{ uniformly in } n \tag{4}$$

holds for every $f \in X_{\mathbb{T}}$, $\lambda > 0$.

A sequence $\mathbb{T} := \{T_j\}$ of positive linear operators of D into $X(I)$ is called an $\mathscr{A}$-summation process on D if $\left\{T_j(f)\right\}$ is $\mathscr{A}$-summable to f (with respect to modular ρ) for every $f \in D$, i.e.,

$$\lim_k \rho\left[\lambda\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j f - f\right)\right] = 0, \text{ uniformly in } n. \tag{5}$$

A different definition is given by, (see [16])

$$\lim_{k}\sum_{j=1}^{\infty}a_{kj}^{(n)}\rho\left[\lambda\left(T_{j}f-f\right)\right]=0,\ \text{uniformly in } n \tag{6}$$

for all $f\in D$ where it is assumed that $\sup_{n,k}\sum_{j=1}^{\infty}a_{kj}^{(n)}<\infty$ holds.

In this paper, we establish a theorem of the Korovkin type with respect to the convergence behavior (5) for a sequence of positive linear operators of D into $X(I)$. So the results of type (5) are extensions of type (6). Also, the following theorem is an extension of Theorem 2 in [24]. Some results concerning summation processes in the space $L_{p}[a,b]$ of Lebesgue integrable functions on a compact interval may be found [24,25].

Throughout the paper we use the test functions e_i defined by

$$e_{i}(x)=x^{i}\quad(i=0,1,2,\ldots).$$

Also, we denote the value of $T_{j}f$ at a point $x\in I$ by $T_{j}\left(f\left(y\right);x\right)$ or, briefly, $T_{j}\left(f;x\right)$.

Theorem 1 *Let $\mathscr{A}=\{A^{n}\}_{n\geq1}$ be a sequence of infinite non-negative real matrices and let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X(I)$, also $\mathscr{B}=(b_{lk})$ be a non-negative regular summability matrix. Let $\mathbb{T}:=\{T_{j}\}$ be a sequence of positive linear operators from D into $X(I)$ satisfying (4) for each $f\in D$. Suppose that*

$$st_{\mathscr{B}}-\lim_{k}\rho\left(\lambda\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_{j}e_{i}-e_{i}\right)\right)=0,\ \textit{uniformly in } n \tag{7}$$

for every $\lambda>0$ and $i=0,1,2$. Now let f be any function belonging to $L^{\rho}(I)$ such that $f-g\in X_{\mathbb{T}}$ for every $g\in C^{\infty}(I)$. Then, we have

$$st_{\mathscr{B}}-\lim_{k}\rho\left(\lambda_{0}\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_{j}f-f\right)\right)=0,\ \textit{uniformly in } n \tag{8}$$

for some $\lambda_0>0$.

Proof We first claim that

$$st_{\mathscr{B}}-\lim_{k}\rho\left(\eta\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_{j}g-g\right)\right)=0\ \textit{uniformly in } n \tag{9}$$

for every $g\in C(I)\cap D$ and every $\eta>0$. To see this assume that g belongs to $C(I)\cap D$. By the continuity of g on I, given $\varepsilon>0$, there exists a number $\delta>0$ such that for all $x,y\in I$ satisfying $|y-x|<\delta$ we have

$$|g(y) - g(x)| < \varepsilon \tag{10}$$

Also we get for all $x, y \in I$ satisfying $|y - x| > \delta$ that

$$|g(y) - g(x)| \leq \frac{2M}{\delta^2}(y - x)^2 \tag{11}$$

where $M := \sup_{x \in I} |g(x)|$ .Combining (10) and (11) we have for $x, y \in I$ that

$$|g(y) - g(x)| < \varepsilon + \frac{2M}{\delta^2}(y - x)^2 .$$

Since T_j is a positive linear operator, we get

$$\begin{aligned}
&\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(g; x) - g(x)\right| \\
&= \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(g(y); x) - \sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(g(x); x) + \sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(g(x); x) - g(x)\right| \\
&= \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(g(y) - g(x); x) + g(x)\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(1; x) - 1\right)\right| \\
&\leq \sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(|g(y) - g(x)|; x) + |g(x)| \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(1; x) - 1\right| \\
&\leq \sum_{j=1}^{\infty} a_{kj}^{(n)} T_j\left(\varepsilon + \frac{2M}{\delta^2}(y - x)^2; x\right) + |g(x)| \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(1; x) - 1\right| \\
&\leq \varepsilon + \varepsilon \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(1; x) - 1\right| + \frac{2M}{\delta^2} \sum_{j=1}^{\infty} a_{kj}^{(n)} T_j\left((y - x)^2; x\right) \\
&\quad + M \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(1; x) - 1\right| \\
&= \varepsilon + (\varepsilon + M) \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(1; x) - 1\right| + \frac{2M}{\delta^2}\left[\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j\left(y^2; x\right) - x^2\right)\right.
\end{aligned}$$

