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Mann type iterative methods for finding a common solution of split feasibility and fixed point problems

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Abstract The purpose of this paper is to study and analyze three different kinds of Mann type iterative methods for finding a common element of the solution set Γ of the split feasibility problem and the set $\mathrm{Fix}(S)$ of fixed points of a nonexpansive mapping S in the setting of infinite-dimensional Hilbert spaces. By combining Mann's iterative method and the extragradient method, we first propose Mann type extragradient-like algorithm for finding an element of the set $\mathrm{Fix}(S) \cap \Gamma$; moreover, we derive the weak convergence of the proposed algorithm under appropriate conditions. Second, we combine Mann's iterative method and the viscosity approximation method to introduce

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Mann type viscosity algorithm for finding an element of the $\mathrm{Fix}(S) \cap \Gamma$; moreover, we derive the strong convergence of the sequences generated by the proposed algorithm to an element of set $\mathrm{Fix}(S) \cap \Gamma$ under mild conditions. Finally, by combining Mann's iterative method and the relaxed CQ method, we introduce Mann type relaxed CQ algorithm for finding an element of the set $\mathrm{Fix}(S) \cap \Gamma$. We also establish a weak convergence result for the sequences generated by the proposed Mann type relaxed CQ algorithm under appropriate assumptions.

Keywords Split feasibility problems · Fixed point problems · Mann type iterative methods · Extragradient method · Viscosity approximation method · Relaxed *CQ* method · Nonexpansive mappings · Averaged mappings · Fixed points · Projection

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1 Introduction

Let C and Q be nonempty closed convex subsets of the infinite-dimensional real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$, where $B(\mathcal{H}_1, \mathcal{H}_2)$ denotes the collection of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. The split feasibility problem (SFP) is to find x^* such that

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \tag{1.1}$$

It was first introduced and considered by Censor and Elfving [7] in the finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. The SFP has also been applied to modeling the intensity-modulated radiation therapy; see, for example [6,8,9] and the references therein. The SFP is also studied in [2,3,20,25,29,30] in the setting of finite-dimensional spaces. The SFP and the relevant project method can be found in [21] for solving image recovery problems. A special case of the SFP is the convex constrained linear inverse problem [11,19]

$$x \in C \quad \text{and} \quad Ax = b. \tag{1.2}$$

It has extensively been investigated in the literature using the projected Landweber iterative method [12,15]. Comparatively, the SFP has received much less attention so far, due to the complexity resulted from the set Q. Therefore, whether various versions of the projected Landweber iterative method can be extended to solve the SFP remains an interesting topic, for example, it is yet not clear if the dual approach to (1.2) of [13] can be extended to the SFP. The original algorithm given in [7] involves the computation of the inverse A^{-1} (assuming the existence of the inverse of A), and thus, does not become popular. A more popular algorithm that solves the SFP seems to be the CQ algorithm of Byrne [2,3] which is found to be a gradient-projection method (GPM) in convex minimization. It is a special case of the proximal forward-backward splitting method [10]. The CQ algorithm only involves the computations of the projections P_C

and P_Q onto the sets C and Q, respectively, and is therefore implementable in the case where P_C and P_Q have closed-form expressions, for example, C and Q are the closed balls or half-spaces. It remains a challenge how to implement the CQ algorithm in the case where the projections P_C and / or P_Q fail to have closed-form expressions though theoretically we can prove (weak) convergence of the algorithm. Very recently, Xu [26] considered the SFP in the framework of infinite-dimensional Hilbert spaces. He gave a continuation of study on the CQ algorithm and its convergence. He applied Mann's algorithm to the SFP and proposed an averaged CQ algorithm which was proved to be weakly convergent to a solution of the SFP.

The purpose of this paper is to study and analyze Mann type three iterative methods for finding a common element of the solution set Γ of the SFP and the set $\mathrm{Fix}(S)$ of fixed points of a nonexpansive mapping S in the setting of infinite-dimensional real Hilbert spaces. In Sect. 2, we mention some known results and definitions which will be used in the sequel. By combining Mann's iterative method and the extragradient method [14,18], we propose Mann type extragradient-like algorithm for finding an element of the set $\mathrm{Fix}(S) \cap \Gamma$; moreover, we derive the weak convergence of this algorithm under appropriate conditions. By combining Mann's iterative method and the viscosity approximation method [4,5,17], in Sect. 4, we introduce Mann type viscosity algorithm for finding an element of the set $\mathrm{Fix}(S) \cap \Gamma$; moreover, we derive the strong convergence of the sequences generated by the proposed algorithm under mild conditions. In the final section, we combine Mann's iterative method and the relaxed CQ method [26,29] to introduce Mann type relaxed CQ algorithm for finding an element of the set $\mathrm{Fix}(S) \cap \Gamma$. In this method, the sets C and Q are level sets of convex functions so that the projections involved in the CQ algorithm are onto half-spaces, which makes the algorithm implementable. We also establish a weak convergence result for Mann type relaxed CQ algorithm under appropriate assumptions. It is worth emphasizing that our results are new and novel in Hilbert spaces. Our results represent the supplement, improvement and extension of the results given in [26] to a great extent.

2 Preliminaries

Let $\{x_n\}$ be a sequence and x be a point in a normed space X. We use $x_n \to x$ and $x_n \rightharpoonup x$ to denote the strong and weak convergence to x of the sequence $\{x_n\}$, respectively. We also use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, namely,

$$\omega_w(x_n) := \left\{x \in X : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\right\}.$$

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty closed convex subset of $\mathcal{H}$. Recall that the (nearest point or metric) projection from $\mathcal{H}$ onto K, denoted by P_K, is defined in such a way that, for each $x \in \mathcal{H}$, $P_K x$ is the unique point in K with the property

$$\|x - P_K x\| = \inf_{y \in K} \|x - y\| =: d(x, K).$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1 *For given* $x \in \mathcal{H}$ *and* $z \in K$.

(i) $z = P_K x$ *if and only if* $\langle x - z, y - z\rangle \leq 0$, *for all* $y \in K$.
(ii) $z = P_K x$ *if and only if* $\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, *for all* $y \in K$.
(iii) *For all* $y \in \mathcal{H}$, $\langle P_K x - P_K y, x - y\rangle \geq \|P_K x - P_K y\|^2$.

Definition 2.1 A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be

(a) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in \mathcal{H};$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in \mathcal{H}.$$

Alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : \mathcal{H} \to \mathcal{H}$ is nonexpansive. Projections are firmly nonexpansive.

Definition 2.2 Let T be a nonlinear operator whose domain is $D(T) \subseteq \mathcal{H}$ and range is $R(T) \subseteq \mathcal{H}$ and let $\beta > 0$ and $\nu > 0$ be given constants. The operator T is said to be

(a) monotone if

$$\langle x - y, Tx - Ty\rangle \geq 0, \quad \forall x, y \in D(T).$$

(b) β-strongly monotone if

$$\langle x - y, Tx - Ty\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in D(T).$$

(c) ν-inverse strongly monotone (ν-ism) if

$$\langle x - y, Tx - Ty\rangle \geq \nu\|Tx - Ty\|^2, \quad \forall x, y \in D(T).$$

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_K is nonexpansive and 1-ism.

Inverse strongly (also referred to as co-coercive) monotone operators have widely been applied to solving practical problems in various fields; for instance, in traffic assignment problems, see, for example, [1,13].

Definition 2.3 A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T = (1 - \alpha)I + \alpha S \tag{2.1}$$

where α is a number in $(0, 1)$ and $S : \mathcal{H} \to \mathcal{H}$ is nonexpansive. More precisely, when (2.1) holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (in particular, projections) are $\frac{1}{2}$-averaged maps.

Proposition 2.2 [3] *Let $T : \mathcal{H} \to \mathcal{H}$ be an operator.*

- (i) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*
- (ii) *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*
- (ii) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Proposition 2.3 [3] *Let $S, T, V : \mathcal{H} \to \mathcal{H}$ be given operators.*

- (i) *If $T = (1-\alpha)S+\alpha V$ for some $\alpha \in (0, 1)$, S is averaged and V is nonexpansive, then T is averaged.*
- (ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*
- (iii) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$, S is firmly nonexpansive and V is nonexpansive, then T is averaged.*
- (iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \circ \cdots \circ T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite $T_1 \circ T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*
- (v) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} Fix(T_i) = Fix(T_1 \circ \cdots \circ T_N).$$

The notation $Fix(T)$ denotes the set of all fixed points of the mapping T, that is, $Fix(T) = \{x \in \mathcal{H} : Tx = x\}$.

The following result is useful to prove the weak convergence of a sequence.

Proposition 2.4 (Proposition 2.6 of [26]) *Let K be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $\{x_n\}$ be a bounded sequence which satisfies the following properties:*

- (i) *every weak limit point of $\{x_n\}$ lies in K;*
- (ii) $\lim_{n\to\infty} \|x_n - x\|$ *exists for every $x \in K$.*

Then, the sequence $\{x_n\}$ converges weakly to a point in K.

