Positivity (2010) 14:859–875
DOI 10.1007/s11117-010-0089-1

Positivity

Discrete stochastic integration in Riesz spaces

Coenraad C. A. Labuschagne · Bruce A. Watson

Received: 2 September 2009 / Accepted: 17 September 2010 / Published online: 15 October 2010

Abstract In this work we continue the developments of Kuo et al. (Indag Math 15:435–451, 2004; J Math Anal Appl 303:509–521, 2005) with the construction of the martingale transform or discrete stochastic integral in a Riesz space (measure-free) setting. The discrete stochastic integral is considered both in terms of a weighted sum of differences and via bilinear vector-valued forms. For this, analogues of the spaces L^2 and Mart^2 on Riesz spaces with a conditional expectation operator and a weak order unit are constructed using the f-algebra structure of the universal completion of the Riesz space and properties of the extension of the conditional expectation to its natural domain.

Keywords Martingale transform · Stochastic integral · Riesz space · f-Algebra · Universal completion

Mathematics Subject Classification (2000) 47B60 · 60H05 · 60G42 · 60G48

C. C. A. Labuschagne was supported in part by the South African National Research Foundation grant FA2007041200015 and B. A. Watson was supported in part by the John Knopfmacher Centre for Applicable Analysis and Number Theory and by the South African National Research Foundation grant FA2007041200006.

C. C. A. Labuschagne
School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, PO Wits 2050, South Africa
e-mail: coenraad.labuschagne@wits.ac.za

B. A. Watson (✉)
School of Mathematics, University of the Witwatersrand, Private Bag 3, PO Wits 2050, South Africa
e-mail: bruce.watson@wits.ac.za

1 Introduction

The stochastic integral plays an important part in the theory and application of stochastic processes. In [9–13] discrete-time martingale theory was extended from the classical L^1 setting to general vector lattices (Riesz spaces), including such aspects as martingale convergence, optional stopping, ergodic theory and the theory of amarts. We refer the reader to [18–20] for another, independent, generalization of stochastic processes to ordered spaces. In this work we continue the developments of [9–11] with the construction of the martingale transform or discrete stochastic integral in a Riesz space (measure-free) setting. As in our previous work, if the underlying Riesz space is taken to be $L^1(\Omega, \mathcal{A}, P)$, where P is a probability measure, then our results yield the classical theory as a special case. For general background in Riesz spaces we refer the reader to [1,14,27].

First, the martingale transform (discrete stochastic integral) is defined in terms of a weighted sum of differences, see [8,17] for the classical version. Next the discrete stochastic integral (martingale transform) is considered via bilinear vector-valued forms, as in [6–8,21]. This requires analogues of the spaces L^2 and Mart^2 for Riesz spaces. The construction of these spaces relies heavily on [9,10], where we considered a conditional expectation operator T on a Dedekind complete Riesz space E with a weak order unit e and its extension $\mathbf{T}$ to its natural domain $\mathrm{dom}(T)$ in the universal completion, E^u, of E. using the fact that E^u is an f-algebra, we consider the space $\mathrm{dom}^2(T)$ of all $x \in \mathrm{dom}(T)$ for which $x^2 \in \mathrm{dom}(T)$. This forms the foundation for the definition of the analogue of L^2 in the Riesz space setting. It is then shown that, as in the classical case, the two approaches to defining the discrete stochastic integral or martingale transform are consistent.

This paper is organized as follows. In Sect. 2 we continue our study of the natural domain of a conditional expectation operator started in [10]. This is needed for the definition of the analogues of spaces $L^p(\Omega, \mathcal{A}, P)$, $p = 1, 2, \infty$ in terms of a Riesz space with conditional expectation operator in Sect. 3. In Sect. 4 the space of square martingales Mart^2 is defined, along with the bilinear vector-valued form on it, needed for the Itô type integral. Finally, in Sect. 5, the discrete stochastic integral is defined and its relationship to Mart^2 considered.

2 Preliminaries

In the probability space setting each conditional expectation operator T can be extended to a conditional expectation $\mathbf{T}$ on the, so called, *natural domain* of T, denoted by $\mathrm{dom}(T)$, see [4,15].