$$-2x\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j\left(y;x\right)-x\right)+x^2\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j\left(1;x\right)-1\right)\Bigg]$$

$$\leq \varepsilon+\left(\varepsilon+M+\frac{2Mc^2}{\delta^2}\right)\left|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j\left(1;x\right)-1\right|+\frac{4Mc}{\delta^2}\left|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j\left(y;x\right)-x\right|$$

$$+\frac{2M}{\delta^2}\left|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j\left(y^2;x\right)-x^2\right|$$

where $c:=\max\{|a|,|b|\}$. So, the last inequality gives, for any $\eta>0$ that

$$\eta\left|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j\left(g;x\right)-g\left(x\right)\right|\leq\Bigg\{\eta\varepsilon+\eta K\left|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j\left(1;x\right)-1\right|$$

$$+\left|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j\left(y;x\right)-x\right|+\left|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j\left(y^2;x\right)-x^2\right|\Bigg\}$$

where $K:=\max\left\{\varepsilon+M+\frac{2Mc^2}{\delta^2},\frac{4Mc}{\delta^2},\frac{2M}{\delta^2}\right\}$. Applying the modular ρ in both-sides of the above inequality, since ρ is monotone, we have

$$\rho\left(\eta\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_jg-g\right)\right)\leq\rho\Bigg(\eta\varepsilon+\eta K\left|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_je_0-e_0\right|$$

$$+\eta K\left|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_je_1-e_1\right|$$

$$+\eta K\left|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_je_2-e_2\right|\Bigg).$$

So, we may write that

$$\rho\left(\eta\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_jg-g\right)\right)$$

$$\leq\rho(4\eta\varepsilon)+\rho\left(4\eta K\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_je_0-e_0\right)\right)$$

$$+\rho\left(4\eta K\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_je_1-e_1\right)\right)+\rho\left(4\eta K\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_je_2-e_2\right)\right).$$

Since ρ is N-quasi semiconvex and strongly finite, we have, assuming $0<\varepsilon\leq 1$

$$\rho\left(\eta\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_jg-g\right)\right)$$
$$\leq N\varepsilon\rho\left(4\eta N\right)+\rho\left(4\eta K\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_je_0-e_0\right)\right)$$
$$+\rho\left(4\eta K\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_je_1-e_1\right)\right)+\rho\left(4\eta K\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_je_2-e_2\right)\right).$$

For a given $r>0$, choose an $\varepsilon\in(0,1]$ such that $N\varepsilon\rho\left(4\eta N\right)<r$. Now define the following sets:

$$S_\eta:=\left\{k:\rho\left(\eta\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_jg-g\right)\right)\geq r\right\},$$
$$S_{\eta,i}:=\left\{k:\rho\left(4\eta K\left(\sum_{j=1}^{\infty}a_{kj}^{(n)}T_je_i-e_i\right)\right)\geq\frac{r-N\varepsilon\rho\left(4\eta N\right)}{3}\right\},$$

where $i=0,1,2$. Then, it is easy to see that $S_\eta\subseteq\bigcup_{i=0}^{2}S_{\eta,i}$. So we can write, for all $l\in\mathbb{N}$, that

$$\sum_{k\in S_\eta}b_{lk}\leq\sum_{k\in S_{\eta,0}}b_{lk}+\sum_{k\in S_{\eta,1}}b_{lk}+\sum_{k\in S_{\eta,2}}b_{lk}. \tag{12}$$