The so-called demiclosedness principle for nonexpansive mappings will often be used.

Lemma 2.1 [12] *Let K be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let $T : K \to K$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If the sequence $\{x_n\} \subseteq K$ weakly converges to x and the sequence $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particular, if $y = 0$, then $x \in Fix(T)$.*

The following elementary result on real sequences is quite well-known.

Lemma 2.2 (Lemma 2.1 of [28]) *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences of real numbers such that

(i) $\{\gamma_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$, *or equivalently,*

$$\prod_{n=0}^{\infty}(1 - \gamma_n) := \lim_{n\to\infty} \prod_{k=0}^{n}(1 - \gamma_k) = 0;$$

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$, *or*
(ii)′ $\sum_{n=0}^{\infty} \gamma_n |\delta_n|$ *is convergent.*

Then, $\lim_{n\to\infty} a_n = 0$.

It is easy to see that the following lemma holds.

Lemma 2.3 [12] *Let $\mathcal{H}$ be a real Hilbert space. Then, for all $x, y \in \mathcal{H}$ and $\lambda \in [0, 1]$*

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Lemma 2.4 [22] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Suppose $x_{n+1} = (1 - \alpha_n)y_n + \alpha_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

3 Mann type extragradient-like algorithm

Throughout the paper, we denote the solution set of the SFP by Γ, that is,

$$\Gamma = \{x \in C : Ax \in Q\} = C \cap A^{-1}Q,$$

and assume that the SFP is consistent, that is, Γ is nonempty, closed and convex.

Recall that the GPM [16] is one of the powerful methods for solving constrained optimization problems. The SFP can be reformulated as an optimization problem so that the GPM is applicable. Indeed, $x \in \Gamma$ means that there is an $x \in C$ such that

$Ax - q = 0$ for some $q \in Q$. This motivates us to consider the distance function $d(Ax, q) = \|Ax - q\|$ and the minimization problem

$$\min_{x \in C, q \in Q} \frac{1}{2}\|Ax - q\|^2.$$

Minimizing with respect to $q \in Q$ motivates us to consider the minimization

$$\min_{x \in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2, \tag{3.1}$$

where the objective function f is continuously differentiable with gradient given by

$$\nabla f(x) = A^*(I - P_Q)Ax, \tag{3.2}$$

where A^* denotes the adjoint of A. Due to the fact that $I - P_Q$ is (firmly) nonexpansive, we find that ∇f is L-Lipschitz continuous with Lipschitz constant $L := \|A\|^2$, namely

$$\|\nabla f(x) - \nabla f(y)\| \leq \|A\|^2\|x - y\|, \quad \forall x, y \in \mathcal{H}_1. \tag{3.3}$$

The GPM is thus applied to solve (3.1). This method with gradient ∇f given as in (3.2) is referred to as the *CQ* algorithm in [2,3] and generates a sequence $\{x_n\}$ via the procedure

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \quad \forall n \geq 0, \tag{3.4}$$

where the initial guess $x_0 \in \mathcal{H}_1$ and $\gamma > 0$ is a parameter. By Proposition 2.5 [26], we immediately get the following convergence result.

Theorem 3.1 [2,3,25] *If* $0 < \gamma < 2/\|A\|^2$*, then the sequence* $\{x_n\}$ *generated by the CQ algorithm* (3.4) *converges weakly to a solution of the SFP.*

On the other hand, let $\gamma > 0$ and assume that $x^* \in \Gamma$. Then $Ax^* \in Q \Rightarrow (I - P_Q)Ax^* = 0 \Rightarrow \gamma A^*(I - P_Q)Ax^* = 0$, hence, we get the fixed point equation $(I - \gamma A^*(I - P_Q)A)x^* = x^*$. Requiring that $x^* \in C$, consider the fixed point equation:

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*. \tag{3.5}$$

The following proposition shows that the solution sets of fixed point equation (3.5) and SFP are the same.

Proposition 3.1 (Proposition 3.2 of [26]) *Given* $x^* \in \mathcal{H}_1$. *Then* x^* *solves the SFP if and only if* x^* *solves the fixed point equation* (3.5).

The following proposition was proved by Takahashi and Toyoda [23].

Proposition 3.2 *Let $\mathcal{H}$ be a real Hilbert space, D be a nonempty closed convex subset of $\mathcal{H}$ and $\{x_n\}$ be a sequence in $\mathcal{H}$. Suppose that, for all $u \in D$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|, \quad \forall n \geq 0.$$

Then, the sequence $\{P_D x_n\}$ converges strongly to some $z \in D$.

By combining Mann's iterative method and the extragradient method [14,18], we propose Mann type extragradient-like method for finding a common element of the set of solutions of the SFP and the set of fixed points of a nonexpansive mapping in the setting of real Hilbert spaces. Motivated by the work in [27], we derive the following weak convergence result.

Theorem 3.2 *Let $S : C \to C$ be a nonexpansive mapping such that $Fix(S) \cap \Gamma \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by the following Mann type extragradient-like algorithm:*

$$\begin{cases} x_0 = x \in \mathcal{H}_1 \text{ chosen arbitrarily}, \\ y_n = (1 - \beta_n)x_n + \beta_n P_C(I - \lambda_n A^*(I - P_Q)A)x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(I - \lambda_n A^*(I - P_Q)A)y_n, \quad \forall n \geq 0, \end{cases} \tag{3.6}$$

where the sequences of parameters $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0, 1]$ *and* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$*;*
(ii) $\{\beta_n\} \subset [0, 1]$ *and* $\liminf_{n\to\infty} \beta_n > 0$*;*
(iii) $\{\lambda_n\} \subset \left(0, \frac{2}{\|A\|^2}\right)$ *and* $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{\|A\|^2}$.

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to $z \in Fix(S) \cap \Gamma$, where

$$z = \|\cdot\| - \lim_{n\to\infty} P_{\mathrm{Fix}(S)\cap\Gamma} x_n.$$

Proof For the sake of simplicity, we may assume that

$$0 < a \leq \lambda_n \leq b < \frac{2}{\|A\|^2} \quad \text{and} \quad 0 < c \leq \alpha_n \leq d < 1, \quad \forall\, n \geq 0,$$

where a, b, c and d are constants.

First, we assert that $P_C(I - \lambda A^*(I - P_Q)A)$ is α-averaged for each $\lambda \in \left(0, \frac{2}{\|A\|^2}\right)$, where

$$\alpha = \frac{2 + \lambda\|A\|^2}{4}.$$

As a matter of fact, we have seen that $A^*(I - P_Q)A$ is $\frac{1}{\|A\|^2}$-ism and $\lambda A^*(I - P_Q)A$ is $\frac{1}{\lambda\|A\|^2}$-ism. Hence, by Proposition 2.2 (iii) the complement $I - \lambda A^*(I - P_Q)A$ is

$\frac{\lambda\|A\|^2}{2}$-averaged. Therefore, noting that P_C is $\frac{1}{2}$-averaged and applying Proposition 2.3 (iv), we know that for each $\lambda \in \left(0, \frac{2}{\|A\|^2}\right)$, $P_C(I - \lambda A^*(I - P_Q)A)$ is α-averaged, with

$$\alpha = \frac{1}{2} + \frac{\lambda\|A\|^2}{2} - \frac{1}{2} \cdot \frac{\lambda\|A\|^2}{2} = \frac{2 + \lambda\|A\|^2}{4} \in (0, 1).$$

Hence, we can write

$$P_C(I - \lambda A^*(I - P_Q)A) = \frac{2 - \lambda\|A\|^2}{4} I + \frac{2 + \lambda\|A\|^2}{4} T_\lambda = (1 - \alpha)I + \alpha T_\lambda, \tag{3.7}$$

where T_λ is nonexpansive and $\alpha = \frac{2+\lambda\|A\|^2}{4} \in \left(\frac{1}{2}, 1\right)$ for each $\lambda \in \left(0, \frac{2}{\|A\|^2}\right)$. In particular, we can write

$$\begin{aligned} P_C(I - \lambda_n A^*(I - P_Q)A) &= \frac{2 - \lambda_n\|A\|^2}{4} I + \frac{2 + \lambda_n\|A\|^2}{4} T_n \\ &= (1 - \gamma_n)I + \gamma_n T_n, \end{aligned} \tag{3.8}$$

where T_n is nonexpansive and $\gamma_n = \frac{2+\lambda_n\|A\|^2}{4} \in [a_1, b_1] \subset (0, 1)$ with $a_1 = (2 + a\|A\|^2)/4$ and $b_1 = (2 + b\|A\|^2)/4 < 1$. Then, from (3.6), we have

$$y_n = (1 - \beta_n)x_n + \beta_n[(1 - \gamma_n)x_n + \gamma_n T_n x_n]. \tag{3.9}$$

