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Domination by ergodic elements in ordered Banach algebras

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Abstract We recall the definition and properties of an algebra cone in an ordered Banach algebra (OBA) and continue to develop spectral theory for the positive elements. An element a of a Banach algebra is called *ergodic* if the sequence of sums $\sum_{k=0}^{n-1} \frac{a^k}{n}$ converges. If a and b are positive elements in an OBA such that $0 \leq a \leq b$ and if b is ergodic, an interesting problem is that of finding conditions under which a is also ergodic. We will show that in a semisimple OBA that has certain natural properties, the condition we need is that the spectral radius of b is a Riesz point (relative to some inessential ideal). We will also show that the results obtained for OBAs can be extended to the more general setting of commutatively ordered Banach algebras (COBAs) when adjustments corresponding to the COBA structure are made.

Keywords Ordered Banach algebra · Commutatively ordered Banach algebra · Positive element · Spectrum · Ergodic element

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1 Introduction

In [3,4,6–12,14,16], spectral theory for positive elements in ordered Banach algebras (OBAs) was developed. The results in these papers were extended to the more general setting of commutatively ordered Banach algebras (COBAs) in [13]. Some

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of the results in [3,7,14,16] were related to the following problem: if a and b are positive elements in an OBA such that $0 \leq a \leq b$, under what conditions are certain properties of b inherited by a? This is a classical problem for positive operators on Banach lattices; a survey of some of the results for the problem is given in [18, Chapter 18]. An element a of a Banach algebra is said to be *ergodic* if the sequence of sums $\sum_{k=0}^{n-1} \frac{a^k}{n}$ converges. In this paper we will obtain results giving conditions under which a dominated positive element in an OBA or COBA is ergodic, given that the dominating element is ergodic. A corresponding problem for uniformly ergodic operators on Banach lattices was studied in [15].

In Sect. 2 we provide some preliminary notation and definitions and in Sect. 3 we review some of the main concepts about OBAs and COBAs. In Sect. 4 the main aim is to establish the ergodic theorem (Theorem 4.10), which will serve as a crucial tool in proving the main results in Sect. 5. This result is a Banach algebra version of part of Dunford's result in [5, Theorem 3.16]. To prove Theorem 4.10 several Banach algebra results, which are of interest in themselves, will be obtained. A number of these results are Banach algebra versions of results in [5] (see Theorems 2.19, 2.20, 2.21, 3.6). While these are typically operator theoretic results, we will obtain ours by purely algebraic means. In Sect. 5 we show that under conditions similar to those of [15, Theorem 4.5], a dominated positive element in an OBA or COBA is ergodic if the dominating element is ergodic. The main results are Theorems 5.1, 5.2 and especially Theorem 5.5.

2 Preliminaries

Throughout A will be a complex Banach algebra with unit 1. The spectrum and spectral radius of an element a in A will be denoted by $\sigma(a)$ and $r(a)$, respectively. The set of isolated points of $\sigma(a)$ will be denoted by **iso** $\sigma(a)$. For any $\alpha \in$ **iso** $\sigma(a)$, we will denote by $p(a, \alpha)$ the spectral idempotent corresponding to a and α. A point $\alpha \in \sigma(a)$ is said to be a *pole* of order k of the resolvent function $\lambda \mapsto R(\lambda, a) = (\lambda 1 - a)^{-1}$, $\lambda \in \mathbb{C} \setminus \sigma(a)$, if $\alpha \in$ **iso** $\sigma(a)$ and k is the smallest natural number such that $(\alpha 1 - a)^k p(a, \alpha) = 0$.

In this paper all the ideals will be assumed to be two-sided. Let F be an ideal in A. A point $\alpha \in$ **iso** $\sigma(a)$ is said to be a *Riesz point* relative to F if the corresponding spectral projection $p(a, \alpha)$ belongs to F. An ideal I in A is called *inessential* whenever the spectrum in A of every element in I is either finite or a sequence converging to zero. We will need the following characterization of Riesz points in a semisimple Banach algebra:

Lemma 2.1 ([10], *Lemma 2.1) Let A be a semisimple Banach algebra, I an inessential ideal of A, and $a \in A$. Then a point α in $\sigma(a)$ is a Riesz point of $\sigma(a)$ relative to I if and only if α is a pole of the resolvent of a and $p(a, \alpha) \in I$.*

A point $\alpha \in \sigma(a)$ is called an *eigenvalue* of a if there exists a $0 \neq u \in A$ such that $au = \alpha u$ or $ua = \alpha u$. Then u is said to be an *eigenvector* corresponding to α.

For part of the material in Sect. 3 we will need to recall the following. Let E be a complex Banach lattice and denote by $\mathcal{L}(E)$ the space of bounded linear operators on E. An operator $T : E \to E$ is *regular* if it can be written as a linear combination over $\mathbb{C}$ of positive operators. The space of all regular operators on E is denoted by $\mathcal{L}^r(E)$ and it is a subspace of $\mathcal{L}(E)$. When $\mathcal{L}^r(E)$ is provided with the r-norm

$$||T||_r = \inf\{||S|| : S \in \mathcal{L}(E), |Tx| \leq S|x| \quad \text{for all } x \in E\},$$

it becomes a Banach algebra which contains the unit of $\mathcal{L}(E)$ ([17, IV §1] and [1]). We denote by $\mathcal{K}^r(E)$ the closure in $\mathcal{L}^r(E)$ of the ideal of finite rank operators on E. This is a closed, inessential ideal.