Taking limit as $l\to\infty$ in (12) and using the hypothesis (7), we get

$$\lim_{l}\sum_{k\in S_\eta}b_{lk}=0,$$

which proves our claim (9). Observe that (9) also holds for every $g\in C^{\infty}(I)$ because of $C^{\infty}(I)\subset C(I)\cap D$. Now let $f\in L^{\rho}(I)$ satisfying $f-g\in X_{\mathbb{T}}$ for every $g\in C^{\infty}(I)$. Since $\mu(I)<\infty$ and ρ is strongly finite and absolutely continuous, we can see that ρ is also absolutely finite on $X(I)$ (see [3]). Using these properties of the modular ρ, it is known from [4,21] that the space $C^{\infty}(I)$ is modularly dense in $L^{\rho}(I)$, i.e., there exists a sequence $\{g_k\}\subset C^{\infty}(I)$ such that

$$\lim_{k}\rho\left[3\lambda_0^{*}\left(g_k-f\right)\right]=0\quad\text{for some }\lambda_0^{*}>0.$$

This means that, for every $\varepsilon>0$, there is a positive number $k_0=k_0(\varepsilon)$ so that

$$\rho\left[3\lambda_0^{*}\left(g_k-f\right)\right]<\varepsilon\quad\text{for every }k\geq k_0. \tag{13}$$

On the other hand, by the linearity and positivity of the operators T_j, we may write that

$$\lambda_0^* \left| \sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(f;x) - f(x) \right| \leq \lambda_0^* \left| \sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(f - g_{k_0};x) \right| + \lambda_0^* \left| \sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(g_{k_0};x) - g_{k_0}(x) \right| + \lambda_0^* \left| g_{k_0}(x) - f(x) \right|$$

holds for every $x \in I$ and $n \in \mathbb{N}$. Applying the modular ρ in the last inequality and using the monotonicity of ρ, we have

$$\rho\left(\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j f - f\right)\right) \leq \rho\left(3\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j\left(f - g_{k_0}\right)\right)\right) + \rho\left(3\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j g_{k_0} - g_{k_0}\right)\right) + \rho\left(3\lambda_0^*\left(g_{k_0} - f\right)\right). \tag{14}$$

Then, it follows from (13) and (14) that

$$\rho\left(\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j f - f\right)\right) \leq \varepsilon + \rho\left(3\lambda_0^* \sum_{j=1}^{\infty} a_{kj}^{(n)} T_j\left(f - g_{k_0}\right)\right) + \rho\left(3\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j g_{k_0} - g_{k_0}\right)\right). \tag{15}$$

So, taking $\mathscr{B}$-statistical limit superior as $k \to \infty$ in the both-sides of (15) and also using the facts that $g_{k_0} \in C^{\infty}(I)$ and $f - g_{k_0} \in X_{\mathbb{T}}$, we obtained from (4) that

$$\begin{aligned} st_{\mathscr{B}} - \limsup_{k} \rho\left(\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j f - f\right)\right) &\leq \varepsilon + P\rho\left(3\lambda_0^*(f - g_{k_0})\right) \\ &\quad + st_{\mathscr{B}} - \limsup_{k} \rho\left(3\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j g_{k_0} - g_{k_0}\right)\right), \end{aligned}$$

which gives

$$st_{\mathscr{B}} - \limsup_k \rho\left(\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j f - f\right)\right)$$
$$\leq \varepsilon(P+1) + st_{\mathscr{B}} - \limsup_k \rho\left(3\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j g_{k_0} - g_{k_0}\right)\right). \tag{16}$$

By (9), since

$$st_{\mathscr{B}} - \lim_k \rho\left(3\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j g_{k_0} - g_{k_0}\right)\right) = 0, \text{ uniformly in } n,$$

we get

$$st_{\mathscr{B}} - \limsup_k \rho\left(3\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j g_{k_0} - g_{k_0}\right)\right) = 0, \text{ uniformly in } n. \tag{17}$$

Combining (16) with (17), we conclude that

$$st_{\mathscr{B}} - \limsup_k \rho\left(\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j f - f\right)\right) \leq \varepsilon(P+1).$$

Since $\varepsilon > 0$ was arbitrary, we find

$$st_{\mathscr{B}} - \limsup_k \rho\left(\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j f - f\right)\right) = 0 \text{ uniformly in } n.$$

Furthermore, since $\rho\left(\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j f - f\right)\right)$ is non-negative for all $k, n \in \mathbb{N}$, we can easily show that

$$st_{\mathscr{B}} - \lim_k \rho\left(\lambda_0^*\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j f - f\right)\right) = 0, \text{ uniformly in } n$$

which completes the proof.

If the modular ρ satisfies the Δ_2-condition, then one can get the following result from Theorem 1 at once.