Put

$$z_n := P_C(I - \lambda_n A^*(I - P_Q)A)y_n \; (= (1 - \gamma_n)y_n + \gamma_n T_n y_n). \tag{3.10}$$

Then, $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sz_n$ for all $n \geq 0$. Now take a fixed $p \in \mathrm{Fix}(S) \cap \Gamma$ arbitrarily. Note that $T_n p = p$ and $Sp = p$. Thus, we have

$$\begin{aligned} \|y_n - p\|^2 &= \|(1 - \beta_n)(x_n - p) + \beta_n[(1 - \gamma_n)x_n + \gamma_n T_n x_n - p]\|^2 \\ &\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n\|(1 - \gamma_n)(x_n - p) + \gamma_n(T_n x_n - p)\|^2 \\ &= (1 - \beta_n)\|x_n - p\|^2 + \beta_n[(1 - \gamma_n)\|x_n - p\|^2 \\ &\quad + \gamma_n\|T_n x_n - p\|^2 - \gamma_n(1 - \gamma_n)\|x_n - T_n x_n\|^2] \\ &\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n[(1 - \gamma_n)\|x_n - p\|^2 \\ &\quad + \gamma_n\|x_n - p\|^2 - \gamma_n(1 - \gamma_n)\|x_n - T_n x_n\|^2] \\ &= \|x_n - p\|^2 - \beta_n\gamma_n(1 - \gamma_n)\|x_n - T_n x_n\|^2, \end{aligned}$$

$$\begin{aligned}\|z_n - p\|^2 &= (1-\gamma_n)\|y_n - p\|^2 + \gamma_n\|T_n y_n - p\|^2 - \gamma_n(1-\gamma_n)\|y_n - T_n y_n\|^2 \\ &\le \|y_n - p\|^2 - \gamma_n(1-\gamma_n)\|y_n - T_n y_n\|^2 \\ &\le \|x_n - p\|^2 - \beta_n\gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 - \gamma_n(1-\gamma_n)\|y_n - T_n y_n\|^2,\end{aligned}$$

and hence,

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|Sz_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sz_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sz_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 - \beta_n\gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 \\ &\quad -\gamma_n(1-\gamma_n)\|y_n - T_n y_n\|^2] - \alpha_n(1-\alpha_n)\|x_n - Sz_n\|^2 \\ &= \|x_n - p\|^2 - (1-\alpha_n)\beta_n\gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 \\ &\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|y_n - T_n y_n\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sz_n\|^2 \\ &\le \|x_n - p\|^2. \end{aligned} \tag{3.11}$$

It follows that the real nonnegative sequence $\{\|x_n - p\|\}$ is nonincreasing. Hence,

$$\lim_{n\to\infty}\|x_n - p\| \quad \text{exists for all} \quad p \in \mathrm{Fix}(S) \cap \Gamma. \tag{3.12}$$

Noting that $\liminf_{n\to\infty}\beta_n > 0$, we may assume that $\beta_n \ge l (\forall n \ge 0)$ for some $l > 0$. From (3.11), it follows that

$$\begin{aligned}&(1-d)la_1(1-b_1)\|x_n - T_n x_n\|^2 \\ &\quad +(1-d)a_1(1-b_1)\|y_n - T_n y_n\|^2 + c(1-d)\|x_n - Sz_n\|^2 \\ &\le (1-\alpha_n)\beta_n\gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 + (1-\alpha_n)\gamma_n(1-\gamma_n)\|y_n - T_n y_n\|^2 \\ &\quad +\alpha_n(1-\alpha_n)\|x_n - Sz_n\|^2 \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.\end{aligned}$$

Consequently, from (3.12), we deduce that

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = \lim_{n\to\infty}\|y_n - T_n y_n\| = \lim_{n\to\infty}\|x_n - Sz_n\| = 0. \tag{3.13}$$

This together with (3.9) and (3.10) implies that

$$\begin{aligned}\lim_{n\to\infty}\|y_n - x_n\| &= \lim_{n\to\infty}\beta_n\gamma_n\|T_n x_n - x_n\| = 0, \\ \lim_{n\to\infty}\|z_n - y_n\| &= \lim_{n\to\infty}\gamma_n\|T_n y_n - y_n\| = 0.\end{aligned}$$

and hence,

$$\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\|z_n - Sz_n\| = 0. \tag{3.14}$$

Next, we show that

$$\omega_w(x_n) \subset \mathrm{Fix}(S) \cap \Gamma. \tag{3.15}$$

Indeed, suppose $\hat{x} \in \omega_w(x_n)$ and $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \hat{x}$. We may assume $\lambda_{n_j} \to \lambda$; then we have $0 < \lambda < 2/\|A\|^2$. Set $T = P_C(I - \lambda A^*(I - P_Q)A)$; then T is nonexpansive. Since $y_{n_j} = (1-\beta_{n_j})x_{n_j} + \beta_{n_j} P_C(I - \lambda_{n_j} A^*(I - P_Q)A)x_{n_j}$ and $\|x_{n_j} - y_{n_j}\| \to 0$, we conclude that

$$\begin{aligned}
\|x_{n_j} - Tx_{n_j}\| &\le \|x_{n_j} - y_{n_j}\| + \|y_{n_j} - Tx_{n_j}\| = \|x_{n_j} - y_{n_j}\| + \|(1-\beta_{n_j})x_{n_j} \\
&\quad + \beta_{n_j} P_C(I - \lambda_{n_j} A^*(I - P_Q)A)x_{n_j} - Tx_{n_j}\| \\
&\le \|x_{n_j} - y_{n_j}\| + (1-\beta_{n_j})\|x_{n_j} - Tx_{n_j}\| + \beta_{n_j}\|P_C(I - \lambda_{n_j} \\
&\quad \times A^*(I - P_Q)A)x_{n_j} - P_C(I - \lambda A^*(I - P_Q)A)x_{n_j}\| \\
&\le \|x_{n_j} - y_{n_j}\| + (1-\beta_{n_j})\|x_{n_j} - Tx_{n_j}\| \\
&\quad + \beta_{n_j}\|(I - \lambda_{n_j} A^*(I - P_Q)A)x_{n_j} - (I - \lambda A^*(I - P_Q)A)x_{n_j}\| \\
&= \|x_{n_j} - y_{n_j}\| + (1-\beta_{n_j})\|x_{n_j} - Tx_{n_j}\| + \beta_{n_j}|\lambda_{n_j} \\
&\quad - \lambda|\|A^*(I - P_Q)A)x_{n_j}\| \le \|x_{n_j} - y_{n_j}\| + (1-\beta_{n_j})\|x_{n_j} \\
&\quad - Tx_{n_j}\| + M|\lambda_{n_j} - \lambda|,
\end{aligned}$$

which hence implies that

$$\begin{aligned}
\|x_{n_j} - Tx_{n_j}\| &\le \frac{1}{\beta_{n_j}}[\|x_{n_j} - y_{n_j}\| + M|\lambda_{n_j} - \lambda|] \\
&\le \frac{1}{l}[\|x_{n_j} - y_{n_j}\| + M|\lambda_{n_j} - \lambda|] \to 0.
\end{aligned}$$

By Lemma 2.1, we obtain $\hat{x} \in \mathrm{Fix}(T)$. But $\mathrm{Fix}(T) = \Gamma$, therefore, we have $\hat{x} \in \Gamma$. Furthermore, since $x_{n_j} \rightharpoonup \hat{x}$ and $\lim_{n\to\infty} \|z_n - x_n\| = \lim_{n\to\infty} \|z_n - Sz_n\| = 0$ (due to (3.14)), it is known that $z_{n_j} \rightharpoonup \hat{x}$ and $\lim_{j\to\infty} \|z_{n_j} - Sz_{n_j}\| = 0$. Consequently, by Lemma 2.1, we get $\hat{x} \in \mathrm{Fix}(S)$. Therefore, we have $\hat{x} \in \mathrm{Fix}(S) \cap \Gamma$. This shows that (3.15) holds.

Finally, by applying Proposition 2.4 to $\mathrm{Fix}(S) \cap \Gamma$, we can see from (3.12) and (3.15) that $\{x_n\}$ converges weakly to a point $z \in \mathrm{Fix}(S) \cap \Gamma$. In the meantime, from $\|x_n - y_n\| \to 0$, it follows that $y_n \rightharpoonup z$. Now, put

$$u_n = P_{\mathrm{Fix}(S)\cap\Gamma} x_n.$$

Let us show that

$$\lim_{n\to\infty} \|u_n - z\| = 0.$$

Indeed, noticing the fact that

$$u_n = P_{\mathrm{Fix}(S)\cap\Gamma} x_n \quad \text{and} \quad z \in \mathrm{Fix}(S) \cap \Gamma,$$

by Proposition 2.1 (i), we have

$$\langle z - u_n, u_n - x_n \rangle \geq 0. \tag{3.16}$$

Utilizing Proposition 3.2, we deduce from (3.11) that $\{u_n\}$ converges strongly to some $z_0 \in \mathrm{Fix}(S) \cap \Gamma$. Then, from (3.16), we have

$$\langle z - z_0, z_0 - z \rangle \geq 0,$$

and hence, $z = z_0$. □

4 Mann type viscosity algorithm

In this section, we modify the Mann type extragradient-like algorithm, proposed in the last section, to obtain the strong convergence of the sequences. Our modification is of viscosity approximation nature [4,5,17].

