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A-distributional summability in topological spaces

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Abstract In the present paper we introduce a new concept of A-distributional convergence in an arbitrary Hausdorff topological space which is equivalent to A-statistical convergence for a degenerate distribution function. We investigate A-distributional convergence as a summability method in an arbitrary Hausdorff topological space. We also study the summability of spliced sequences, in particular, for metric spaces and give the Bochner integral representation of A-limits of the spliced sequences for Banach spaces.

Keywords A-distributional convergence · A-statistical convergence · Hausdorff spaces · Matrix summability

Mathematics Subject Classification (2000) Primary 54A20 · 40J05; Secondary 40A05 · 40G15 · 11B05

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1 Preliminaries

The main motivation of using summability theory has always been to make a non-convergent sequence to converge to a desirable limit [7,16]. This was the idea behind Fejer's theorem showing Cesàro method being effective in making the Fourier series of a continuous periodic function to converge [22]. The classical summability theory has been investigated thoroughly mostly in linear spaces. It has also been studied in some other spaces either by relaxing the linearity condition but keeping the metric structure or relaxing the metric structure and introducing binary addition operation in the context of topological groups [4–6,17–20].

Statistical convergence, although a special case of convergence in measure, is one of the newer methods of summability theory and has been studied for scalar sequences by several authors [2,3,8,10–12,21]. Recently Khan and Orhan [13] provided a matrix characterization of A-statistical convergence on a proper class of sequences, thereby linking this form of summability with the classical form of matrix summability.

Since the classical matrix summability methods cannot be used in arbitrary topological spaces, several authors have, therefore, restricted the scope by assuming either the topological space to have a group structure or a linear structure. One can introduce a summability concept directly into abstract topological spaces through A-statistical convergence [4,9]. More precisely, let X be a topological space and let $A = (a_{nk})$ be a nonnegative regular summability matrix such that each row adds up to one. Then a sequence $x = (x_k)$ in X is said to be A-statistically convergent to $\alpha \in X$ if for any open set U that contains α,

$$\lim_n \sum_{k: x_k \notin U} a_{nk} = 0.$$

As shown in [4] A-statistical convergence is a regular summability method. One can extend this notion by introducing I-convergence in a straightforward way, by allowing any ideal I instead of the ideal of the A-density zero sets. One of the aims of this paper is to explore further properties of statistical convergence as a summability notion for arbitrary topological spaces, similar to the results of the papers by Cakalli and Khan [4] and Maio and Kočinac [9]. Throughout the paper we will assume that X is a Hausdorff topological space with at least two elements, to avoid trivialities.

The second aim of the paper is to use the classical notion of distributional convergence as a summability method and then provide its relationship with A-statistical convergence as well as characterize the resulting limits for spliced sequences. To introduce a limit of a sequence we use a distribution on the Borel sigma field, $\sigma(\tau)$, of subsets of (X, τ), where τ is its topology. Consider a set function $F : \sigma(\tau) \to [0, 1]$ such that $F(X) = 1$ and if $G_1, G_2, \ldots$ are disjoint sets in $\sigma(\tau)$ then

$$F\left(\bigcup_{j=1}^{\infty} G_j\right) = \sum_{j=1}^{\infty} F\left(G_j\right).$$

Such a function is called a probability measure, or a distribution. More precisely, let $A = (a_{nk})$ be a nonnegative regular summability matrix whose each row adds up to one. Let F be a probability measure on $\sigma(\tau)$. Then the sequence $x = (x_k)$ in X is said to be A-distributionally convergent to F if for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$ we have

$$\lim_{n\to\infty} \sum_{k:x_k\in G} a_{nk} = F(G)$$

where ∂G is the boundary of G.

2 *A*-distributional and *A*-statistical convergences

In this section we observe that A-statistical convergence is a very special case of A-distributional convergence in a Hausdorff topological space by giving a characterization for A-statistical convergence. For this purpose we recall the following theorem (see [1], Theorem 2.1).