Theorem 2 *Let $\mathscr{A} = \{A^n\}_{n\geq 1}$ be a sequence of infinite non-negative real matrices, $\mathscr{B} = (b_{lk})$ be a non-negative regular summability matrix and $\mathbb{T} := \{T_j\}$, ρ be the same as in Theorem 1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

(a) $st_{\mathscr{B}} - \lim_k \rho\left(\lambda\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j e_i - e_i\right)\right) = 0$ *uniformly in* n *for every* $\lambda > 0$ *and* $i = 0, 1, 2,$

(b) $st_{\mathscr{B}} - \lim_k \rho\left(\lambda\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j f - f\right)\right) = 0$ *uniformly in* n *for every* $\lambda > 0$ *provided that* f *is any function belonging to* $L^{\rho}(I)$ *such that* $f - g \in X_{\mathbb{T}}$ *for every* $g \in C^{\infty}(I)$.

If one replaces the matrices $\mathscr{B}$ and A^n $(n \geq 1)$ by the identity matrix, then the condition (4) reduces to

$$st - \limsup_k \rho\left(\lambda\left(T_j h\right)\right) \leq P\rho\left(\lambda h\right) \tag{18}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant P. In this case, the next results which were obtained by Bardaro and Mantellini [5] immediately follows from our Theorems 1 and 2.

Corollary 1 [5] *Let* ρ *be a monotone, strongly finite, absolutely continuous and* N*-quasi semiconvex modular on* $X(I)$. *Let* $\mathbb{T} := \{T_j\}$ *be a sequence of positive linear operators from D into* $X(I)$ *satisfying (18). If* $\{T_j e_i\}$ *is strongly convergent to* e_i *for each* $i = 0, 1, 2,$ *then* $\{T_j f\}$ *is modularly convergent to* f *provided that* f *is any function belonging to* $L^{\rho}(I)$ *such that* $f - g \in X_{\mathbb{T}}$ *for every* $g \in C^{\infty}(I)$.

Corollary 2 [5] $\mathbb{T} := \{T_j\}$ *and* ρ *be the same as in Corollary 1. If* ρ *satisfies the* Δ_2*-condition, then the following statements are equivalent:*

(a) $\{T_j e_i\}$ *is strongly convergent to* e_i *for each* $i = 0, 1, 2,$

(b) $\{T_j f\}$ *is strongly convergent to* f *provided that* f *is any function belonging to* $L^{\rho}(I)$ *such that* $f - g \in X_{\mathbb{T}}$ *for every* $g \in C^{\infty}(I)$.

3 Application

In this section we give an example of positive linear operators which satisfy the conditions of Theorem 1.

Example 1 Take $I = [0, 1]$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function for which the following conditions hold:

- φ *is convex,*
- $\varphi(0) = 0$, $\varphi(u) > 0$ *for* $u > 0$ *and* $\lim_{u\to\infty} \varphi(u) = \infty$.

Hence, consider the functional ρ^{φ} *on* $X(I)$ *defined by*

$$\rho^{\varphi}(f) := \int_0^1 \varphi\left(|f(x)|\right) dx \quad \textit{for } f \in X(I). \tag{19}$$

In this case, ρ^{φ} *is a convex modular on* $X(I)$, *which satisfies all assumptions listed in Sect.* 1 (see [5]). *Consider the Orlicz space generated by* φ *as follows:*

$$L_{\varphi}^{\rho}(I) := \left\{f \in X(I) : \rho^{\varphi}(\lambda f) < +\infty \quad \textit{for some } \lambda > 0\right\}.$$

Then consider the following classical Bernstein-Kantorovich operator $\mathbf{U} := \{U_j\}$ *on the space* $L_{\varphi}^{\rho}(I)$ *(see* [5]*which is defined by:*

$$U_j(f; x) := \sum_{k=0}^{j} \binom{j}{k} x^k (1-x)^{j-k} (j+1) \int\limits_{k/(j+1)}^{(k+1)/(j+1)} f(t)\, dt \text{ for } x \in I.$$

Observe that the operators U_j *map the Orlicz space* $L_{\varphi}^{\rho}(I)$ *into itself.* Moreover, property (18) is satisfied with the choice of $X_{\mathbf{U}} := L_{\varphi}^{\rho}(I)$. Then, by Corollary 1, we know that, for any function $f \in L_{\varphi}^{\rho}(I)$ such that $f - g \in X_{\mathbf{U}}$ for every $g \in C^{\infty}(I)$, $\{U_j f\}$ is modularly convergent to f.