A Riesz space E is said to be *universally complete* if E is Dedekind complete and every subset of E which consists of mutually disjoint elements, has a supremum in E. A Riesz space E^u is a *universal completion* of E, if E^u is universally complete and E^u contains E as an order dense Riesz subspace.

Every Archimedean Riesz space has (up to a Riesz isomorphism) a unique universal completion and if e is a weak order unit for E then e is a weak order unit for E^u, see [26]. In addition, the multiplication on E^e (the space of e bounded elements of E)

can be uniquely extended to E^u giving E^u an f-algebra structure in which e is both multiplicative unit and weak order unit, see [2,3,5,16], and, for a proof that does not rely on function spaces representations [23,24]. Since the multiplication on an Archimedean f-algebra is order continuous [26, Theorems 139.4 and 141.1], E^u has an order continuous multiplication.

Let E be a Dedekind complete Riesz space with a weak order unit e and T be a conditional expectation on E with $T(e) = e$. Let

$$D(\tau) := \{x \in E^u_+ \mid \exists\, (x_\alpha) \subset E_+, 0 \le x_\alpha \uparrow x, (Tx_\alpha) \text{ order bounded in } E^u\},$$

and for $x \in D(\tau)$ define

$$\tau(x) = \sup_\alpha T(x_\alpha),$$

where $(x_\alpha) \subset E_+$ with $x_\alpha \uparrow x$ and (Tx_α) order bounded in E^u. Then $\tau : D(\tau) \to E^u_+$ is a well defined increasing additive order continuous map. Define

$$\mathrm{dom}(T) := D(\tau) - D(\tau)$$

and $\mathbf{T} : \mathrm{dom}(T) \to E^u$ by

$$\mathbf{T}f := \tau(f^+) - \tau(f^-) \quad \text{for} \quad f \in \mathrm{dom}(T).$$

Then $\mathrm{dom}(T)$ is an order dense order ideal of E^u containing E and e is a weak order unit for $\mathrm{dom}(T)$. In addition, $\mathbf{T}$ is a projection (with $R(\mathbf{T}) \subset \mathrm{dom}(T)$) and is the unique, order continuous positive linear extension of T to $\mathrm{dom}(T)$. These properties of $\mathbf{T}$ along with the following theorem were proved in [10].

Theorem 2.1 *Let E be a Dedekind complete Riesz space with a weak order unit and T a conditional expectation on E. The extension $\mathbf{T} : \mathrm{dom}(T) \to \mathrm{dom}(T)$ is a conditional expectation and an averaging operator on $\mathrm{dom}(T)$, i.e.*

$$\mathbf{T}(gf) = g\mathbf{T}(f) \quad \textit{for} \quad g \in R(\mathbf{T}),\ f, gf \in \mathrm{dom}(T).$$

Define

$$\mathrm{dom}^2(T) := \{x \in \mathrm{dom}(T) | x^2 \in \mathrm{dom}(T)\}.$$

Lemma 2.2 *The set $\mathrm{dom}^2(T)$ is an order ideal of $\mathrm{dom}(T)$ (and is thus Dedekind complete) and pairwise products of elements of $\mathrm{dom}^2(T)$ give elements of $\mathrm{dom}(T)$.*

Proof From the definition of $\mathrm{dom}^2(T)$, it is clear that $\mathrm{dom}^2(T)$ is homogeneous.

Let $x, y \in \mathrm{dom}^2(T)$ then

$$0 \le (x - y)^2 = x^2 + y^2 - 2xy.$$

Hence $2xy \le x^2 + y^2$ and, similarly, $-2xy \le (-x)^2 + y^2$, which give

$$2|xy| \le x^2 + y^2 \in \mathrm{dom}(T).$$

Since $\mathrm{dom}(T)$ is an order ideal in E^u, it now follows that $xy \in \mathrm{dom}(T)$.