Theorem A *Let X be a topological space, let $A = (a_{nk})$ be a nonnegative regular summability matrix such that each row adds up to one, let F be a distribution function and let $x = (x_k)$ be a sequence in X. Then the following statements are equivalent:*

i) *x is A-distributionally convergent to F,*

ii) $\limsup_n \sum_{k:x_k\in V} a_{nk} \leq F(V)$ *for all closed subsets V,*

iii) $\liminf_n \sum_{k:x_k\in U} a_{nk} \geq F(U)$ *for all open subsets U.*

The following proposition gives the characterization of A-statistical convergence. For the sake of completeness we give its straightforward proof.

Proposition 1 *Let X be a Hausdorff topological space, let $A = (a_{nk})$ be a nonnegative regular summability matrix such that each row adds up to one and let $x = (x_k)$ be a sequence in X. Then x is A-statistically convergent to $\alpha \in X$ if and only if it is A-distributionally convergent to $F : \sigma(\tau) \to [0, 1]$ defined by*

$$F(G) = \begin{cases} 0, & if\ \alpha \notin G \\ 1, & if\ \alpha \in G. \end{cases}$$

Proof It is easy to see that F is a distribution function. Assume that x is A-statistically convergent to α and let U be an open set.

Case I: If $\alpha \in U$ then $F(U) = 1$ and as x is A-statistically convergent to α we get

$$\lim_{n\to\infty} \sum_{k:x_k\notin U} a_{nk} = 0$$

which implies

$$\liminf_n \sum_{k:x_k\in U} a_{nk} = 1 = F(U).$$

Case II: If $\alpha \notin U$ then $F(U) = 0$. Now we trivially get

$$\liminf_n \sum_{k:x_k\in U} a_{nk} \ \geq \ 0 \ = \ F(U).$$

Hence it follows from Theorem A that x is A-distributionally convergent to F.

Conversely assume that x is A-distributionally convergent to F and let U be an open set that contains α. Then $V := U^c$ is a closed set that does not contain α. Therefore we can write $F(V) = 0$. As V is closed we have $\partial V \subseteq V$ which implies $\alpha \notin \partial V$. Hence we get $F(\partial V) = 0$. Since x is A-distributionally convergent to F and $F(\partial V) = 0$ we have

$$\lim_{n\to\infty} \sum_{k:x_k\in V} a_{nk} \ = \ F(V) \ = \ 0.$$

Hence x is A-statistically convergent to α. □

This proposition shows that A-distributional convergence is more general than A-statistical convergence. Several characterizations of distributional convergence can be found in [1]. To characterize the limits of A-distributional convergence we need the concept of splices.

3 Splices

In [15], Osikiewicz investigated which nonnegative regular matrices will sum complex spliced sequences and to what value. In this section we are concerned with the A-distributional convergence of a spliced sequence in an arbitrary Hausdorff topological space. The next three definitions are given in [15].

Definition 1 Let M be a fixed positive integer. An M-partition of $\mathbb{N}$ consists of infinite sets $K_i = \{\vartheta_i(j)\}$ for $i = 1, 2, ..., M$ such that $\bigcup_{i=1}^{M} K_i = \mathbb{N}$ and $K_i \cap K_j = \varnothing$ for all $i \neq j$, where $\mathbb{N}$ is the set of all positive integers. An ∞-partition on $\mathbb{N}$ consists of a countably infinite number of infinite sets $K_i = \{\vartheta_i(j)\}$ for $i \in \mathbb{N}$ such that $\bigcup_{i=1}^{\infty} K_i = \mathbb{N}$ and $K_i \cap K_j = \varnothing$ for all $i \neq j$.

Definition 2 Let $\{K_i : i = 1, 2, ..., M\}$ be a fixed M-partition of $\mathbb{N}$, let $x^{(i)} = \left(x_j^{(i)}\right)$ be a sequence in X with $\lim_{j\to\infty} x_j = \alpha_i,\ i = 1, 2, ..., M$. If $k \in K_i$, then $k = \vartheta_i(j)$

for some j. Define $x = (x_k)$ by $x_k = x_{\vartheta_i(j)} = x_j^{(i)}$. Then x is called an M-splice over $\{K_i : i = 1, 2, ..., M\}$ with limit points $\alpha_1, \alpha_2, ..., \alpha_M$.