Theorem 4.1 *Let $f : C \to C$ be a ρ-contraction with $\rho \in [0, 1)$ and $S : C \to C$ be a nonexpansive mapping such that $Fix(S) \cap \Gamma \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by the following Mann type viscosity algorithm:*

$$\begin{cases} x_0 = x \in \mathcal{H}_1 \text{ chosen arbitrarily}, \\ y_n = P_C(I - \lambda_n A^*(I - P_Q)A)x_n, \\ z_n = P_C(I - \lambda_n A^*(I - P_Q)A)y_n, \\ x_{n+1} = \theta_n f(y_n) + \mu_n x_n + \nu_n z_n + \delta_n S z_n, \quad \forall n \geq 0, \end{cases} \tag{4.1}$$

where the sequences of parameters $\{\theta_n\}, \{\mu_n\}, \{\nu_n\}, \{\delta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset \left(0, \frac{2}{\|A\|^2}\right)$ satisfy the following conditions:

(i) $\theta_n + \mu_n + \nu_n + \delta_n = 1$;
(ii) $\lim_{n\to\infty} \theta_n = 0$ *and* $\sum_{n=0}^{\infty} \theta_n = \infty$;
(iii) $\liminf_{n\to\infty} \delta_n > 0$;
(iv) $\lim_{n\to\infty} \left(\frac{\nu_{n+1}}{1-\mu_{n+1}} - \frac{\nu_n}{1-\mu_n} \right) = 0$;
(v) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{\|A\|^2}$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* \in Fix(S) \cap \Gamma$ which is also a unique solution of the variational inequality (VI)

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in Fix(S) \cap \Gamma.$$

In other words, x^* is a unique fixed point of the contraction $P_{\mathrm{Fix}(S)\cap\Gamma} f$, $x^* = (P_{\mathrm{Fix}(S)\cap\Gamma} f)x^*$.

Proof We divide the proof into several steps.
STEP 1. The sequence $\{x_n\}$ is bounded.

Indeed, repeating the same argument as in the proof of Theorem 3.2, we can obtain that for each $n \geq 0$

$$V_n := P_C(I - \lambda_n A^*(I - P_Q)A) = (1 - \gamma_n)I + \gamma_n T_n, \tag{4.2}$$

where T_n is nonexpansive and $\gamma_n = \frac{2+\lambda_n\|A\|^2}{4}$. Then, V_n is nonexpansive, and Mann type viscosity algorithm (4.1) can be rewritten as

$$\begin{cases} x_0 = x \in \mathcal{H}_1 \text{ chosen arbitrarily,} \\ y_n = V_n x_n, \\ z_n = V_n y_n, \\ x_{n+1} = \theta_n f(y_n) + \mu_n x_n + \nu_n z_n + \delta_n S z_n, \quad \forall n \geq 0. \end{cases}$$

Moreover, we have $V_n p = p$ for all $p \in \Gamma$. Consequently, $T_n p = p$ for all $p \in \Gamma$. Now take a fixed $\bar{x} \in \mathrm{Fix}(S) \cap \Gamma$ arbitrarily. Then, we have

$$\begin{aligned} \|y_n - \bar{x}\|^2 &= (1-\gamma_n)\|x_n - \bar{x}\|^2 + \gamma_n\|T_n x_n - \bar{x}\|^2 - \gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 \\ &\leq \|x_n - \bar{x}\|^2 - \gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 \\ &\leq \|x_n - \bar{x}\|^2, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \|z_n - \bar{x}\|^2 &= (1-\gamma_n)\|y_n - \bar{x}\|^2 + \gamma_n\|T_n y_n - \bar{x}\|^2 - \gamma_n(1-\gamma_n)\|y_n - T_n y_n\|^2 \\ &\leq \|y_n - \bar{x}\|^2 - \gamma_n(1-\gamma_n)\|y_n - T_n y_n\|^2 \\ &\leq \|x_n - \bar{x}\|^2 - \gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 - \gamma_n(1-\gamma_n)\|y_n - T_n y_n\|^2 \\ &\leq \|x_n - \bar{x}\|^2, \end{aligned} \tag{4.4}$$

and hence,

$$\begin{aligned} \|x_{n+1} - \bar{x}\| &\leq \theta_n\|f(y_n) - \bar{x}\| + \mu_n\|x_n - \bar{x}\| + \nu_n\|z_n - \bar{x}\| + \delta_n\|S z_n - \bar{x}\| \\ &\leq \theta_n[\|f(y_n) - f(\bar{x})\| + \|f(\bar{x}) - \bar{x}\|] + \mu_n\|x_n - \bar{x}\| + (\nu_n + \delta_n)\|z_n - \bar{x}\| \\ &\leq \theta_n\rho\|y_n - \bar{x}\| + \theta_n\|f(\bar{x}) - \bar{x}\| + \mu_n\|x_n - \bar{x}\| + (\nu_n + \delta_n)\|z_n - \bar{x}\| \\ &\leq \theta_n\rho\|x_n - \bar{x}\| + \theta_n\|f(\bar{x}) - \bar{x}\| + \mu_n\|x_n - \bar{x}\| + (\nu_n + \delta_n)\|x_n - \bar{x}\| \\ &= \theta_n\rho\|x_n - \bar{x}\| + \theta_n\|f(\bar{x}) - \bar{x}\| + (\mu_n + \nu_n + \delta_n)\|x_n - \bar{x}\| \\ &= \theta_n\rho\|x_n - \bar{x}\| + \theta_n\|f(\bar{x}) - \bar{x}\| + (1-\theta_n)\|x_n - \bar{x}\| \\ &= (1-(1-\rho)\theta_n)\|x_n - \bar{x}\| + \theta_n\|f(\bar{x}) - \bar{x}\| \\ &\leq \max\left\{\|x_n - \bar{x}\|, \frac{1}{1-\rho}\|f(\bar{x}) - \bar{x}\|\right\}. \end{aligned}$$

So, an induction argument shows that

$$\|x_n - \bar{x}\| \leq \max\left\{\|x_0 - \bar{x}\|, \frac{1}{1-\rho}\|f(\bar{x}) - \bar{x}\|\right\}, \quad \forall n \geq 0.$$

Thus, $\{x_n\}$ is bounded, and so are $\{y_n\}$ and $\{z_n\}$.

STEP 2. $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Indeed, observe that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|V_{n+1}x_{n+1}-V_nx_n\|\\ &\le \|V_{n+1}x_{n+1}-V_{n+1}x_n\|+\|V_{n+1}x_n-V_nx_n\|\\ &\le \|x_{n+1}-x_n\|\\ &\quad +\|P_C(I-\lambda_{n+1}A^*(I-P_Q)A)x_n-P_C(I-\lambda_nA^*(I-P_Q)A)x_n\|\\ &\le \|x_{n+1}-x_n\|\\ &\quad +\|(I-\lambda_{n+1}A^*(I-P_Q)A)x_n-(I-\lambda_nA^*(I-P_Q)A)x_n\|\\ &\quad = \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|A^*(I-P_Q)Ax_n\|,\end{aligned}$$

and hence,

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|V_{n+1}y_{n+1}-V_ny_n\|\\ &\le \|V_{n+1}y_{n+1}-V_{n+1}y_n\|+\|V_{n+1}y_n-V_ny_n\|\\ &\le \|y_{n+1}-y_n\|\\ &\quad +\|P_C(I-\lambda_{n+1}A^*(I-P_Q)A)y_n-P_C(I-\lambda_nA^*(I-P_Q)A)y_n\|\\ &\le \|y_{n+1}-y_n\|\\ &\quad +\|(I-\lambda_{n+1}A^*(I-P_Q)A)y_n-(I-\lambda_nA^*(I-P_Q)A)y_n\|\\ &= \|y_{n+1}-y_n\|+|\lambda_{n+1}-\lambda_n|\|A^*(I-P_Q)Ay_n\|\\ &\le \|x_{n+1}-x_n\|\\ &\quad +|\lambda_{n+1}-\lambda_n|(\|A^*(I-P_Q)Ax_n\|+\|A^*(I-P_Q)Ay_n\|).\end{aligned}$$