3 Ordered Banach algebras

In [16, Sect. 3] an algebra cone C of a Banach algebra A was defined and it was shown that C induces an ordering on A which is compatible with the algebraic structure of A. Such a Banach algebra is called an ordered Banach algebra (OBA). In [13, Sect. 3], we defined an algebra c-cone, which is more general than an algebra cone, and showed that a Banach algebra can be ordered by an algebra c-cone, and is then called a commutatively ordered Banach algebra (COBA). We now recall those definitions and the additional properties that algebra cones and algebra c-cones may have.

A nonempty subset C of a Banach algebra A is called a *cone* if C satisfies the following:

(i) $C + C \subseteq C$,
(ii) $\lambda C \subseteq C$ for all $\lambda \geq 0$.

If C also satisfies the property $C \cap -C = \{0\}$, then it is called a *proper* cone. We say that C is *closed* if it is a closed subset of A.

Every cone C in a Banach algebra A induces an ordering $\leq$ defined by $a \leq b$ if and only if $b - a \in C$, for $a, b \in A$. This ordering is reflexive and transitive. In addition, C is proper if and only if the ordering is antisymmetric. In view of the fact that C induces an ordering on A, we find that $C = \{a \in A : a \geq 0\}$. Therefore the elements of C are called *positive.*

A cone C in a Banach algebra A is called an *algebra cone* if it satisfies the following:

(i) $ab \in C$ for all $a, b \in C$,
(ii) $1 \in C$, where 1 is the unit of A.

Following [13], we call a cone C that satisfies the weaker conditions $ab \in C$ for all $a, b \in C$ such that $ab = ba$ and $1 \in C$ an *algebra c-cone.* Obviously, every algebra cone is an algebra c-cone.

A Banach algebra ordered by an algebra cone (algebra c-cone) is called an *ordered Banach algebra* (OBA) (*commutatively ordered Banach algebra* (COBA)). Clearly, every OBA is a COBA. We will denote by (A, C) a Banach algebra A ordered by an algebra cone (algebra c-cone) C.

The following result, which can easily be established by induction, plays an important role in Sect. 5:

Proposition 3.1 *Let* (A, C) *be an OBA and let* $a, b \in A$. *If* $0 \leq a \leq b$, *then* $0 \leq a^n \leq b^n$ *for any* $n \in \mathbb{N}$.

In order to obtain the COBA analogue of Proposition 3.1, the additional assumption $ab = ba$ is required. As the following counter example shows, this condition cannot be dropped.

Example 3.2 ([13], Example 3.13) Let $A = M_2(\mathbb{C})$ and $C = \{a \in A : a = a^* \text{ and } \sigma(a) \subseteq [0, \infty)\}$, where a^* denotes the complex conjugate transpose of a. Then C is a closed algebra c-cone of A. If $a = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \in A$, then $0 \leq a \leq b$ but $a^2 \leq b^2$ does not hold.

The following result, which can be deduced from the proof of [14, Theorem 3.2], will also be a useful tool.

Proposition 3.3 *Let* A *be an OBA with a closed algebra cone* C *and let* $0 \neq a \in C$ *such that* $r(a) > 0$. *If* $r(a)$ *is a simple pole of the resolvent of* a, *then* $p = p(a, r(a)) \in C$, $ap = pa = r(a)p$ *and* $apa = r(a)^2 p$.

The result in Proposition 3.3 also holds in a COBA and has the same proof.

An algebra cone or algebra c-cone C is said to be *inverse-closed* if it has the property that if $a \in C$ and a is invertible, then $a^{-1} \in C$. We will need the following result, which is a COBA analogue of [4, Proposition 4.2] and has the same proof:

Proposition 3.4 *Let* (A, C) *be a COBA with* C *closed and inverse-closed. If* $a \in C$, *then* $0 \leq a \leq r(a)1$.

Suppose that (A, C) is an OBA or COBA. We say that C is *normal* if there exists a scalar $\alpha > 0$ such that $0 \leq a \leq b$ relative to C implies that $||a|| \leq \alpha||b||$. If C has the weaker property that there exists a scalar $\alpha > 0$ such that $0 \leq a \leq b$ and $ab = ba$ imply that $||a|| \leq \alpha||b||$, then C is said to be *c-normal*. Clearly, every normal algebra cone or c-cone is c-normal. Also, it can easily be shown that a normal or c-normal algebra cone or c-cone is necessarily proper.

If $0 \leq a \leq b$ relative to C implies that $r(a) \leq r(b)$ then we say that the spectral radius is *monotone* w.r.t. C. If we have the weaker property that $0 \leq a \leq b$ relative to C and $ab = ba$ imply that $r(a) \leq r(b)$, then we say that the spectral radius is *c-monotone* w.r.t. C. Obviously, monotonicity implies c-monotonicity.