Definition 3 Let $\{K_i : i \in \mathbb{N}\}$ be a fixed ∞-partition of $\mathbb{N}$, let $x^{(i)} = \left(x_j^{(i)}\right)$ be a sequence in X with $\lim_{j\to\infty} x_j = \alpha_i$, $i \in \mathbb{N}$. If $k \in K_i$, then $k = \vartheta_i(j)$ for some j. Define $x = (x_k)$ by $x_k = x_{\vartheta_i(j)} = x_j^{(i)}$. Then x is called an ∞-splice over $\{K_i : i \in \mathbb{N}\}$ with limit points $\alpha_1, \alpha_2, ..., \alpha_M, ...$

Theorem 1 *Let X be a Hausdorff topological space, let $A = (a_{nk})$ be a non-negative regular summability matrix such that each row adds up to one and let $\{K_i = \{\vartheta_i(j)\} : i = 1, 2, ..., M\}$ be an M-partition of $\mathbb{N}$. Then the following statements are equivalent:*

i) $\delta_A(K_i)$ *exists for all* $i = 1, 2, ..., M$.
ii) *There exist* $p_1, p_2, ..., p_M \in [0, 1]$ *such that* $\sum_{i=1}^{M} p_i = 1$ *and any M-spliced sequence over* $\{K_i : i = 1, 2, ..., M\}$ *with limit points* $\alpha_1, \alpha_2, ..., \alpha_M$ *is A-distributionally convergent to the distribution* $F : \sigma(\tau) \to [0, 1]$ *where*

$$F(G) = \sum_{\substack{1\le i\le M \\ \alpha_i\in G}} p_i, \quad \text{for all } G \in \sigma(\tau).$$

iii) *There exist* $p_1, p_2, ..., p_M \in [0, 1]$ *such that* $\sum_{i=1}^{M} p_i = 1$ *and the M-splice of* $x^{(1)}, x^{(2)}, ..., x^{(M)}$ *over* $\{K_i : i = 1, 2, ..., M\}$ *where* $x^{(i)} = (\alpha_i, \alpha_i, ...)$ *being a constant sequence, is A-distributionally convergent to the distribution* $F : \sigma(\tau) \to [0, 1]$ *and*

$$F(G) = \sum_{\substack{1\le i\le M \\ \alpha_i\in G}} p_i, \quad \text{for all } G \in \sigma(\tau).$$

Proof $i \Longrightarrow ii$: Assume that $\delta_A(K_i)$ exists for all $i = 1, 2, ..., M$. Let $p_i = \delta_A(K_i)$ for $i = 1, 2, ..., M$. Since $\{K_i : i = 1, 2, ..., M\}$ is an M-partition of $\mathbb{N}$ we get

$$1 = \sum_{i=1}^{M} \delta_A(K_i) = \sum_{i=1}^{M} p_i.$$

Now let x be any M-splice of $x^{(1)}, x^{(1)}, ..., x^{(M)}$ over $\{K_i : i = 1, 2, ..., M\}$ with some limit points $\alpha_1, \alpha_2, ..., \alpha_M$ and let V be a closed set.

Case I: If $\alpha_i \notin V$ for all $i = 1, 2, ..., M$ then $F(V) = 0$ and we get

$$\sum_{k:x_k\in V} a_{nk} = \sum_{i=1}^{M} \sum_{\substack{k:x_k\in V \\ k\in K_i}} a_{nk}$$
$$= \sum_{i=1}^{M} \sum_{j:x_j^{(i)}\in V} a_{n,\vartheta_i(j)}$$
$$= \sum_{i=1}^{M} \sum_{j:x_j^{(i)}\notin V^c} a_{n,\vartheta_i(j)}.$$

As $x^{(i)}$ is convergent to α_i, the sum $\sum_{j:x_j^{(i)}\notin V^c} a_{n,\vartheta_i(j)}$ is a finite sum for all $i = 1, 2, ..., M$ and since A is regular, $\lim_n a_{n,\vartheta_i(j)} = 0$ for all j. Hence the right hand side of the last equality goes to zero as $n \to \infty$. Therefore we have

$$\limsup_n \sum_{k:x_k\in V} a_{nk} = F(V) = 0.$$

Case II: If $\alpha_{m(1)}, \alpha_{m(2)}, ..., \alpha_{m(S)} \in V$ and $\alpha_{l(1)}, \alpha_{l(2)}, ..., \alpha_{l(R)} \notin V$ where