Now, since $xy \in \mathrm{dom}(T)$,

$$(x+y)^2 = x^2 + 2xy + y^2 \in \mathrm{dom}(T),$$

from which the additivity of $\mathrm{dom}^2(T)$ follows, making it a vector subspace of $\mathrm{dom}(T)$. Also, from the f-algebra structure of E^u,

$$|x|^2 = x^2 \in \mathrm{dom}(T),$$

giving $|x| \in \mathrm{dom}^2(T)$. Hence $\mathrm{dom}^2(T)$ is a Riesz subspace of $\mathrm{dom}(T)$.

It remains only to show that $\mathrm{dom}^2(T)$ is solid. Let $|g| \le |f|$ where $g \in \mathrm{dom}(T)$ and $f \in \mathrm{dom}^2(T)$. Then $|g^2| \in E^u$ and

$$|g^2| = |g|^2 \le |f|^2 = |f^2| \in \mathrm{dom}(T).$$

Since $\mathrm{dom}(T)$ is an ideal in E^u, we have that $|g|^2 \in \mathrm{dom}(T)$ and hence $g \in \mathrm{dom}^2(T)$. □

Note that if $f \in E^u_+$ then there exists a unique $g \in E^u_+$ with $g^2 = f$. This g will be denoted $\sqrt{f}$.

Lemma 2.3 *If* $f_\alpha, f \in \mathrm{dom}^2(T)$ *and* $0 \le f_\alpha \le f \in \mathrm{dom}^2(T)$, *then* $\sup f_\alpha \in \mathrm{dom}^2(T)$; *in fact,*

$$(\sup f_\alpha)^2 = \sup f_\alpha^2.$$

Proof Since $\mathrm{dom}(T)$ is Dedekind complete, $\sup f_\alpha \in \mathrm{dom}(T)$ and $\sup f_\alpha^2 \in \mathrm{dom}(T)$. It follows from $0 \le f_\alpha \le \sup f_\alpha \le f \in \mathrm{dom}^2(T)$, that $0 \le f_\alpha^2 \le (\sup f_\alpha)^2 \le f^2 \in \mathrm{dom}(T)$. Hence $\sup f_\alpha^2 \le (\sup f_\alpha)^2$ and $\sup f_\alpha \in \mathrm{dom}^2(T)$.

To show $(\sup f_\alpha)^2 = \sup f_\alpha^2$, let $h \in E^u_+$ with $h \ge f_\alpha^2$ for all α, then $\sqrt{h} \ge f_\alpha$ for all α. Hence, $\sqrt{h} \ge \sup f_\alpha$, from which we get $h \ge (\sup f_\alpha)^2$. Since $\sup f_\alpha^2$ is the least upper bound of the set $\{f_\alpha^2 \mid \text{ for all } \alpha\}$, taking $h = \sup f_\alpha$, it follows that $(\sup f_\alpha)^2 \le \sup f_\alpha^2$. □

In fact this proof can be adapted to show that multiplication is jointly order-continuous.

Definition 2.4 Let E be a Dedekind complete Riesz space and T a strictly positive conditional expectation on E. The space E is *universally complete* with respect to T (or T-universally complete), if for each increasing net (f_α) in E_+ with (Tf_α) order bounded in E^u, we have that (f_α) is order convergent in E to f. In this case $\mathrm{dom}(T) = E$.

If E is a Dedekind complete Riesz space with a weak order unit and T is a conditional expectation on E, then $\mathrm{dom}(T)$ quotient the absolute kernel of $\mathbf{T}$ will also be referred to as the T-universal completion of E. Here the absolute kernel of T is given by $|Ker|(T) := \{x \in E \ : \ T|x| = 0\}$ and, since $|Ker|(T)$ is a band (as T is order continuous), $\mathrm{dom}(T)/|Ker|(T)$ is a Dedekind complete Riesz space with respect to the canonical ordering. In addition, $\mathbf{T}$ is a strictly positive conditional expectation on this space, see [10].

We now give a version of the "Conditional Variance Identity".

Theorem 2.5 (Conditional Variance Identity) *If E is a Dedekind complete Riesz space with a weak order unit and T is a conditional expectation on E, then*

$$\mathbf{T}(y - \mathbf{T}y)^2 = \mathbf{T}y^2 - (\mathbf{T}y)^2 \quad \textit{for all } y, \mathbf{T}y \in \mathrm{dom}^2(T)$$

and $(\mathbf{T}y)^2 \in R(\mathbf{T})$.