It is well known that if C is a normal algebra cone in a Banach algebra A, then the spectral radius in (A, C) is monotone (see [16, Theorem 4.1]). Similarly, if C is a c-normal algebra c-cone in a Banach algebra A, then the spectral radius is c-monotone in (A, C) (see [13, Theorem 4.2]).

If (A, C) is an OBA, F a closed ideal in A and $\pi : A \to A/F$ the canonical homomorphism, then $(A/F, \pi C)$ is an OBA. However, if (A, C) is a COBA, πC is in general only a cone and not an algebra c-cone. Therefore although A/F can be ordered by πC in the usual way, the structure $(A/F, \pi C)$ is not a COBA. The cone πC however does have the property that it contains the unit of A/F as well as all powers of elements of πC. Such a cone is called an *algebra c'-cone*. A Banach algebra ordered by an algebra c'-cone is called a C'OBA. Obviously, every COBA is a C'OBA.

We will say that the spectral radius in an OBA or C'OBA $(A/F, \pi C)$ is monotone if $0 \leq a \leq b$ in A relative to C implies that $r(a + F, A/F) \leq r(b + F, A/F)$. If we have the weaker property that $0 \leq a \leq b$ relative to C and $ab = ba$ imply that $r(a + F, A/F) \leq r(b + F, A/F)$, we will say that the spectral radius in $(A/F, \pi C)$ is c-monotone.

The following are examples of an OBA, COBA and C'OBA.

Example 3.5 ([10], Example 3.2) Let E be a Dedekind complete Banach lattice, $C = \{x \in E : x \geq 0\}$ and $K = \{T \in \mathcal{L}(E) : TC \subseteq C\}$. Then $(\mathcal{L}^r(E), K)$ is an OBA with a closed, normal algebra cone and $(\mathcal{L}^r(E)/\mathcal{K}^r(E), \pi K)$ is an OBA such that the spectral radius in $(\mathcal{L}^r(E)/\mathcal{K}^r(E), \pi K)$ is monotone.

Example 3.6 ([13], Examples 3.2 and 3.17) Let A be a C^*-algebra and $C = \{a \in A : a = a^* \text{ and } \sigma(a) \subseteq [0, \infty)\}$. Then C is a closed, normal algebra c-cone of A. Therefore (A, C) is a COBA. Suppose that F is a closed ideal in A and for each $a + F$ in A/F, define $(a + F)^* = a^* + F$. Then πC is a normal algebra c'-cone in A/F. Therefore $(A/F, \pi C)$ is a C'OBA.

If A in Example 3.6 is commutative, then (A, C) is an OBA.

For more examples of OBAs and COBAs see [3,7–11,13,16].

We will need the following result, which follows immediately from [10, Theorem 4.4].

Corollary 3.7 *Let (A, C) be an OBA with C closed and the spectral radius in (A, C) monotone. Let I be a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a, b \in A$ with $0 \leq a \leq b$ and $r(a) = r(b)$. If $r(b)$ is a Riesz point of $\sigma(b)$, then $r(a)$ is a Riesz point of $\sigma(a)$.*

If (A, C) is a COBA, we can establish the analogue of Corollary 3.7 with the weaker property of c-monotonicity in (A, C) and in the C'OBA $(A/I, \pi C)$, provided that $ab = ba$ is also assumed. If we replace c-monotonicity with monotonicity in the C'OBA $(A/I, \pi C)$, then the condition $ab = ba$ can be dropped.

4 The ergodic theorem

Let (f_n) be the sequence of functions defined by $f_n(\lambda) = \sum_{k=0}^{n-1} \frac{\lambda^k}{n}$ $(\lambda \in \mathbb{C})$. If a is an element of a Banach algebra, then the terms of the sequence $(f_n(a))$ are called *ergodic sums* of a, and a is said to be *ergodic* if its sequence of ergodic sums converges.

The aim of this section is to establish the ergodic theorem (Theorem 4.10), which will be useful in the proofs of results in the next section. This result is a generalization of part of a result of Dunford ([5, Theorem 3.16]) to Banach algebras. The results from Theorem 4.1 through Proposition 4.9, some of which are Banach algebra versions of results in [5], lead to Theorem 4.10. The proofs of Dunford's results rely partly on operator theory, while our proofs are completely algebraic.

We start with the following theorem, which gives conditions under which a Banach algebra element obtained through the holomorphic functional calculus will be the zero element.

Theorem 4.1 *Let A be a Banach algebra and let $a \in A$. Suppose that f is a complex valued function analytic on a neighbourhood Ω of $\sigma(a)$ such that*

(i) *for every pole λ of the resolvent of a of order k, $f^{(j)}(\lambda) = 0$ $(j = 0, 1, \ldots, k-1)$ and*
(ii) *$\sigma(a)$ contains at least one pole λ_1 of the resolvent of a, and there exists a neighbourhood U of $\sigma(a)\backslash\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for some $n \geq 1$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are poles of the resolvent of a, such that $f(\lambda) = 0$ for all $\lambda \in U$.*

Then $f(a) = 0$.