$$\{m(t)\}_{t=1}^{S} \cup \{l(t)\}_{t=1}^{R} = \{1, 2, ..., M\}, \quad for\ some\ S, R$$

then $F(V) = \sum_{t=1}^{S} \delta_A(K_{m(t)})$ and we get

$$\sum_{k:x_k\in V} a_{nk} = \sum_{t=1}^{S} \sum_{\substack{k:x_k\in V \\ k\in K_{m(t)}}} a_{nk} + \sum_{t=1}^{R} \sum_{\substack{k:x_k\in V \\ k\in K_{l(t)}}} a_{nk}$$
$$= \sum_{t=1}^{S} \sum_{\substack{k:x_k\in V \\ k\in K_{m(t)}}} a_{nk} + \sum_{t=1}^{R} \sum_{j:x_j^{(l(t))}\in V} a_{n,\vartheta_{l(t)}(j)}$$
$$\leq \sum_{t=1}^{S} \sum_{k\in K_{m(t)}} a_{nk} + \sum_{t=1}^{R} \sum_{j:x_j^{(l(t))}\notin V^c} a_{n,\vartheta_{l(t)}(j)}.$$

The first part of the right hand side of the last inequality tends to $\sum_{t=1}^{S} \delta_A(K_{m(t)})$ as $n \to \infty$ and similar to Case I, the sum $\sum_{j:x_j^{(l(t))}\notin V^c} a_{n,\vartheta_{l(t)}(j)}$ is a finite sum for all $t = 1, 2, ..., R$ and since A is regular, $\lim_n a_{n,\vartheta_{l(t)}(j)} = 0$ for all j. Hence the second term of the right hand side of the last inequality goes to zero as $n \to \infty$. Therefore we have

$$\limsup_n \sum_{k:x_k\in V} a_{nk} \leq \sum_{t=1}^{S} \delta_A\left(K_{m(t)}\right) = F(V).$$

Case III: If $\alpha_i \in V$ for all $i = 1, 2, ..., M$ then $F(V) = 1$ and we trivially have

$$\limsup_n \sum_{k:x_k \in V} a_{nk} \leq 1 = F(V).$$

Thus, in all cases we get for any closed set V that

$$\limsup_n \sum_{k:x_k \in V} a_{nk} \leq F(V).$$

Hence from Theorem A we have that x is A-distributionally convergent to F.
$ii \implies iii$: As $x^{(i)} = (\alpha_i, \alpha_i, ...)$ is convergent for all $i = 1, 2, ..., M$ the proof follows immediately.
$iii \implies i$: Assume that x is the M-splice of the sequences $x^{(1)}, x^{(2)}, ..., x^{(M)}$ over $\{K_1, K_2, ..., K_M\}$ where for $i = 1, 2, ...M$, $x^{(i)} = (x_k^{(i)})$ is defined by $x_k^{(i)} = \alpha_i$ for all $k \in \mathbb{N}$. Then from the hypothesis x is A-distributionally convergent to F. As X is a Hausdorff topological space, for a fixed i there exists an open set U_i such that $\alpha_i \in U$ and $\alpha_j \notin U$ for all $j \neq i$. Since $F(U_i) = p_i$ we also get from Theorem A that

$$\liminf_n \sum_{k:x_k \in U_i} a_{nk} \geq p_i$$

which implies

$$\liminf_n \sum_{k \in K_i} a_{nk} \geq p_i. \tag{3.1}$$

Now let $V_i = \{\alpha_i\}$. Since X is a Hausdorff topological space V_i is closed and as $\alpha_i \in V_i$ and $\alpha_j \notin V_i$ for all $j \neq i$ then $F(V) = p_i$. Here we use that X has at least two distinct points, and that α_i's are distinct. Hence by Theorem A we get

$$\limsup_n \sum_{k:x_k \in V_i} a_{nk} \leq p_i$$

which implies

$$\limsup_n \sum_{k \in K_i} a_{nk} \leq p_i. \tag{3.2}$$

Thus we get from (3.1) and (3.2) that

$$p_i \leq \liminf_n \sum_{k \in K_i} a_{nk} \leq \limsup_n \sum_{k \in K_i} a_{nk} \leq p_i$$

which implies $\delta_A(K_i)$ exists and is equal to p_i. □

It is a well known fact that a density introduced by a regular summability matrix does not have sigma additivity property. The next result deals with the sigma additivity of densities of an infinite partition. Its proof will need the result of the last theorem.