Now, define $x_{n+1}=\mu_nx_n+(1-\mu_n)w_n$ for all $n\ge 0$. It then follows that

$$\begin{aligned}w_{n+1}-w_n &= \frac{x_{n+2}-\mu_{n+1}x_{n+1}}{1-\mu_{n+1}}-\frac{x_{n+1}-\mu_nx_n}{1-\mu_n}\\ &= \frac{\theta_{n+1}f(y_{n+1})+\nu_{n+1}z_{n+1}+\delta_{n+1}Sz_{n+1}}{1-\mu_{n+1}}-\frac{\theta_nf(y_n)+\nu_nz_n+\delta_nSz_n}{1-\mu_n}\\ &= \frac{\theta_{n+1}f(y_{n+1})}{1-\mu_{n+1}}-\frac{\theta_nf(y_n)}{1-\mu_n}+\frac{\nu_{n+1}(z_{n+1}-z_n)+\delta_{n+1}(Sz_{n+1}-Sz_n)}{1-\mu_{n+1}}\\ &\quad +\left(\frac{\nu_{n+1}}{1-\mu_{n+1}}-\frac{\nu_n}{1-\mu_n}\right)z_n+\left(\frac{\delta_{n+1}}{1-\mu_{n+1}}-\frac{\delta_n}{1-\mu_n}\right)Sz_n,\end{aligned}$$

which hence implies that

$$\begin{aligned}\|w_{n+1}-w_n\| \le{}& \frac{\theta_{n+1}}{1-\mu_{n+1}}\|f(y_{n+1})\|+\frac{\theta_n}{1-\mu_n}\|f(y_n)\|\\ &+\frac{\|\nu_{n+1}(z_{n+1}-z_n)+\delta_{n+1}(Sz_{n+1}-Sz_n)\|}{1-\mu_{n+1}}\\ &+\left|\frac{\nu_{n+1}}{1-\mu_{n+1}}-\frac{\nu_n}{1-\mu_n}\right|\|z_n\|+\left|\frac{\delta_{n+1}}{1-\mu_{n+1}}-\frac{\delta_n}{1-\mu_n}\right|\|Sz_n\|\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\theta_{n+1}}{1-\mu_{n+1}}(\|f(y_{n+1})\|+\|Sz_n\|)+\frac{\theta_n}{1-\mu_n}(\|f(y_n)\|+\|Sz_n\|)\\
&\quad+\frac{\nu_{n+1}+\delta_{n+1}}{1-\mu_{n+1}}\|z_{n+1}-z_n\|+\left|\frac{\nu_{n+1}}{1-\mu_{n+1}}-\frac{\nu_n}{1-\mu_n}\right|(\|z_n\|+\|Sz_n\|)\\
&\le \frac{\theta_{n+1}}{1-\mu_{n+1}}(\|f(y_{n+1})\|+\|Sz_n\|)+\frac{\theta_n}{1-\mu_n}(\|f(y_n)\|+\|Sz_n\|)\\
&\quad+\|z_{n+1}-z_n\|+\left|\frac{\nu_{n+1}}{1-\mu_{n+1}}-\frac{\nu_n}{1-\mu_n}\right|(\|z_n\|+\|Sz_n\|)\\
&\le \frac{\theta_{n+1}}{1-\mu_{n+1}}(\|f(y_{n+1})\|+\|Sz_n\|)+\frac{\theta_n}{1-\mu_n}(\|f(y_n)\|+\|Sz_n\|)\\
&\quad+\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|(\|A^*(I-P_Q)Ax_n\|+\|A^*\\
&\quad\times(I-P_Q)Ay_n\|)+\left|\frac{\nu_{n+1}}{1-\mu_{n+1}}-\frac{\nu_n}{1-\mu_n}\right|(\|z_n\|+\|Sz_n\|),
\end{aligned}
\tag{4.5}
$$

Consequently, it follows from (4.5) and conditions (ii), (iv), (v) that

$$
\begin{aligned}
&\limsup_{n\to\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|)\\
&\le \limsup_{n\to\infty}\left\{\frac{\theta_{n+1}}{1-\mu_{n+1}}(\|f(y_{n+1})\|+\|Sz_n\|)+\frac{\theta_n}{1-\mu_n}(\|f(y_n)\|+\|Sz_n\|)\right.\\
&\quad+|\lambda_{n+1}-\lambda_n|(\|A^*(I-P_Q)Ax_n\|+\|A^*(I-P_Q)Ay_n\|)\\
&\quad\left.+\left|\frac{\nu_{n+1}}{1-\mu_{n+1}}-\frac{\nu_n}{1-\mu_n}\right|(\|z_n\|+\|Sz_n\|)\right\}=0.
\end{aligned}
$$

Hence, by Lemma 2.4, we get $\lim_{n\to\infty}\|w_n-x_n\|=0$. Thus,

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\mu_n)\|w_n-x_n\|=0. \tag{4.6}
$$

STEP 3. $\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|y_n-z_n\|=\lim_{n\to\infty}\|z_n-Sz_n\|=0$.
Indeed, from (4.1), we get

$$
\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &= \langle\theta_n(f(y_n)-\bar{x})+\mu_n(x_n-\bar{x})+\nu_n(z_n-\bar{x})+\delta_n(Sz_n-\bar{x}),x_{n+1}-\bar{x}\rangle\\
&=\theta_n\langle f(y_n)-\bar{x},x_{n+1}-\bar{x}\rangle+\mu_n\langle x_n-\bar{x},x_{n+1}-\bar{x}\rangle\\
&\quad+\langle\nu_n(z_n-\bar{x})+\delta_n(Sz_n-\bar{x}),x_{n+1}-\bar{x}\rangle\\
&\le\theta_n\langle f(y_n)-\bar{x},x_{n+1}-\bar{x}\rangle+\mu_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|\\
&\quad+\|\nu_n(z_n-\bar{x})+\delta_n(Sz_n-\bar{x})\|\|x_{n+1}-\bar{x}\|\\
&\le\theta_n\langle f(y_n)-\bar{x},x_{n+1}-\bar{x}\rangle+\mu_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|\\
&\quad+(\nu_n+\delta_n)\|z_n-\bar{x}\|\|x_{n+1}-\bar{x}\|\\
&\le\theta_n\langle f(y_n)-\bar{x},x_{n+1}-\bar{x}\rangle+\frac{\mu_n}{2}\left(\|x_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2\right)\\
&\quad+\frac{\nu_n+\delta_n}{2}\left(\|z_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2\right),
\end{aligned}
$$

that is,

$$\|x_{n+1}-\bar{x}\|^2 \le \frac{2\theta_n}{1+\theta_n}\langle f(y_n)-\bar{x}, x_{n+1}-\bar{x}\rangle + \frac{\mu_n}{1+\theta_n}\|x_n-\bar{x}\|^2 + \frac{\nu_n+\delta_n}{1+\theta_n}\|z_n-\bar{x}\|^2. \tag{4.7}$$

By combining (4.4) and (4.7), we have

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\le \frac{2\theta_n}{1+\theta_n}\|f(y_n)-\bar{x}\|\|x_{n+1}-\bar{x}\| + \frac{\mu_n}{1+\theta_n}\|x_n-\bar{x}\|^2 + \frac{\nu_n+\delta_n}{1+\theta_n} \\
&\times\|z_n-\bar{x}\|^2 \le \frac{2\theta_n}{1+\theta_n}\|f(y_n)-\bar{x}\|\|x_{n+1}-\bar{x}\| + \frac{\mu_n}{1+\theta_n}\|x_n-\bar{x}\|^2 \\
&+\frac{\nu_n+\delta_n}{1+\theta_n}\left[\|x_n-\bar{x}\|^2-\gamma_n(1-\gamma_n)\|x_n-T_nx_n\|^2-\gamma_n(1-\gamma_n)\|y_n-T_ny_n\|^2\right] \\
&= \frac{2\theta_n}{1+\theta_n}\|f(y_n)-\bar{x}\|\|x_{n+1}-\bar{x}\| + \frac{\mu_n+\nu_n+\delta_n}{1+\theta_n}\|x_n-\bar{x}\|^2 \\
&-\frac{(\nu_n+\delta_n)\gamma_n(1-\gamma_n)}{1+\theta_n}(\|x_n-T_nx_n\|^2+\|y_n-T_ny_n\|^2) \\
&\le \frac{2\theta_n}{1+\theta_n}\|f(y_n)-\bar{x}\|\|x_{n+1}-\bar{x}\| + \|x_n-\bar{x}\|^2 \\
&-\frac{(\nu_n+\delta_n)\gamma_n(1-\gamma_n)}{1+\theta_n}(\|x_n-T_nx_n\|^2+\|y_n-T_ny_n\|^2).
\end{aligned}$$

It immediately follows that

$$\begin{aligned}
&\frac{(\nu_n+\delta_n)\gamma_n(1-\gamma_n)}{1+\theta_n}(\|x_n-T_nx_n\|^2+\|y_n-T_ny_n\|^2) \\
&\le \frac{2\theta_n}{1+\theta_n}\|f(y_n)-\bar{x}\|\|x_{n+1}-\bar{x}\| + \|x_n-\bar{x}\|^2 - \|x_{n+1}-\bar{x}\|^2 \\
&\le \frac{2\theta_n}{1+\theta_n}\|f(y_n)-\bar{x}\|\|x_{n+1}-\bar{x}\| + \|x_n-x_{n+1}\|(\|x_n-\bar{x}\|+\|x_{n+1}-\bar{x}\|).
\end{aligned}$$