Proof If Γ is a smooth contour surrounding $\sigma(a)$, then $f(a) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda 1 - a)^{-1}d\lambda$. We may take Γ to be the union of Γ_1 and $C_1, C_2, \ldots, C_n$, where Γ_1 is a smooth contour contained in U and surrounding $\sigma(a)\backslash\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, and C_i is a small circle centered at the pole λ_i of the resolvent of a and separating λ_i from the rest of $\sigma(a)$. Therefore

$$f(a) = \frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)(\lambda 1 - a)^{-1}d\lambda + \sum_{i=1}^{n} \frac{1}{2\pi i}\int_{C_i} f(\lambda)(\lambda 1 - a)^{-1}d\lambda.$$

From assumption (ii) we have that $\frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)(\lambda 1 - a)^{-1}d\lambda = 0$. Now, since $f^{(j)}(\lambda_i) = 0$ for $i = 1, 2, \ldots, n$ and for $j = 0, 1, \ldots, k_i - 1$ (with k_i the order of the pole λ_i) by assumption (i), there exist analytic functions g_i $(i = 1, 2, \ldots, n)$ on Ω such that $f(\lambda) = (\lambda - \lambda_i)^{k_i} g_i(\lambda)$ on a neighbourhood of λ_i including C_i. Since λ_i is a pole of order k_i of the resolvent of a, this resolvent has a Laurent series expansion $(\lambda 1 - a)^{-1} = \frac{a_{-k_i}}{(\lambda-\lambda_i)^{k_i}} + \frac{a_{-k_i+1}}{(\lambda-\lambda_i)^{k_i-1}} + \cdots + a_0 + a_1(\lambda - \lambda_i) + \cdots$, so that $f(\lambda)(\lambda 1 - a)^{-1} = (\lambda - \lambda_i)^{k_i} g_i(\lambda)(\lambda 1 - a)^{-1} = g_i(\lambda)[a_{-k_i} + a_{-k_i+1}(\lambda - \lambda_i) + \cdots + a_0(\lambda - \lambda_i)^{k_i} + a_1(\lambda - \lambda_i)^{k_i+1} + \cdots]$, on a deleted neighbourhood of λ_i which includes C_i. Since g_i is analytic on a neighbourhood of $\sigma(a)$, it has no singularities on or inside C_i, and so $f(\lambda)(\lambda 1 - a)^{-1}$ has no singularities on or inside C_i. It follows from Cauchy's theorem that $\frac{1}{2\pi i}\int_{C_i} f(\lambda)(\lambda 1 - a)^{-1}d\lambda = 0$ for $i = 1, 2, \ldots, n$. Hence $f(a) = 0$. □

The following result is an immediate consequence of Theorem 4.1.

Corollary 4.2 *Let A be a Banach algebra and let $a \in A$. Suppose that f, g are complex valued functions analytic on a neighbourhood of $\sigma(a)$ such that*

(i) *for every pole λ of the resolvent of a of order k, $f^{(j)}(\lambda) = g^{(j)}(\lambda)$ $(j = 0, 1, \ldots, k-1)$ and*
(ii) *$\sigma(a)$ contains at least one pole λ_1 of the resolvent of a, and there exists a neighbourhood U of $\sigma(a)\backslash\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for some $n \geq 1$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are poles of the resolvent of a, such that $f(\lambda) = g(\lambda)$ for all $\lambda \in U$.*

Then $f(a) = g(a)$.

The following theorem is a very important application of Corollary 4.2, and forms the basis for the rest of our work.

Theorem 4.3 *Let A be a Banach algebra and let $a \in A$. Suppose that f is a complex valued function analytic on a neighbourhood of $\sigma(a)$. If α is a pole of order k of the resolvent of a, then $f(a) = f(a)(1 - p) + \sum_{n=0}^{k-1} \frac{(a-\alpha 1)^n}{n!} f^{(n)}(\alpha)p$, where $p = p(a, \alpha)$.*

Proof Since α is an isolated point in $\sigma(a)$, we can take two open sets U_0 and U_1 such that $\sigma(a)\backslash\{\alpha\} \subseteq U_0$, $\{\alpha\} \subseteq U_1$, $U_0 \cap U_1 = \emptyset$ and f is analytic on $U = U_0 \cup U_1$. Let $\chi : U \to \mathbb{C}$ be the function defined by $\chi(\lambda) = 0$ if $\lambda \in U_0$ and $\chi(\lambda) = 1$ if $\lambda \in U_1$. Then χ is analytic on U and $p = \chi(a)$. Now let $g : U \to \mathbb{C}$ be the function defined by $g(\lambda) = f(\lambda)(1 - \chi(\lambda)) + \sum_{n=0}^{k-1} \frac{(\lambda-\alpha)^n}{n!} f^{(n)}(\alpha)\chi(\lambda)$. We show that f and g satisfy conditions (i) and (ii) of Corollary 4.2. If $\lambda \in U_0$ then $\chi(\lambda) = 0$ and so $g(\lambda) = f(\lambda)$. Hence f and g satisfy condition (ii) of Corollary 4.2. Since $g(\lambda) = f(\lambda)$ for all $\lambda \in U_0$, we have that $g^{(j)}(\lambda_i) = f^{(j)}(\lambda_i)$ $(j = 0, 1, \ldots, k_i - 1)$ for every pole $\lambda_i \in U_0$ of order k_i of the resolvent of a. We show that $g^{(j)}(\alpha) = f^{(j)}(\alpha)$ for $j = 0, 1, \ldots, k - 1$. We restrict the functions f and g to the set U_1. For $j = 0$, it is clear that $f(\alpha) = g(\alpha)$. For $j = 1$, we have that

$$g'(\lambda) = f'(\alpha) + \frac{2(\lambda-\alpha)}{2!} f''(\alpha) + \frac{3(\lambda-\alpha)^2}{3!} f'''(\alpha) + \cdots + \frac{(k-1)(\lambda-\alpha)^{k-2}}{(k-1)!} f^{(k-1)}(\alpha).$$