Theorem 2 *Let X be a Hausdorff topological space, let $A = (a_{nk})$ be a non-negative regular summability matrix such that each row adds up to one and let $\{K_i = \{\vartheta_i(j)\} : i \in \mathbb{N}\}$ be an ∞-partition of $\mathbb{N}$. Then $\delta_A(K_i)$ exists for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \delta_A(K_i) = 1$ if and only if there exist $p_i \in [0, 1]$ for $i \in \mathbb{N}$ such that $\sum_{i=1}^{\infty} p_i = 1$ and any ∞-splice sequence over $\{K_i : i \in \mathbb{N}\}$ with limit points $\alpha_1, \alpha_2, ...$ is A-distributionally convergent to the distribution $F : \sigma(\tau) \to [0, 1]$ where*

$$F(G) = \sum_{\alpha_i \in G} p_i, \quad for\ all\ G \in \sigma(\tau).$$

Proof Assume that $\delta_A(K_i)$ exists for all $i \in \mathbb{N}$. Take $p_i = \delta_A(K_i)$ for $i \in \mathbb{N}$ so that

$$1 = \sum_{i=1}^{\infty} \delta_A(K_i) = \sum_{i=1}^{\infty} p_i.$$

Now let x be any ∞-splice of $x^{(1)}, x^{(2)}, ...$ over $\{K_i : i \in \mathbb{N}\}$ with limit points $\alpha_1, \alpha_2, ...$ and let V be a closed set.

Case I: If $\alpha_i \notin V$ for all $i \in \mathbb{N}$ then $F(V) = 0$ and we get

$$\sum_{k:x_k \in V} a_{nk} = \sum_{i=1}^{\infty} \sum_{\substack{k:x_k \in V \\ k \in K_i}} a_{nk} = \sum_{i=1}^{\infty} \sum_{j:x_j^{(i)} \notin V^c} a_{n,\vartheta_i(j)}.$$

Let $f_n(i) := \sum_{j:x_j^{(i)} \notin V^c} a_{n,\vartheta_i(j)}$ and $g_n(i) := \sum_{k \in K_i} a_{nk}$. Then for any $i \in \mathbb{N}$

$$g(i) := \lim_n g_n(i) = \lim_n \sum_{k \in K_i} a_{nk} = \delta_A(K_i).$$

If μ represents the counting measure over $\mathbb{N}$, we have as in [15]

$$\lim_n \int_{\mathbb{N}} g_n(i) d\mu = \lim_n \sum_{i=1}^{\infty} \sum_{k \in K_i} a_{nk} = \lim_n \sum_{k=1}^{\infty} a_{nk}$$
$$= 1 = \sum_{k=1}^{\infty} \delta_A(K_i) = \sum_{k=1}^{\infty} g(i) = \int_{\mathbb{N}} g(i) d\mu. \tag{3.3}$$

And for any $n \in \mathbb{N}$

$$|f_n(i)| = \sum_{j: x_j^{(i)} \notin V^c} a_{n,\vartheta_i(j)} = \sum_{\substack{k: x_k \notin V^c \\ k \in K_i}} a_{nk} \le \sum_{k \in K_i} a_{nk} = g_n(i). \tag{3.4}$$

Then (3.3) and (3.4) and the Lebesgue Dominated Convergence Theorem yield

$$\begin{aligned} \lim_n \sum_{k: x_k \in V} a_{nk} &= \lim_n \sum_{i=1}^{\infty} \sum_{j: x_j^{(i)} \notin V^c} a_{n,\vartheta_i(j)} \\ &= \sum_{i=1}^{\infty} \lim_n \sum_{j: x_j^{(i)} \notin V^c} a_{n,\vartheta_i(j)}. \end{aligned} \tag{3.5}$$