Since $\theta_n \to 0$, $\|x_n-x_{n+1}\| \to 0$, $\frac{1}{2} < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$ and $\liminf_{n\to\infty}(\nu_n+\delta_n) > 0$, we have

$$\lim_{n\to\infty}\|x_n-T_nx_n\| = \lim_{n\to\infty}\|y_n-T_ny_n\| = 0. \tag{4.8}$$

This implies that

$$\begin{aligned}
\lim_{n\to\infty}\|y_n-x_n\| &= \lim_{n\to\infty}\gamma_n\|T_nx_n-x_n\| = 0, \\
\lim_{n\to\infty}\|z_n-y_n\| &= \lim_{n\to\infty}\gamma_n\|T_ny_n-y_n\| = 0,
\end{aligned}$$

and so,

$$\limsup_{n\to\infty}\|z_n - x_n\| \le \limsup_{n\to\infty}(\|z_n - y_n\| + \|y_n - x_n\|) = 0.$$

Note that

$$\begin{aligned}\delta_n\|Sz_n - z_n\| &= \|x_{n+1} - z_n - \theta_n(f(y_n) - z_n) - \mu_n(x_n - z_n)\| \\ &\le \|x_{n+1} - z_n\| + \theta_n\|f(y_n) - z_n\| + \mu_n\|x_n - z_n\| \\ &\le \|x_{n+1} - x_n\| + \|x_n - z_n\| + \theta_n\|f(y_n) - z_n\| + \mu_n\|x_n - z_n\| \\ &= \|x_{n+1} - x_n\| + (1 + \mu_n)\|x_n - z_n\| + \theta_n\|f(y_n) - z_n\|.\end{aligned}$$

Since $\theta_n \to 0$, $\liminf_{n\to\infty}\delta_n > 0$, $\|x_n - z_n\| \to 0$ and $\|x_n - x_{n+1}\| \to 0$, we obtain

$$\lim_{n\to\infty}\|Sz_n - z_n\| = 0.$$

STEP 4. $\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \le 0$ where $x^* = (P_{\mathrm{Fix}(S)\cap\Gamma} f)x^*$.
Indeed, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle f(x^*) - x^*, x_{n_i} - x^*\rangle. \tag{4.9}$$

Without loss of generality, we may assume that $x_{n_i} \rightharpoonup w$ and $\lambda_{n_i} \to \lambda \in \left(0, \frac{2}{\|A\|^2}\right)$, due to condition (v). First, it is clear from $\|x_n - z_n\| \to 0$ that $z_{n_i} \rightharpoonup w$. Hence, by Lemma 2.1, we deduce from $\|z_n - Sz_n\| \to 0$ that $w \in \mathrm{Fix}(S)$. Second, let us show that $w \in \Gamma$. Set $V = P_C(I - \lambda A^*(I - P_Q)A)$. Notice that V is nonexpansive and $\mathrm{Fix}(V) = \Gamma$. It turns out that

$$\begin{aligned}\|x_{n_i} - Vx_{n_i}\| &\le \|x_{n_i} - V_{n_i}x_{n_i}\| + \|V_{n_i}x_{n_i} - Vx_{n_i}\| \\ &= \|x_{n_i} - y_{n_i}\| \\ &\quad + \|P_C(I - \lambda_{n_i}A^*(I - P_Q)A)x_{n_i} - P_C(I - \lambda A^*(I - P_Q)A)x_{n_i}\| \\ &\le \|x_{n_i} - y_{n_i}\| \\ &\quad + \|(I - \lambda_{n_i}A^*(I - P_Q)A)x_{n_i} - (I - \lambda A^*(I - P_Q)A)x_{n_i}\| \\ &= \|x_{n_i} - y_{n_i}\| + |\lambda_{n_i} - \lambda|\|A^*(I - P_Q)Ax_{n_i}\| \to 0.\end{aligned}$$

By Lemma 2.1, we get $w \in \mathrm{Fix}(V)$, and hence, $w \in \Gamma$. Therefore, $w \in \mathrm{Fix}(S) \cap \Gamma$. This together with (4.9) and the property of metric projection implies that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle &= \lim_{i\to\infty}\langle f(x^*) - x^*, x_{n_i} - x^*\rangle \\ &= \langle f(x^*) - x^*, w - x^*\rangle \\ &\le 0.\end{aligned} \tag{4.10}$$

STEP 5. $\lim_{n\to\infty}\|x_n - x^*\| = \lim_{n\to\infty}\|y_n - x^*\| = 0$.

Indeed, from (4.4) and (4.7), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \frac{2\theta_n}{1+\theta_n}\langle f(y_n)-x^*, x_{n+1}-x^*\rangle + \frac{\mu_n}{1+\theta_n}\|x_n-x^*\|^2 \\ &\quad + \frac{\nu_n+\delta_n}{1+\theta_n}\|x_n-x^*\|^2 = \frac{2\theta_n}{1+\theta_n}\langle f(x_n)-f(x^*), x_{n+1}-x^*\rangle \\ &\quad + \frac{2\theta_n}{1+\theta_n}[\langle f(x^*)-x^*, x_{n+1}-x^*\rangle + \langle f(y_n)-f(x_n), x_{n+1}-x^*\rangle] \\ &\quad + \frac{1-\theta_n}{1+\theta_n}\|x_n-x^*\|^2 \le \frac{2\theta_n\rho}{1+\theta_n}\|x_n-x^*\|\|x_{n+1}-x^*\| \\ &\quad + \frac{2\theta_n}{1+\theta_n}[\langle f(x^*)-x^*, x_{n+1}-x^*\rangle \\ &\quad + \rho\|y_n-x_n\|\|x_{n+1}-x^*\|] + \frac{1-\theta_n}{1+\theta_n}\|x_n-x^*\|^2 \le \frac{\theta_n\rho}{1+\theta_n} \\ &\quad \times(\|x_n-x^*\|^2 + \|x_{n+1}-x^*\|^2) + \frac{2\theta_n}{1+\theta_n}[\langle f(x^*)-x^*, x_{n+1}-x^*\rangle \\ &\quad + \rho\|y_n-x_n\|\|x_{n+1}-x^*\|] + \frac{1-\theta_n}{1+\theta_n}\|x_n-x^*\|^2 \\ &= \frac{1-(1-\rho)\theta_n}{1+\theta_n}\|x_n-x^*\|^2 + \frac{\theta_n\rho}{1+\theta_n}\|x_{n+1}-x^*\|^2 \\ &\quad + \frac{2\theta_n}{1+\theta_n}[\langle f(x^*)-x^*, x_{n+1}-x^*\rangle + \rho\|y_n-x_n\|\|x_{n+1}-x^*\|],\end{aligned}$$

which hence implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \frac{1-(1-\rho)\theta_n}{1+(1-\rho)\theta_n}\|x_n-x^*\|^2 + \frac{2\theta_n}{1+(1-\rho)\theta_n} \\ &\quad \times[\langle f(x^*)-x^*, x_{n+1}-x^*\rangle + \rho\|y_n-x_n\|\|x_{n+1}-x^*\|] \\ &\le (1-(1-\rho)\theta_n)\|x_n-x^*\|^2 + \frac{2\theta_n}{1+(1-\rho)\theta_n} \\ &\quad \times[\langle f(x^*)-x^*, x_{n+1}-x^*\rangle + \rho\|y_n-x_n\|\|x_{n+1}-x^*\|] \\ &= (1-(1-\rho)\theta_n)\|x_n-x^*\|^2 \\ &\quad +(1-\rho)\theta_n\cdot\frac{2}{(1+(1-\rho)\theta_n)(1-\rho)}[\langle f(x^*)-x^*, x_{n+1}-x^*\rangle \\ &\quad +\rho\|y_n-x_n\|\|x_{n+1}-x^*\|]. \end{aligned} \tag{4.11}$$

Note that $\theta_n \to 0$ and $\sum_{n=0}^{\infty}(1-\rho)\theta_n = \infty$ due to condition (ii). Since

$$\limsup_{n\to\infty}\frac{2}{(1+(1-\rho)\theta_n)(1-\rho)}[\langle f(x^*)-x^*, x_{n+1}-x^*\rangle + \rho\|y_n-x_n\| \\ \|x_{n+1}-x^*\|] \le 0$$

due to (4.10), an application of Lemma 2.2 to (4.11) yields $\|x_n - x^*\| \to 0$. Consequently, $\|y_n - x^*\| \to 0$ by using $\|x_n - y_n\| \to 0$. □

5 Mann type relaxed *CQ* algorithm

As pointed out in [26], the *CQ* algorithm (3.4) involves two projections P_C and P_Q and hence might be hard to be implemented in the case where one of them fails to have a closed-form expression. Thus, in [26] it was shown that if C and Q are level sets of convex functions, then the projections onto half-spaces are just needed to make the *CQ* algorithm implementable in this case. Inspired by his relaxed *CQ* algorithm, we propose Mann type relaxed *CQ* algorithm via projections onto half-spaces.