Therefore $g'(\alpha) = f'(\alpha)$. Next,

$$g''(\lambda) = f''(\alpha) + (\lambda-\alpha) f'''(\alpha) + \frac{3\cdot 4(\lambda-\alpha)^2}{4!} f^{(4)}(\alpha) + \cdots + \frac{(k-2)(k-1)(\lambda-\alpha)^{k-3}}{(k-1)!} f^{(k-1)}(\alpha).$$

Hence $g''(\alpha) = f''(\alpha)$. Continuing in this way, we obtain

$$g^{(k-1)}(\lambda) = \frac{1\cdot 2\cdots(k-1)}{(k-1)!} f^{(k-1)}(\alpha) = f^{(k-1)}(\alpha),$$

and so $g^{(k-1)}(\alpha) = f^{(k-1)}(\alpha)$. Therefore condition (i) of Corollary 4.2 is satisfied and so the result follows. □

The following three corollaries are consequences of Theorem 4.3.

Corollary 4.4 *Let A be a Banach algebra and let $a \in A$. Let (f_n) be a sequence of complex valued functions analytic on a neighbourhood of $\sigma(a)$. Suppose that $\alpha \neq 0$ is a pole of order k of the resolvent of a such that $f_n(\alpha) \to 1$ and $f_n^{(j)}(\alpha) \to 0$ $(j = 1, 2, \ldots, k-1)$ as $n \to \infty$. If $(\alpha 1 - a) f_n(a) \to 0$ as $n \to \infty$ then $f_n(a) \to p$ as $n \to \infty$, where $p = p(a, \alpha)$.*

Proof Since α is an isolated point in $\sigma(a)$, we can take two open sets U_0 and U_1 such that $\sigma(a)\backslash\{\alpha\} \subseteq U_0$, $\{\alpha\} \subseteq U_1$, $U_0 \cap U_1 = \emptyset$ and f_n is analytic on $U = U_0 \cup U_1$ for all $n \in \mathbb{N}$. By [2, Theorem 3.3.4], we have that $\sigma((1-p)a) = (\sigma(a) \cap U_0) \cup \{0\}$. Since $\alpha \neq 0$, it follows that $\alpha \notin \sigma((1-p)a)$, so that $b = \alpha 1 - (1-p)a$ is invertible. Then

$$\begin{aligned} f_n(a)(1-p) &= f_n(a)(1-p)(\alpha 1 - (1-p)a)b^{-1} = f_n(a)(1-p)(\alpha 1 - a)b^{-1} \\ &= f_n(a)(\alpha 1 - a)(1-p)b^{-1}, \end{aligned}$$

since $(1-p)p = 0$. From the assumption $(\alpha 1 - a)f_n(a) \to 0$ as $n \to \infty$, it follows that $f_n(a)(1-p) \to 0$ as $n \to \infty$. Also, from Theorem 4.3, we get that $f_n(a) = f_n(a)(1-p) + \sum_{j=0}^{k-1} \frac{(a-\alpha 1)^j}{j!} f_n^{(j)}(\alpha)p$. Together with the assumptions $f_n(\alpha) \to 1$ and $f_n^{(j)}(\alpha) \to 0$ $(j = 1, 2, \ldots, k-1)$ as $n \to \infty$, it follows that $f_n(a) \to p$ as $n \to \infty$. □

In the case where α in Corollary 4.4 is a simple pole, we get the following result.

Corollary 4.5 *Let A be a Banach algebra and $a \in A$. Let (f_n) be a sequence of complex valued functions analytic on a neighbourhood of $\sigma(a)$. Suppose that $\alpha \neq 0$ is a simple pole of the resolvent of a such that $f_n(\alpha) \to 1$ as $n \to \infty$. Then $f_n(a) \to p$ as $n \to \infty$ if and only if $(\alpha 1 - a)f_n(a) \to 0$ as $n \to \infty$, where $p = p(a, \alpha)$.*

Proof Suppose that $f_n(a) \to p$ as $n \to \infty$. Then $(\alpha 1 - a)f_n(a) \to (\alpha 1 - a)p$ as $n \to \infty$. Since α is a simple pole of the resolvent of a, we have that $(\alpha 1 - a)p = 0$. Hence $(\alpha 1 - a)f_n(a) \to 0$ as $n \to \infty$. The converse follows from Corollary 4.4. □

In Corollary 4.4 we made the assumption that α is a pole of arbitrary order k. The following result shows that the other conditions then force α to be a simple pole.