Since $x^{(i)}$ is convergent to α_i the sum $\sum_{j: x_j^{(i)} \notin V^c} a_{n,\vartheta_i(j)}$ consists of finitely many terms, and since A is regular, $\lim_n a_{n,\vartheta_i(j)} = 0$ for all j. Hence from (3.5) we get

$$\lim_n \sum_{k: x_k \in V} a_{nk} = 0$$

which implies

$$\limsup_n \sum_{k: x_k \in V} a_{nk} = 0 = F(V).$$

Case II: If $\alpha_{m(1)}, \alpha_{m(2)}, \ldots \in V$ and $\alpha_{l(1)}, \alpha_{l(2)}, \ldots \notin V$ where

$$\{m(t)\}_{t=1}^{\infty} \cup \{l(t)\}_{t=1}^{\infty} = \mathbb{N}$$

then $F(V) = \sum_{t=1}^{\infty} \delta_A(K_{m(t)})$ and we get

$$\begin{aligned} \sum_{k: x_k \in V} a_{nk} &= \sum_{t=1}^{\infty} \sum_{\substack{k: x_k \in V \\ k \in K_{m(t)}}} a_{nk} + \sum_{t=1}^{\infty} \sum_{\substack{k: x_k \in V \\ k \in K_{l(t)}}} a_{nk} \\ &= \sum_{t=1}^{\infty} \sum_{\substack{k: x_k \in V \\ k \in K_{m(t)}}} a_{nk} + \sum_{t=1}^{\infty} \sum_{j: x_j^{(l(t))} \in V} a_{n,\vartheta_{l(t)}(j)} \\ &\le \sum_{t=1}^{\infty} \sum_{k \in K_{m(t)}} a_{nk} + \sum_{t=1}^{\infty} \sum_{j: x_j^{(l(t))} \notin V^c} a_{n,\vartheta_{l(t)}(j)}. \end{aligned} \tag{3.6}$$

Let $f_n(t) := \sum_{k\in K_{m(t)}} a_{nk}$ and $g_n(t) := \sum_{k\in K_t} a_{nk}$. Then for any $t \in \mathbb{N}$,

$$g(t) := \lim_n g_n(t) = \lim_n \sum_{k\in K_t} a_{nk} = \delta_A(K_t).$$

If μ represents the counting measure, as in Case I we have

$$\lim_n \int_{\mathbb{N}} g_n(t)d\mu = \int_{\mathbb{N}} g(t)d\mu.$$

Since

$$|f_n(t)| = \sum_{k\in K_{m(t)}} a_{nk} \le \sum_{k\in K_{m(t)}} a_{nk} + \sum_{k\in K_{l(t)}} a_{nk} = \sum_{k\in K_t} a_{nk} = g_n(t)$$

we have from the Lebesgue Dominated Convergence Theorem that

$$\begin{aligned}\lim_n \sum_{t=1}^{\infty} \sum_{k\in K_{m(t)}} a_{nk} &= \sum_{t=1}^{\infty} \lim_n \sum_{k\in K_{m(t)}} a_{nk} \\ &= \sum_{t=1}^{\infty} \delta_A(K_{m(t)}).\end{aligned}$$

Also as in Case I using the Lebesgue Dominated Convergence Theorem, convergence of $x^{(i)}$ and the regularity of A it is easy to see that the second part of the right hand side of inequality (3.6) goes to zero as $n \to \infty$. Therefore we have

$$\limsup_n \sum_{k:x_k\in V} a_{nk} \le \sum_{t=1}^{\infty} \delta_A\left(K_{m(t)}\right) = F(V).$$

Case III: If $\alpha_i \in V$ for all $i \in \mathbb{N}$ then $F(V) = 1$ and as in Theorem 1 we have that

$$\limsup_n \sum_{k:x_k\in V} a_{nk} \le F(V).$$

Thus we get for any closed subset V that

$$\limsup_n \sum_{k:x_k\in V} a_{nk} \le F(V).$$

Hence from Theorem A we have that x is A-distributionally convergent to F.