Proof By Theorem 2.1 we get $\mathbf{T}(y\mathbf{T}y) = (\mathbf{T}y)^2$, giving $(\mathbf{T}y)^2 \in R(\mathbf{T})$ and thus $\mathbf{T}(\mathbf{T}y)^2 = (\mathbf{T}y)^2$. Hence

$$\begin{aligned}
\mathbf{T}(y - \mathbf{T}y)^2 &= \mathbf{T}[y^2 - 2y\mathbf{T}y + (\mathbf{T}y)^2] \\
&= \mathbf{T}y^2 - 2\mathbf{T}(y\mathbf{T}y) + \mathbf{T}(\mathbf{T}y)^2 \\
&= \mathbf{T}y^2 - (\mathbf{T}y)^2.
\end{aligned}$$

□

3 $\mathcal{L}^p(T_i)$

Let E be a Dedekind complete Riesz space with a weak order unit e and let (T_i) be a filtration on E for which $T_i(e) = e$ for all $i \in \mathbb{N}$, and with T_1 strictly positive.

Consider the sequence $(\mathbf{T}_i)$ of extensions of (T_i) where $\mathbf{T}_i$ has domain in E^u, $\mathrm{dom}(T_i)$. We now show that $\mathrm{dom}(T_i) \uparrow_i$ since $T_i T_j = T_i$ for all $i \le j$.

Prior to proving this a couple of lemmas are proved.

Lemma 3.1 *Let E be a Dedekind complete Riesz space with a weak order unit e. If (x_α) is an increasing net in E_+ and is not bounded in E^u_+, then there is a non-zero band projection P with*

$$kPe \wedge Px_\alpha \uparrow_\alpha kPe, \quad \textit{for all} \quad k \in \mathbb{N}. \tag{3.1}$$

Proof Let (x_α) be an increasing net in E_+ which is not bounded in E^u_+. Suppose that (3.1) fails for all non-zero band projections P. Let

$$\begin{aligned}
f_k &= \bigvee_\alpha \left(e \wedge \frac{x_\alpha}{k}\right), \quad k \in \mathbb{N}, \\
P_k &= I - P_{(e - f_k)}.
\end{aligned}$$

Then as (f_k) is a decreasing sequence, so is (P_k) and the sequence (P_k) converges to a band projection Q which is also the infimum of the sequence.

If $Q \neq 0$, then as $P_k \geq Q$ we have $P_k Q = Q$, giving

$$Qe = P_k Qe = Qe - P_{(e-f_k)} Qe.$$

Thus $P_{Q(e-f_\alpha)} e = P_{(e-f_\alpha)} Qe = 0$ from which it follows that

$$Qe = Qf_k = \bigvee_\alpha \left(Qe \wedge \frac{Qx_\alpha}{k} \right),$$

and after multiplication by k, this gives that (3.1) holds for $P = Q > 0$, a contradiction of our hypothesis. Hence $Q = 0$.

Setting $P_0 := I$ it follows that, since $P_k \downarrow 0$,

$$\sum_{k=1}^{\infty} (P_{k-1} - P_k) = I.$$

Let

$$e_k := k(P_{k-1} - P_k)e$$

then $e_k \wedge e_j = 0$ for all $k \neq j$ and consequently

$$\gamma := \sum_{k=1}^{\infty} e_k = \bigvee_{k=1}^{\infty} e_k \in E_+^u.$$

But

$$(P_{k-1} - P_k)x_\alpha \leq e_k, \quad \text{for all } \alpha.$$

Hence

$$x_\alpha \leq \gamma, \quad \text{for all } \alpha$$

making (x_α) bounded in E^u, a contradiction. □

Lemma 3.2 *Let E be a Dedekind complete Riesz space with a weak order unit e and let T be a strictly positive conditional expectation operator on E with $T(e) = e$. If (x_α) is an increasing net in E_+ with (Tx_α) order bounded in E^u, then (x_α) is bounded in E^u.*