If $\varphi(x) = x^p$ *for* $1 \le p < \infty$, $x \ge 0$, *then* $L_{\varphi}^{\rho}(I) = L_p(I)$ *Moreover we have*

$$\rho^{\varphi}(f) = \|f\|_{L_p(I)}^{p}.$$

Now take $\mathscr{B} = C_1 = (c_{kj})$, *the Ces áro matrix of order one. In this case, we know that* C_1 *-statistical convergence coincides with statistical convergence, and its limit is denoted by* $st - \lim$*Assume that* $\mathscr{A} := \{A^n\}_{n \ge 1} = \left\{ \left(a_{kj}^{(n)} \right)_{k,j \in \mathbb{N}} \right\}_{n \ge 1}$ *is a sequence of infinite matrices defined by* $a_{kj}^{(n)} = \frac{1}{k+1}$ *if* $n \le j \le n+k$, $(n = 1, 2, \ldots)$ *and* $a_{kj}^{(n)} = 0$ *otherwise. Since, for positive constant* C, $\left\| U_j(f; x) \right\|_{L_p} \le C \|f\|_{L_p}$ [9], *we can easily see that*

$$st - \limsup_k \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} U_j f \right\|_{L_p}^{p} \le C \|f\|_{L_p}^{p}, \text{ uniformly in } n.$$

We now claim that

$$st - \lim_k \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} U_j e_i - e_i \right\|_{L_p}^{p} = 0, \text{ uniformly in } n, \quad i = 0, 1, 2. \tag{20}$$

Observe that $U_j(e_0; x) = e_0$, $U_j(e_1; x) = \frac{jx}{j+1} + \frac{1}{2(j+1)}$ and $U_j(e_2; x) = \frac{j(j-1)x^2}{(j+1)^2} + \frac{2jx}{(j+1)^2} + \frac{1}{3(j+1)^2}$. *So, we can see,*

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} U_j(e_0; x) - e_0(x) \right\|_{L_p} &= \left\| \sum_{j=n}^{n+k} \frac{1}{k+1} U_j(e_0; x) - e_0(x) \right\|_{L_p} \\ &= \|1 - 1\|_{L_p} = 0, \end{aligned}$$

we get

$$st-\lim_k \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} U_j e_0 - e_0 \right\|_{L_p}^{p} = 0, \text{ uniformly in } n.$$

which guarantees that (20) holds true for $i=0$. Also, we have

$$\begin{aligned}
\left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} U_j\left(e_1; x\right) - e_1(x) \right\|_{L_p} &= \left\| \sum_{j=n}^{n+k} \frac{1}{k+1} U_j\left(e_1; x\right) - e_1(x) \right\|_{L_p} \\
&= \left\| x \frac{1}{k+1} \sum_{j=n}^{n+k} \frac{j}{j+1} + \frac{1}{2\left(k+1\right)} \sum_{j=n}^{n+k} \frac{1}{j+1} - x \right\|_{L_p} \\
&\leq \left(\frac{1}{k+1} \sum_{j=n}^{n+k} \frac{j}{j+1} - 1 \right) \left\| e_1(x) \right\|_{L_p} \\
&\quad + \left(\frac{1}{2\left(k+1\right)} \sum_{j=n}^{n+k} \frac{1}{j+1} \right) \left\| e_0(x) \right\|_{L_p} \\
&= \frac{1}{\left(p+1\right)^{1/p}} \left(-\frac{1}{k+1} \sum_{j=n}^{n+k} \frac{1}{j+1} \right) \\
&\quad + \left(\frac{1}{2\left(k+1\right)} \sum_{j=n}^{n+k} \frac{1}{j+1} \right) \\
&= \left(\frac{1}{k+1} \sum_{j=n}^{n+k} \frac{1}{j+1} \right) \left(\frac{1}{2} - \frac{1}{\left(p+1\right)^{1/p}} \right)
\end{aligned}$$

Since $st-\lim_k \left(\sup_n \frac{1}{k+1} \sum_{j=n}^{n+k} \frac{1}{j+1} \right) = 0$, *we have,*

$$\begin{aligned}
& st-\lim_k \left(\sup_n \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} U_j\left(e_1; x\right) - e_1 \right\|_{L_p} \right) \\
& \quad \leq \left(\frac{1}{2} - \frac{1}{\left(p+1\right)^{1/p}} \right) st-\lim_k \left(\sup_n \frac{1}{k+1} \sum_{j=n}^{n+k} \frac{1}{j+1} \right)
\end{aligned}$$

which gives

$$st-\lim_{k}\left\Vert \sum_{j=1}^{\infty}a_{kj}^{(n)}U_{j}e_{1}-e_{1}\right\Vert _{L_{p}}=0,\ uniformly\ in\, n.$$