To prove sufficiency let M be a fixed positive integer, let $\alpha_i \in X, i = 1, 2, ..., M$, let $x^{(i)}$ be a convergent sequence to α_i, $i = 1, 2, ..., M-1$ and let $x^{(i)} = (\alpha_M, \alpha_M, ...)$, $i = M, M+1...$, where α_M is distinct from α_i, $i < M$. Then from the hypotheses the sequence x that is the ∞-splice of $x^{(1)}, x^{(2)}, ...$ over $\{K_i : i \in \mathbb{N}\}$ is A-distributionally convergent to the distribution $F_M : \sigma(\tau) \to [0, 1]$ where for all $G \in \sigma(\tau)$

$$F_M(G) = \sum_{\alpha_i \in G} p_i.$$

On the other hand we have

$$\sum_{\alpha_i \in G} p_i = \begin{cases} \sum_{\substack{1 \le i \le M-1 \\ \alpha_i \in G}} p_i, & \alpha_M \notin G, \\ \sum_{\substack{1 \le i \le M-1 \\ \alpha_i \in G}} p_i + \sum_{i=M}^{\infty} p_i, & \alpha_M \in G \end{cases}$$
$$= \sum_{\substack{1 \le i \le M-1 \\ \alpha_i \in G}} p_i + I_{\{\alpha_M \in G\}} p_M^*.$$

where $p_M^* = 1 - \sum_{i=1}^{M-1} p_i$. Now define a finite partition of $\mathbb{N}$ as

$$\left\{ K_1, K_2, ..., K_{M-1}, K = \bigcup_{j=M}^{\infty} K_j \right\}.$$

Then the M-splice of $x^{(1)}, x^{(2)}, ..., x^{(M)}$ over $\{K_1, K_2, ..., K_{M-1}, K\}$ is again the sequence x. Since x is A-distributionally convergent to F_M, from Theorem 1 we have for all $i = 1, 2, ..., M-1$ that $\delta_A(K_i)$ exists and is equal to p_i. As M is arbitrary $\delta_A(K_i)$ exists for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \delta_A(K_i) = 1$. □

4 Summability in metric spaces

In this section we will give some summability results in metric spaces, and in particular Banach spaces. Due to the concept of a distance, we can introduce the classical matrix summability structure applied to the distances. The resulting type of convergence is called the strong summability.

Definition 4 Let (X, d) be a metric space and let $A = (a_{nk})$ be a nonnegative summability matrix. Then a sequence $x = (x_k)$ in X is said to be A-uniformly integrable if

$$\lim_{c \to \infty} \sup_n \sum_{k : d(x_k, \alpha) \ge c} d(x_k, \alpha) a_{nk} = 0$$

for some $\alpha \in X$. Note that we can replace the statement "for some $\alpha \in X$" with "for any $\alpha \in X$".

Definition 5 Let (X, d) be a metric space and let $A = (a_{nk})$ be a nonnegative summability matrix. Then a sequence $x = (x_k)$ in X is said to be A-strongly convergent to $\alpha \in X$ if

$$\lim_n \sum_{k=1}^{\infty} d(x_k, \alpha) a_{nk} = 0.$$

Definition 6 Let $(X, \|.\|)$ be a normed space, let $A = (a_{nk})$ be an infinite matrix and let $x = (x_k)$ be a sequence in X. If $Ax := \left\{(Ax)_n\right\}$ exists and is $convergent$ to L then we say that x is $A - summable$ to L where $for\ all\ n = 1, 2, ...$

$$(Ax)_n := \sum_k x_k a_{nk}.$$

We also say that Ax is the A-transformation of x and L is the A-limit of x.

Recently Khan and Orhan [14] have given a characterization for A-strong convergence of sequences by proving that a sequence is A-strongly convergent if and only if it is A-statistically convergent and A-uniformly integrable. The same result also holds in a metric space. In particular, this result and Proposition 1 give the following corollary in normed spaces immediately.

Corollary 1 *Let* $(X, \|.\|)$ *be a normed space, let* $A = (a_{nk})$ *be a nonnegative regular summability matrix such that each row adds up to one, let* $x = (x_k)$ *be a sequence in* X *and let* $\alpha \in X$. *Then the following statements are equivalent:*

i) *x is A-uniformly integrable and A-distributionally convergent to the distribution* $F : \sigma(\tau) \to [0, 1]$ *where F is defined by*

$$F(G) = \begin{cases} 0, & if\ \alpha \notin G \\ 1, & if\ \alpha \in G. \end{cases}$$

ii) *x is A-statistically convergent to α and A-uniformly integrable.*
iii) *x is A-strongly convergent to α.*

Furthermore any one of these three statements implies Ax exists and converges to α.