Corollary 4.6 *Let A be a Banach algebra, $a \in A$ and suppose that α is a pole of order at most $k \geq 1$ of the resolvent of a. Suppose that (f_n) is a sequence of complex valued functions analytic on a neighbourhood of $\sigma(a)$. If $(a - \alpha 1)f_n(a) \to 0$ and $f_n(\alpha) \to 1$ as $n \to \infty$, then α is a simple pole of the resolvent of a.*

Proof Let $p = p(a, \alpha)$. By Theorem 4.3 we may write

$$f_n(a) = f_n(a)(1-p) + \sum_{j=0}^{k-1} \frac{(a-\alpha 1)^j}{j!} f_n^{(j)}(\alpha)p. \tag{*}$$

For $k = 1$, the result is trivial. For $k = 2$ we have from (*) that $f_n(a) = f_n(a)(1-p) + f_n(\alpha)p + f_n'(\alpha)(a - \alpha 1)p$. Therefore

$$(a - \alpha 1)f_n(a) = (a - \alpha 1)f_n(a)(1-p) + (a - \alpha 1)f_n(\alpha)p + (a - \alpha 1)^2 f_n'(\alpha)p. \tag{**}$$

Since $k = 2$, we have that α is a pole of order at most 2 of the resolvent of a. Therefore $(a - \alpha 1)^2 f_n'(\alpha)p = 0$. Using the assumptions $(a - \alpha 1)f_n(a) \to 0$ and $f_n(\alpha) \to 1$

as $n \to \infty$, it then follows from (**) that $(a - \alpha 1)p = 0$. Hence α is a simple pole of the resolvent of a. For any $k > 2$, the general procedure is as follows: In the first step multiply both sides of (*) by $(a - \alpha 1)^{k-1}$. Since α is a pole of order at most k, this makes all but the first two terms of the expression on the right hand side of (*) zero. On the resulting equation, we take limits as $n \to \infty$ and use the assumptions $(a - \alpha 1) f_n(a) \to 0$ and $f_n(\alpha) \to 1$ as $n \to \infty$ to obtain that $(a - \alpha 1)^{k-1} p = 0$. In the second step multiply both sides of (*) by $(a - \alpha 1)^{k-2}$. Using arguments similar to the first step and the fact that $(a - \alpha 1)^{k-1} p = 0$, we get that $(a - \alpha 1)^{k-2} p = 0$. After $k - 1$ steps it follows that $(a - \alpha 1)p = 0$. □

In the following proposition we establish that if $\sigma(a)$ contains only one element, then a stronger form of the reverse implication in Corollary 4.5 holds. This result is not used further; it is included for the sake of interest.

Proposition 4.7 *Let A be a Banach algebra and $a \in A$. Suppose that $\sigma(a) = \{\alpha\}$ and α is a simple pole of the resolvent of a. If (f_n) is a sequence of complex valued functions analytic on a neighbourhood of $\sigma(a)$ and if $f_n(\alpha) \to 1$, then $f_n(a) \to p(a, \alpha)$.*

Proof Let Γ be a small circle centred at α. Then $f_n(a) = \frac{1}{2\pi i} \int_\Gamma f_n(\lambda)(\lambda 1 - a)^{-1} d\lambda$. Since α is a simple pole of the resolvent of a, we obtain the Laurent series expansion $(\lambda 1 - a)^{-1} = \frac{a_{-1}}{\lambda - \alpha} + a_0 + a_1(\lambda - \alpha) + a_2(\lambda - \alpha)^2 + \cdots$ $(a_j \in A,\ j = -1, 0, 1, 2, \ldots)$ on a deleted neighbourhood N_0 of α which contains Γ. Let $S(\lambda)$ be the sum of the power series $a_0 + a_1(\lambda - \alpha) + a_2(\lambda - \alpha)^2 + \cdots$. Then S is analytic on N_0 and then $(\lambda 1 - a)^{-1} = S(\lambda) + \frac{a_{-1}}{\lambda - \alpha}$. Clearly, $f_n S$ is analytic on N_0. Since Γ is contained in N_0, it follows that

$$f_n(a) = \frac{1}{2\pi i} \int_\Gamma f_n(\lambda) S(\lambda) d\lambda + \frac{a_{-1}}{2\pi i} \int_\Gamma \frac{f_n(\lambda)}{\lambda - \alpha} d\lambda = f_n(\alpha) a_{-1}.$$

Since $a_{-1} = p(a, \alpha)$ and $f_n(\alpha) \to 1$, it follows that $f_n(a) \to p(a, \alpha)$. □

In the rest of this section, we will consider only sequences of analytic functions of the form $f_n(\lambda) = \sum_{k=0}^{n-1} \frac{\lambda^k}{n}$ $(\lambda \in \mathbb{C})$, as they are of particular interest for the problem at hand. Note that $f_n(1) = 1$ for all $n \in \mathbb{N}$. The next proposition shows that if we take this sequence of functions and $\alpha = 1$ in Corollary 4.5, then we obtain a stronger form of the forward implication in Corollary 4.5. To prove this result we will use the following lemma.