So, we have

$$st-\lim_{k}\left\Vert \sum_{j=1}^{\infty}a_{kj}^{(n)}U_{j}e_{1}-e_{1}\right\Vert _{L_{p}}^{p}=0,\ uniformly\ in\, n.$$

Finally, since

$$\begin{aligned}
&\left\Vert \sum_{j=1}^{\infty}a_{kj}^{(n)}U_{j}\left(e_{2};x\right)-e_{2}\left(x\right)\right\Vert _{L_{p}}\\
&=\left\Vert \sum_{j=n}^{n+k}\frac{1}{k+1}U_{j}\left(e_{2};x\right)-e_{2}\left(x\right)\right\Vert _{L_{p}}\\
&=\left\Vert \sum_{j=n}^{n+k}\frac{1}{k+1}\left(\frac{j\left(j-1\right)x^{2}}{\left(j+1\right)^{2}}+\frac{2jx}{\left(j+1\right)^{2}}+\frac{1}{3\left(j+1\right)^{2}}\right)-x^{2}\right\Vert _{L_{p}}\\
&\leq\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{j\left(j-1\right)}{\left(j+1\right)^{2}}-1\right)\left\Vert e_{2}(x)\right\Vert _{L_{p}}+\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{2j}{\left(j+1\right)^{2}}\left\Vert e_{1}(x)\right\Vert _{L_{p}}\\
&\quad+\frac{1}{k+1}\sum_{j=1}^{n+k}\frac{1}{3\left(j+1\right)^{2}}\left\Vert e_{0}(x)\right\Vert _{L_{p}}\\
&=\frac{1}{\left(2p+1\right)^{1/p}}\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{j\left(j-1\right)}{\left(j+1\right)^{2}}-1\right)\\
&\quad+\frac{1}{\left(p+1\right)^{1/p}}\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{2j}{\left(j+1\right)^{2}}\right)+\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{1}{3\left(j+1\right)^{2}}\\
&=\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{j}{\left(j+1\right)^{2}}\right)\left(\frac{2}{\left(p+1\right)^{1/p}}-\frac{3}{\left(2p+1\right)^{1/p}}\right)\\
&\quad+\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{1}{\left(j+1\right)^{2}}\right)\left(\frac{1}{3}-\frac{1}{\left(2p+1\right)^{1/p}}\right),
\end{aligned}$$

Since $st-\lim_{k}\left(\sup_{n}\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{j}{(j+1)^{2}}\right)=0$ and $st-\lim_{k}\left(\sup_{n}\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{1}{(j+1)^{2}}\right)=0$, *we have,*

$$st-\lim_{k}\left(\sup_{n}\left\|\sum_{j=1}^{\infty}a_{kj}^{(n)}U_{j}\left(e_{2};x\right)-e_{2}\left(x\right)\right\|_{L_{p}}\right)$$
$$\leq\left(\frac{2}{(p+1)^{1/p}}-\frac{3}{(2p+1)^{1/p}}\right)st-\lim_{k}\left(\sup_{n}\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{j}{(j+1)^{2}}\right)$$
$$+\left(\frac{1}{3}-\frac{1}{(2p+1)^{1/p}}\right)st-\lim_{k}\left(\sup_{n}\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{1}{(j+1)^{2}}\right)$$

which gives

$$st-\lim_{k}\left(\left\|\sum_{j=1}^{\infty}a_{kj}^{(n)}U_{j}e_{2}-e_{2}\right\|_{L_{p}}\right)=0, \textit{uniformly in}\, n.$$

We get

$$st-\lim_{k}\left\|\sum_{j=1}^{\infty}a_{kj}^{(n)}U_{j}e_{2}-e_{2}\right\|_{L_{p}}^{p}=0\,\textit{uniformly in } n.$$

So, our claim (20) holds true for each $i = 0, 1, 2$. $\{U_j\}$ *satisfies all hypothesis of Theorem 1 and we immediately see that,*

$$st-\lim_{k}\left\|\sum_{j=1}^{\infty}a_{kj}^{(n)}U_{j}f-f\right\|_{L_{p}}^{p}=0,\ \textit{uniformly in}\, n\ \textit{on}\ [0,1]\ \textit{for all}\ f\in L_{p}(I).$$

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