Define the closed convex sets C and Q as the level sets:

$$C = \{x \in \mathcal{H}_1 : c(x) \le 0\} \quad \text{and} \quad Q = \{y \in \mathcal{H}_2 : q(y) \le 0\}, \tag{5.1}$$

where $c : \mathcal{H}_1 \to \mathbf{R}$ and $q : \mathcal{H}_2 \to \mathbf{R}$ are the convex functions. We assume that c and q are subdifferentiable on C and Q, respectively, namely, the subdifferentials

$$\partial c(x) = \{z \in \mathcal{H}_1 : c(u) \ge c(x) + \langle u - x, z\rangle,\ \forall u \in \mathcal{H}_1\} \ne \emptyset$$

for all $x \in C$, and

$$\partial q(y) = \{w \in \mathcal{H}_2 : q(v) \ge q(y) + \langle v - y, w\rangle,\ \forall v \in \mathcal{H}_2\} \ne \emptyset$$

for all $y \in Q$. We also assume that c and q are bounded on bounded sets. Note that this condition is automatically satisfied if $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional. This assumption guarantees that if $\{x_n\}$ is a bounded sequence in $\mathcal{H}_1$ (respectively, $\mathcal{H}_2$) and $\{x_n^*\}$ is another sequence in $\mathcal{H}_1$ (respectively, $\mathcal{H}_2$) such that $x_n^* \in \partial c(x_n)$ (respectively, $x_n^* \in \partial q(x_n)$) for each $n \ge 0$, then $\{x_n^*\}$ is bounded.

Let $S : \mathcal{H}_1 \to \mathcal{H}_1$ be a nonexpansive mapping. Assume that the sequences of parameters $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0, 1]$ and $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\{\beta_n\} \subset [0, 1]$ and $\liminf_{n\to\infty} \beta_n > 0$;
(iii) $\{\lambda_n\} \subset \left(0, \frac{2}{\|A\|^2}\right)$ and $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \frac{2}{\|A\|^2}$.

Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by the following Mann type relaxed *CQ* algorithm:

$$\begin{cases} x_0 = x \in \mathcal{H}_1 \text{ chosen arbitrarily}, \\ y_n = (1 - \beta_n)x_n + \beta_n P_{C_n}(I - \lambda_n A^*(I - P_{Q_n})A)x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_{C_n}(I - \lambda_n A^*(I - P_{Q_n})A)y_n, \quad \forall n \ge 0, \end{cases} \tag{5.2}$$

where $\{C_n\}$ and $\{Q_n\}$ are the sequences of closed convex sets constructed as follows:

$$C_n = \{x \in \mathcal{H}_1 : c(x_n) + \langle \xi_n, x - x_n\rangle \le 0\}, \tag{5.3}$$

where $\xi_n \in \partial c(x_n)$, and

$$Q_n = \{y \in \mathcal{H}_2 : q(Ax_n) + \langle \eta_n, y - Ax_n \rangle \leq 0\}, \tag{5.4}$$

where $\eta_n \in \partial q(Ax_n)$.

It can be easily seen that $C \subset C_n$ and $Q \subset Q_n$ for all $n \geq 0$. Also note that C_n and Q_n are half-spaces; thus, the projections P_{C_n} and P_{Q_n} have closed-form expressions.

The relaxed *CQ* algorithm was introduced in [29] (see also [24]) in the finite-dimensional setting. In [26], Xu derived the weak convergence of this algorithm in the infinite-dimensional setting. The following theorem establishes the weak convergence of Mann type relaxed *CQ* algorithm in the infinite-dimensional setting. Our proof is very different from that of Xu's Theorem 4.1 [26], and also needs the Hilbert space technique and averaged mapping expression technique.

Theorem 5.1 *Suppose that $Fix(S) \cap \Gamma \neq \emptyset$. The sequences $\{x_n\}$ and $\{y_n\}$ generated by algorithm (5.2) converge weakly to $z \in Fix(S) \cap \Gamma$, where*

$$z = \|\cdot\| - \lim_{n\to\infty} P_{\mathrm{Fix}(S)\cap\Gamma} x_n.$$

Proof For the sake of simplicity, we may assume that

$$0 < a \leq \lambda_n \leq b < \frac{2}{\|A\|^2} \quad \text{and} \quad 0 < c \leq \alpha_n \leq d < 1$$

for all $n \geq 0$, where a, b, c and d are constants.

Repeating the same argument as in the proof of Theorem 3.2, we can write

$$\begin{aligned} \widehat{V}_n &:= P_{C_n}(I - \lambda_n A^*(I - P_{Q_n})A) \\ &= \frac{2 - \lambda_n\|A\|^2}{4} I + \frac{2 + \lambda_n\|A\|^2}{4} \widehat{T}_n \\ &= (1 - \gamma_n)I + \gamma_n \widehat{T}_n, \end{aligned} \tag{5.5}$$

where $\widehat{T}_n$ is nonexpansive and $\gamma_n = \frac{2+\lambda_n\|A\|^2}{4} \in [a_1, b_1] \subset (0, 1)$ with $a_1 = (2 + a\|A\|^2)/4$ and $b_1 = (2 + b\|A\|^2)/4 < 1$. Then $\widehat{V}_n$ is nonexpansive and the Mann type relaxed *CQ* algorithm (5.2) can be rewritten as

$$\begin{cases} x_0 = x \in \mathcal{H}_1 \text{ chosen arbitrarily}, \\ y_n = (1 - \beta_n)x_n + \beta_n \widehat{V}_n x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S \widehat{V}_n y_n, \quad \forall n \geq 0. \end{cases}$$

Moreover, we have $\widehat{V}_n x^* = x^*$ for all $x^* \in \Gamma$. In the meantime, we have

$$y_n = (1 - \beta_n)x_n + \beta_n[(1 - \gamma_n)x_n + \gamma_n \widehat{T}_n x_n]. \tag{5.6}$$

Put

$$z_n := \widehat{V}_n y_n \ (= (1-\gamma_n)y_n + \gamma_n \widehat{T}_n y_n). \tag{5.7}$$

Then, $x_{n+1} = \alpha_n x_n + (1-\alpha_n)Sz_n$ for all $n \geq 0$. Now take a fixed $p \in \mathrm{Fix}(S) \cap \Gamma$ arbitrarily. Note that $\widehat{T}_n p = p$ and $Sp = p$. Thus, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n[(1-\gamma_n)x_n + \gamma_n \widehat{T}_n x_n - p]\|^2 \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n \|(1-\gamma_n)(x_n - p) + \gamma_n(\widehat{T}_n x_n - p)\|^2 \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n[(1-\gamma_n)\|x_n - p\|^2 \\
&\quad + \gamma_n \|\widehat{T}_n x_n - p\|^2 - \gamma_n(1-\gamma_n)\|x_n - \widehat{T}_n x_n\|^2] \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n[(1-\gamma_n)\|x_n - p\|^2 \\
&\quad + \gamma_n \|x_n - p\|^2 - \gamma_n(1-\gamma_n)\|x_n - \widehat{T}_n x_n\|^2] \\
&= \|x_n - p\|^2 - \beta_n \gamma_n (1-\gamma_n)\|x_n - \widehat{T}_n x_n\|^2,
\end{aligned}$$

$$\begin{aligned}
\|z_n - p\|^2 &= (1-\gamma_n)\|y_n - p\|^2 + \gamma_n \|\widehat{T}_n y_n - p\|^2 - \gamma_n(1-\gamma_n)\|y_n - \widehat{T}_n y_n\|^2 \\
&\leq \|y_n - p\|^2 - \gamma_n(1-\gamma_n)\|y_n - \widehat{T}_n y_n\|^2 \\
&\leq \|x_n - p\|^2 - \beta_n \gamma_n(1-\gamma_n)\|x_n - \widehat{T}_n x_n\|^2 - \gamma_n(1-\gamma_n)\|y_n - \widehat{T}_n y_n\|^2,
\end{aligned}$$

and hence,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|Sz_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sz_n\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sz_n\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 - \beta_n \gamma_n(1-\gamma_n)\|x_n - \widehat{T}_n x_n\|^2 \\
&\quad - \gamma_n(1-\gamma_n)\|y_n - \widehat{T}_n y_n\|^2] - \alpha_n(1-\alpha_n)\|x_n - Sz_n\|^2 \\
&= \|x_n - p\|^2 - (1-\alpha_n)\beta_n \gamma_n (1-\gamma_n)\|x_n - \widehat{T}_n x_n\|^2 \\
&\quad - (1-\alpha_n)\gamma_n(1-\gamma_n)\|y_n - \widehat{T}_n y_n\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sz_n\|^2 \\
&\leq \|x_n - p\|^2.
\end{aligned} \tag{5.8}$$