Lemma 4.8 *Let A be a Banach algebra and $a \in A$. Then $(1 - a) \sum_{k=0}^{n-1} \frac{a^k}{n} \to 0$ if and only if $\frac{a^n}{n} \to 0$.*

Proof We have that $(1 - a) \sum_{k=0}^{n-1} \frac{a^k}{n} = \sum_{k=0}^{n-1} \frac{a^k}{n} - \sum_{k=0}^{n-1} \frac{a^{k+1}}{n} = \frac{1}{n} - \frac{a^n}{n}$. Now, $\frac{1}{n} - \frac{a^n}{n} \to 0$ if and only if $\frac{a^n}{n} \to 0$. □

Proposition 4.9 *Let A be a Banach algebra and $a \in A$. If $(f_n(a))$ converges, where $f_n(a) = \sum_{k=0}^{n-1} \frac{a^k}{n}$, then $(1 - a) f_n(a) \to 0$.*

Proof Suppose that $(f_n(a))$ converges, say $f_n(a) \to b$. We have that $(n+1)f_{n+1}(a) - nf_n(a) = a^n$, and so $\frac{a^n}{n} = \frac{1}{n}((n+1)f_{n+1}(a) - nf_n(a)) = \frac{n+1}{n}f_{n+1}(a) - f_n(a) \to b - b = 0$. It follows from Lemma 4.8 that $(1-a)f_n(a) \to 0$. □

The following theorem, which we will call the ergodic theorem, is our key result corresponding to part of [5, Theorem 3.16].

Theorem 4.10 *Let A be a Banach algebra and $a \in A$. Suppose that $1 \in \mathbf{iso}\,\sigma(a)$. Let (f_n) be the sequence of functions $f_n(\lambda) = \sum_{k=0}^{n-1} \frac{\lambda^k}{n}$ ($\lambda \in \mathbb{C}$). Then the following statements are equivalent:*

(i) *$(f_n(a))$ converges, i.e. a is ergodic, with $f_n(a) \to p(a, 1)$.*
(ii) *$(1-a)f_n(a) \to 0$ as $n \to \infty$ and 1 is a simple pole of the resolvent of a.*

Proof (i) ⇒ (ii): Suppose that $f_n(a) \to p(a, 1)$. Then by Proposition 4.9, we have that $(1-a)f_n(a) \to 0$. Since $(1-a)f_n(a) \to (1-a)p(a, 1)$, by uniqueness of limits, $(1-a)p(a, 1) = 0$. Therefore 1 is a simple pole of the resolvent of a.

(ii) ⇒ (i): If $(1-a)f_n(a) \to 0$ as $n \to \infty$ and 1 is a simple pole of the resolvent of a, then since $f_n(1) = 1$, it follows from Corollary 4.5 that $(f_n(a))$ is convergent, with $f_n(a) \to p(a, 1)$. □

5 Domination by ergodic elements

In this section we will obtain results for the problem of domination by ergodic elements. As stated earlier, a corresponding problem for uniformly ergodic operators on Banach lattices was studied in [15] and a result was obtained (see [15, Theorem 4.5]). The proof of this result depends on a theorem by Dunford (see [5, Theorem 3.16]). The results we will obtain here are OBA and COBA analogues (although not generalizations in the strict sense) of [15, Theorem 4.5] and these are Theorems 5.1, 5.2 and 5.5. For a Dedekind complete Banach lattice Theorem 5.5 generalizes [15, Theorem 4.5] in the special case of the dominated operator being positive. A key result that will be used to establish these theorems is Theorem 4.10.

If a is an element of a Banach algebra A, then, in this section, $f_n(a)$ will always indicate the element $\sum_{k=0}^{n-1} \frac{a^k}{n} \in A$.

We start with the following result.

Theorem 5.1 *Let (A, C) be an OBA with C closed and normal, and let $a, b \in A$ such that $0 \leq 1 \leq a \leq b$. Suppose that $r(b) = 1 \in \mathbf{iso}\,\sigma(a)$ and that 1 is a simple pole of the resolvent of b. If b is ergodic with $f_n(b) \to p(b, 1)$ and if $p(a, 1) = p(b, 1)$, then a is ergodic, with $f_n(a) \to p(a, 1)$.*

Proof Since 1 is a simple pole of the resolvent of b, by Proposition 3.3, it is an eigenvalue of b with positive corresponding eigenvector $p(b, 1)$. Since $p(a, 1) = p(b, 1)$, the assumption $0 \leq 1 \leq a \leq b$ implies that $0 \leq (a-1)p(a, 1) \leq (b-1)p(b, 1) = 0$. From the fact that C is normal, and hence proper, it follows that $(1-a)p(a, 1) = 0$, so that 1 is a simple pole of the resolvent of a. Now since b is ergodic, Proposition 4.9 and Lemma 4.8 imply that $\frac{b^n}{n} \to 0$ as $n \to \infty$. It follows from Proposition 3.1 and the normality of C that $\frac{a^n}{n} \to 0$ as $n \to \infty$. Lemma 4.8 then implies that $(1-a)f_n(a) \to 0$ as $n \to \infty$. From Theorem 4.10, it follows that a is ergodic, with $f_n(a) \to p(a, 1)$. □

The remaining results about domination by ergodic elements will be proved under conditions similar to those of [15, Theorem 4.5]. We start with Theorem 5.2, which is the basic result from which the others will be obtained.