Proof Suppose that (x_α) is not bounded in E_+^u, then by Lemma 3.1 there is a non-zero band projection P with

$$Px_\alpha \wedge kPe \uparrow_\alpha kPe, \quad \text{for all } k \in \mathbb{N}.$$

Let $z \in E^u_+$ be a bound for (Tx_α). Then

$$z \geq Tx_\alpha \geq T(Px_\alpha \wedge kPe) \uparrow_\alpha kTPe,$$

giving $z \geq kTPe$ for all $k \in \mathbb{N}$. Since E^u is Archimedean, the above family of inequalities imply $Pe = 0$, contradicting $P \neq 0$. □

We are now in a position to consider the relative sizes of the domains of the universal extensions of conditional expectations from a filtration.

Theorem 3.3 *Let E be a Dedekind complete Riesz space with a weak order unit e, (T_i) a filtration on E with T_1 strictly positive and $T_i(e) = e$, for all $i \in \mathbb{N}$, then*

$$\mathrm{dom}(T_i) \subset \mathrm{dom}(T_j), \quad \textit{for all} \quad i \leq j.$$

Proof We observe that since T_1 is strictly positive, so are $T_i, i \in \mathbb{N}$. Let $i \leq j, x \in \mathrm{dom}(T_i)$ and $(x_\alpha) \subset E_+$ with $x_\alpha \uparrow x$. Now $T_i x_\alpha \uparrow \mathbf{T}_i x$ in E^u_+, hence

$$T_i(T_j x_\alpha) = T_i x_\alpha \leq \mathbf{T}_i x.$$

Thus, by Lemma 3.2, $(T_j x_\alpha)$ is bounded in E^u so by the definition of $\mathrm{dom}(T_j)$, $x \in \mathrm{dom}(T_j)$. □

Set

$$\mathcal{L}^1(T_1) = \mathrm{dom}(T_1)$$
$$\mathcal{L}^2(T_1) = \{x \in \mathcal{L}^1(T_1) \mid x^2 \in \mathcal{L}^1(T_1)\}$$

and

$$\mathcal{L}^\infty(T_1) = E^e := \bigcup_{n \in \mathbb{N}} [-ne, ne].$$

It should be observed that $\mathcal{L}^\infty(T_1)$ is a ring and a vector space over $\mathbb{R}$, while $\mathcal{L}^1(T_1)$ and $\mathcal{L}^2(T_1)$ are modules over $\mathcal{L}^\infty(T_1)$.

Theorem 3.4 *Let E be a Dedekind complete Riesz space with a weak order unit e, (T_i) a filtration on E with T_1 strictly positive and $T_i(e) = e$, for all $i \in \mathbb{N}$. The restriction, $(\tilde{\mathbf{T}}_i)$, of the extension $(\mathbf{T}_i)$, of (T_i), to $\mathcal{L}^1(T_1)$ is a filtration on $\mathcal{L}^1(T_1)$.*

Proof From [10] it follows that $\mathbf{T}_i e = e$ and that $\mathbf{T}_i$ is positive and order continuous, and hence these properties hold for $\tilde{\mathbf{T}}_i$.

We now show that $R(\tilde{\mathbf{T}}_i) \subset \mathcal{L}^1(T_1)$. Let $x \in \mathcal{L}^1(T_1)_+$. Then there is a net (x_α) in E_+ with $x_\alpha \uparrow x$ and by definition $\mathbf{T}_1 x = \lim_\alpha T_1 x_\alpha$. Now

$$T_1(T_i x_\alpha) = T_1 x_\alpha \leq \mathbf{T}_1 x,$$

showing that $(T_1(T_i x_\alpha))_\alpha$ is bounded in E^u. Thus

$$\tilde{\mathbf{T}}_i x = \lim_\alpha T_i x_\alpha \in \mathcal{L}^1(T_1).$$

Hence $\tilde{\mathbf{T}}_i : \mathcal{L}^1(T_1) \to \mathcal{L}^1(T_1)$, and since $\mathbf{T}_i$ is a projection it follows that $\tilde{\mathbf{T}}_i$ is a projection.