Our next result characterizes the limit L of A-transformation for some A-distributionally convergent sequences over Banach spaces. In the following $\{K_i = \{\vartheta_i(j)\} : i = 1, 2, ..., \}$ will stand for an ∞-partition of $\mathbb{N}$, for which the A-densities $\delta_A(K_i)$ exist for all i and $\sum_{i=1}^{\infty} \delta_A(K_i) = 1$. Consider a normed linear space $(X, \|\cdot\|)$ and let x be any bounded ∞-spliced sequence in X over the partition. It is not difficult to show that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} x_k\, a_{nk} \; = \; \sum_{i=1}^{\infty} \alpha_i\ \delta_A(K_i),$$

where $\alpha_1, \alpha_2, \ldots$ are the respective limit points of the ∞-spliced sequence. The following result shows that, when X is a Banach space, the limit is naturally linked to A-distributional convergence via Bochner integrals.

Proposition 2 *Let $(X, \|\cdot\|)$ be a Banach space, let $A = (a_{nk})$ be a nonnegative regular summability matrix such that each row adds up to one and let $\{K_i = \{\vartheta_i(j)\} : i \in \mathbb{N}\}$ be an ∞-partition of $\mathbb{N}$. If $\delta_A(K_i)$ exists for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \delta_A(K_i) = 1$ then for any bounded ∞-spliced sequence $x = (x_k)$ over $\{K_i : i \in \mathbb{N}\}$*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} x_k a_{nk} = \int_X t dF \tag{4.1}$$

where F is a distribution defined by

$$F(G) = \sum_{\alpha_i \in G} \delta_A(K_i), G \in \sigma(\tau).$$

and the integral in (4.1) is the Bochner integral.

Proof Let $f : X \to X$ be the identity and let $s : X \to X$ be defined by

$$s(t) = \begin{cases} \alpha_i, & t = \alpha_i,\ i \in \mathbb{N} \\ \theta, & otherwise. \end{cases}$$

We will show that $\int_X s\, dF$ exists and equals (4.1). Observe that $f = s$ almost everywhere with respect to F. Thus we have

$$\int_X t dF = \int_X s(t) dF. \tag{4.2}$$

Consider a sequence of simple functions (s_m),

$$s_m(t) = \begin{cases} \alpha_i, & t = \alpha_i,\ i = 1, 2, ..., m \\ \theta, & otherwise. \end{cases}$$

It is easy to see that for all m

$$\|s_m(t) - s(t)\| = \begin{cases} \|\alpha_i\|, & t = \alpha_i,\ i > m \\ 0, & otherwise. \end{cases}$$

Thus for all $t \in X$, $\lim_{m\to\infty} \|s_m(t) - s(t)\| = 0$. On the other hand since the spliced sequence is bounded there exists an $H > 0$ such that

$$\sup_{t\in X} \|s_m(t) - s(t)\| \leq \sup_{i>m} \|\alpha_i\| < H.$$

Then from the Bounded Convergence Theorem we have

$$\lim_{m\to\infty}\int\limits_X \|s_m(t)-s(t)\|\,dF=\int\limits_X \lim_{m\to\infty}\|s_m(t)-s(t)\|\,dF=0$$

which implies

$$\begin{aligned}\int\limits_X s(t)dF &= \lim_{m\to\infty}\int\limits_X s_m(t)dF\\ &= \lim_{m\to\infty}\int\limits_X \left(\sum_{i=1}^{m} I_{\{\alpha_i\}}(t)\alpha_i\right)dF\\ &= \lim_{m\to\infty}\sum_{i=1}^{m} F\left(\{\alpha_i\}\right)\alpha_i\\ &= \lim_{m\to\infty}\sum_{i=1}^{m}\delta_A(K_i)\alpha_i\\ &= \sum_{i=1}^{\infty}\delta_A(K_i)\alpha_i. \end{aligned}\tag{4.3}$$

Hence from (4.2) and (4.3) we get that

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}x_k a_{nk} \;=\; \int\limits_X t\,dF.$$

□

One can relax the assumption of boundedness of the ∞-spliced sequence a bit by introducing A-uniform integrability.

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