Theorem 5.2 *Let A be an OBA with a closed, normal algebra cone C and let $a, b \in A$ such that $0 \le a \le b$. Suppose that $1 \in$ **iso** $\sigma(a)$ is a pole of the resolvent of a. If b is ergodic, then a is ergodic.*

Proof From Proposition 3.1 we have that $0 \le \frac{a^n}{n} \le \frac{b^n}{n}$ for all $n \in \mathbb{N}$. Since b is ergodic, $\frac{b^n}{n} \to 0$ as $n \to \infty$ by Proposition 4.9 and Lemma 4.8, and since C is normal, $\frac{a^n}{n} \to 0$ as $n \to \infty$. Lemma 4.8 then implies that $(1-a)f_n(a) \to 0$ as $n \to \infty$. Since 1 is a pole of the resolvent of a, it follows from Corollary 4.6 that 1 is a simple pole of the resolvent of a. From Theorem 4.10 it follows that a is ergodic, with $f_n(a) \to p(a, 1)$ as $n \to \infty$. □

To prove our main theorem, the following two lemmas will be required in addition to the theory we have developed so far.

Lemma 5.3 *Let A be a Banach algebra and $a \in A$. If a is ergodic, then $r(a) \le 1$.*

Proof Since a is ergodic, $(f_n(a))$ converges. It follows from Proposition 4.9 that $(1-a)f_n(a) \to 0$ as $n \to \infty$. Lemma 4.8 then implies that $\frac{||a^n||}{n} \to 0$ as $n \to \infty$. Therefore there exists a constant $c > 0$ such that $\frac{||a^n||}{n} \le c$ for all $n \in \mathbb{N}$, so that $||a^n|| \le cn$ for all $n \in \mathbb{N}$. Hence $r(a) = \lim_{n\to\infty} ||a^n||^{\frac{1}{n}} \le \lim_{n\to\infty} c^{\frac{1}{n}} n^{\frac{1}{n}} = 1$. □

Lemma 5.4 *Let A be a Banach algebra and $a \in A$. If $\frac{a^n}{n} \to 0$ as $n \to \infty$ and $1 \notin \sigma(a)$, then $\sum_{k=0}^{n-1} \frac{a^k}{n}$ converges to 0 as $n \to \infty$.*

Proof If $\frac{a^n}{n} \to 0$ as $n \to \infty$, then by Lemma 4.8 $(1-a)\sum_{k=0}^{n-1} \frac{a^k}{n} \to 0$ as $n \to \infty$. If also $1 \notin \sigma(a)$, then $1-a$ is invertible, which yields the result. □

Theorem 5.5 *Let (A, C) be a semisimple OBA with C closed and normal, and let $a, b \in A$ such that $0 \le a \le b$. Let I be a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. If b is ergodic and if $r(b)$ is a Riesz point of $\sigma(b)$, then a is ergodic.*

Proof Since C is normal, the spectral radius in (A, C) is monotone. This, together with Lemma 5.3 and the ergodicity of b, implies that $r(a) \le r(b) \le 1$. Then we have four cases: $r(a) < r(b) < 1$, $r(a) < r(b) = 1$, $r(a) = r(b) < 1$ and $r(a) = r(b) = 1$. Now from Proposition 3.1, we have that $0 \le \frac{a^n}{n} \le \frac{b^n}{n}$ for all $n \in \mathbb{N}$. Since b is ergodic, $\frac{b^n}{n} \to 0$ as $n \to \infty$ by Proposition 4.9 and Lemma 4.8, and then, by the normality of C, $\frac{a^n}{n} \to 0$ as $n \to \infty$. In the first three cases, we get that $1 \notin \sigma(a)$. Therefore $\sum_{k=0}^{n-1} \frac{a^k}{n} \to 0$ as $n \to \infty$ by Lemma 5.4, so that a is ergodic. To deal with the last case suppose that $r(a) = r(b) = 1$. Since $r(b)$ is a Riesz point of $\sigma(b)$, by Corollary 3.7, we have that $r(a)$ is a Riesz point of $\sigma(a)$. Lemma 2.1 then implies that $r(a)$ is a pole of the resolvent of a. From Theorem 5.2, it follows that a is ergodic. □

To obtain COBA versions of Theorems 5.1, 5.2 and 5.5, the additional condition $ab = ba$ is assumed, since this assumption is needed in the COBA version of Proposition 3.1, and normality can be replaced with c-normality. For the quotient algebra in Theorem 5.5, we then work with the C'OBA $(A/F, \pi C)$. With $ab = ba$, monotonicity in $(A/F, \pi C)$ can be replaced with c-monotonicity. Whether the assumption $ab = ba$ can be dropped under any circumstances in the COBA case remains an open problem. Example 3.2 shows that it is essential for our proof.

We end this section with the following observation:

Proposition 5.6 *Let A be a COBA with a closed and inverse-closed algebra c-cone C. If $a \in C$ and a is ergodic, then $a \leq 1$.*

Proposition 5.6 follows from Proposition 3.4 and Lemma 5.3.

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