It follows that the real nonnegative sequence $\{\|x_n - p\|\}$ is nonincreasing. Hence,

$$\lim_{n\to\infty} \|x_n - p\| \text{ exists for all} p \in \mathrm{Fix}(S) \cap \Gamma. \tag{5.9}$$

Noting that $\liminf_{n\to\infty} \beta_n > 0$, we may assume that $\beta_n \geq l (\forall n \geq 0)$ for some $l > 0$. From (5.8) it follows that

$$\begin{aligned}
&(1-d)la_1(1-b_1)\|x_n - \widehat{T}_n x_n\|^2 \\
&\quad + (1-d)a_1(1-b_1)\|y_n - \widehat{T}_n y_n\|^2 + c(1-d)\|x_n - Sz_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n)\beta_n\gamma_n(1-\gamma_n)\|x_n-\widehat{T}_n x_n\|^2+(1-\alpha_n)\gamma_n(1-\gamma_n)\|y_n-\widehat{T}_n y_n\|^2\\
&\quad +\alpha_n(1-\alpha_n)\|x_n-Sz_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2.
\end{aligned}
$$

Consequently, from (5.9) we deduce that

$$
\lim_{n\to\infty}\|x_n-\widehat{T}_n x_n\|=\lim_{n\to\infty}\|y_n-\widehat{T}_n y_n\|=\lim_{n\to\infty}\|x_n-Sz_n\|=0. \tag{5.10}
$$

This together with (5.6) and (5.7) implies that

$$
\begin{aligned}
\lim_{n\to\infty}\|y_n-x_n\| &= \lim_{n\to\infty}\beta_n\gamma_n\|\widehat{T}_n x_n-x_n\|=0,\\
\lim_{n\to\infty}\|z_n-y_n\| &= \lim_{n\to\infty}\gamma_n\|\widehat{T}_n y_n-y_n\|=0.
\end{aligned}
$$

and hence,

$$
\lim_{n\to\infty}\|z_n-x_n\|=\lim_{n\to\infty}\|z_n-Sz_n\|=0. \tag{5.11}
$$

Next let us show that

$$
\omega_w(x_n)\subset \mathrm{Fix}(S)\cap\Gamma. \tag{5.12}
$$

Indeed, assume that $\hat{x}\in\omega_w(x_n)$ and $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_j}\rightharpoonup\hat{x}$. Utilizing (5.11), we have $z_{n_j}\rightharpoonup\hat{x}$ and $\lim_{j\to\infty}\|z_{n_j}-Sz_{n_j}\|=0$. By Lemma 2.1, we have $\hat{x}\in\mathrm{Fix}(S)$. Furthermore, since $x_{n_j+1}\in C_{n_j}$, we obtain

$$
c(x_{n_j})+\langle\xi_{n_j},x_{n_j+1}-x_{n_j}\rangle\le 0.
$$

Thus,

$$
c(x_{n_j})\le-\langle\xi_{n_j},x_{n_j+1}-x_{n_j}\rangle\le\xi\|x_{n_j+1}-x_{n_j}\|,
$$

where ξ satisfies $\|\xi_n\|\le\xi$ for all $n\ge 0$. By virtue of the lower semicontinuity of c, we get from (5.8)

$$
c(\hat{x})\le\liminf_{j\to\infty}c(x_{n_j})\le 0.
$$

Therefore, $\hat{x}\in C$.

Now we show that $\|(I-P_{Q_n})Ax_n\|\to 0$. Indeed, we need more accurate estimates on $\|z_n-p\|$ as follows. Using the nonexpansivity of projections, we get

$$\begin{aligned}\|z_n - p\|^2 &\le \|(I - \lambda_n A^*(I - P_{Q_n})A)y_n - p\|^2\\ &= \|(y_n - p) - \lambda_n A^*(I - P_{Q_n})Ay_n\|^2\\ &= \|y_n - p\|^2 + \lambda_n^2\|A^*(I - P_{Q_n})Ay_n\|^2\\ &\quad -2\lambda_n\langle y_n - p, A^*(I - P_{Q_n})Ay_n\rangle\\ &\le \|y_n - p\|^2 + \lambda_n^2\|A\|^2\|(I - P_{Q_n})Ay_n\|^2\\ &\quad -2\lambda_n\langle Ay_n - Ap, (I - P_{Q_n})Ay_n\rangle. \end{aligned} \tag{5.13}$$

Since $Ap \in Q \subset Q_n$, we have

$$\langle (I - P_{Q_n})Ay_n, Ap - P_{Q_n}Ay_n\rangle \le 0.$$

This implies that

$$\begin{aligned}\langle (I - P_{Q_n})Ay_n, Ay_n - Ap\rangle &= \langle (I - P_{Q_n})Ay_n, Ay_n - P_{Q_n}Ay_n\rangle\\ &\quad +\langle (I - P_{Q_n})Ay_n, P_{Q_n}Ay_n - Ap\rangle\\ &\ge \|(I - P_{Q_n})Ay_n\|^2. \end{aligned} \tag{5.14}$$

By combining (5.13) and (5.14), we get

$$\|z_n - p\|^2 \le \|y_n - p\|^2 - \lambda_n(2 - \lambda_n\|A\|^2)\|(I - P_{Q_n})Ay_n\|^2, \tag{5.15}$$

which hence implies that

$$\begin{aligned}\lambda_n(2 - \lambda_n\|A\|^2)\|(I - P_{Q_n})Ay_n\|^2 &\le \|y_n - p\|^2 - \|z_n - p\|^2\\ &\le (\|y_n - p\| + \|z_n - p\|)\|z_n - y_n\|. \end{aligned}$$

Since $\lim_{n\to\infty}\|z_n - y_n\| = 0$ and $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < \frac{2}{\|A\|^2}$, we have

$$\lim_{n\to\infty}\|(I - P_{Q_n})Ay_n\| = 0. \tag{5.16}$$

Thus it follows that

$$\begin{aligned}\|(I - P_{Q_n})Ax_n\| &\le \|(I - P_{Q_n})Ay_n\| + \|(I - P_{Q_n})Ax_n - (I - P_{Q_n})Ay_n\|\\ &\le \|(I - P_{Q_n})Ay_n\| + \|Ax_n - Ay_n\|\\ &\le \|(I - P_{Q_n})Ay_n\| + \|A\|\|x_n - y_n\| \to 0, \end{aligned}$$

due to $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Next let us show that $A\hat{x} \in Q$. To see this, set $w_n = Ax_n - P_{Q_n}Ax_n \to 0$ and let η be such that $\|\eta_n\| \le \|\eta\|$ for all $n \ge 0$. Then, since $Ax_{n_j} - w_{n_j} = P_{Q_{n_j}}Ax_{n_j} \in Q_{n_j}$, we get

$$q(Ax_{n_j}) + \langle \eta_{n_j}, (Ax_{n_j} - w_{n_j}) - Ax_{n_j} \rangle \le 0.$$

Hence,

$$q(Ax_{n_j}) \le \langle \eta_{n_j}, w_{n_j} \rangle \le \eta \|w_{n_j}\| \to 0.$$

By the weak lower semicontinuity of q and the fact that $Ax_{n_j} \to A\hat{x}$ weakly, we arrive at the following conclusion

$$q(A\hat{x}) \le \liminf_{j\to\infty} q(Ax_{n_j}) \le 0.$$

Namely, $A\hat{x} \in Q$.

Therefore, $\hat{x} \in \Gamma$. This shows that (5.12) holds. Now we can apply Proposition 2.4 to $K := \mathrm{Fix}(S) \cap \Gamma$ to get that the full sequence $\{x_n\}$ converges weakly to a point $z \in \mathrm{Fix}(S) \cap \Gamma$. In the meantime, from $\|x_n - y_n\| \to 0$ it follows that $y_n \rightharpoonup z$. Now, put

$$u_n = P_{\mathrm{Fix}(S)\cap\Gamma} x_n.$$

Let us show that

$$\lim_{n\to\infty} \|u_n - z\| = 0.$$

Indeed, noticing the fact that

$$u_n = P_{\mathrm{Fix}(S)\cap\Gamma} x_n \quad \text{and} \quad z \in \mathrm{Fix}(S) \cap \Gamma,$$

in terms of Proposition 2.1 (i) we have

$$\langle z - u_n, u_n - x_n \rangle \ge 0. \tag{5.17}$$

Utilizing Proposition 3.2, we deduce from (5.8) that $\{u_n\}$ converges strongly to some $z_0 \in \mathrm{Fix}(S) \cap \Gamma$. Then, we have from (5.17)

$$\langle z - z_0, z_0 - z \rangle \ge 0,$$

and hence $z = z_0$. □

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