Since $\tilde{\mathbf{T}}_i$ is a projection and $R(\tilde{\mathbf{T}}_i) \subset \mathcal{L}^1(T_1)$ it follows that $R(\tilde{\mathbf{T}}_i) = R(\mathbf{T}_i) \cap \mathcal{L}^1(T_1)$, which as the intersection of two Dedekind complete spaces, is Dedekind complete. Thus $\tilde{\mathbf{T}}_i$ is a conditional expectation on $\mathcal{L}^1(T_1)$.

Finally, as before let $x \in \mathcal{L}^1(T_1)_+$ and (x_α) be a net in E_+ with $x_\alpha \uparrow x$. Let $i \le j$ then by the order continuity of $\tilde{\mathbf{T}}_i$ and $\tilde{\mathbf{T}}_j$ we have

$$\tilde{\mathbf{T}}_i \tilde{\mathbf{T}}_j x = \lim_\alpha T_i T_j x_\alpha = \lim_\alpha T_i x_\alpha = \tilde{\mathbf{T}}_i x$$

and similarly $\tilde{\mathbf{T}}_j \tilde{\mathbf{T}}_i x = \tilde{\mathbf{T}}_i x$. Thus $(\tilde{\mathbf{T}}_i)$ is a filtration on $\mathcal{L}^1(T_1)$. □

It should be noted that $\tilde{\mathbf{T}}_i$ could equally have been constructed as the unique conditional expectation having $\tilde{\mathbf{T}}_i \mathbf{T}_1 = \mathbf{T}_1 = \mathbf{T}_1 \tilde{\mathbf{T}}_i$ in $\mathcal{L}^1(T_1)$ having range $\mathcal{L}^1(T_1) \cap R(\mathbf{T}_i)$, see [25].

The following lemmas are needed in the construction of the stochastic integral.

Lemma 3.5 *Let E be a Dedekind complete Riesz space with a weak order unit e, (T_i) a filtration on E with T_1 strictly positive and $T_i(e) = e$, for all $i \in \mathbb{N}$. If $a, b \ge 0$ with $a \in R(\mathbf{T}_n)$, $b, a^2 \mathbf{T}_n b \in \mathcal{L}^1(T_1)$, then $a^2 b \in \mathcal{L}^1(T_1)$.*

Proof Let (a_α) be a net in $E^e_+ \cap R(T_n)$ with $a_\alpha \uparrow a$. Then $a_\alpha^2 \in E^e$ and hence $a_\alpha^2 \in R(T_n)$ and, by Theorem 2.1, $T_n a_\alpha^2 = a_\alpha T_n a_\alpha = a_\alpha^2$. Also $a_\alpha^2 b \in \mathcal{L}^1(T_1)$ so from Theorems 3.3 and 3.4,

$$\mathbf{T}_1(a_\alpha^2 b) = \mathbf{T}_1 \mathbf{T}_n(a_\alpha^2 b) = \mathbf{T}_1(a_\alpha^2 \mathbf{T}_n b) \le \mathbf{T}_1(a^2 \mathbf{T}_n b).$$

Taking (b_β) an increasing net in E^e_+ with $b_\beta \uparrow b$, we have that $(a_\alpha^2 b_\beta)_{(\alpha,\beta)}$ is an increasing net where $(\alpha, \beta) \le (\alpha', \beta')$ if and only if $\alpha \le \alpha'$ and $\beta \le \beta'$. Here $\lim_{(\alpha,\beta)} a_\alpha^2 b_\beta = a^2 b$ and $T_1(a_\alpha^2 b_\beta) \uparrow \mathbf{T}_1(a^2 b)$, giving $a^2 b \in \mathcal{L}^1(T_1)$. □

4 The square of a martingale

We have already seen that if $x \in \mathcal{L}^p(T_1)$, then $\tilde{\mathbf{T}}_j x \in \mathcal{L}^p(T_1)$, $p = 1, \infty$. If $x \in \mathcal{L}^2(T_1)$ then $x^2 \in \mathcal{L}^1(T_1) \subset \mathrm{dom}(T_j)$ and $x \in \mathcal{L}^1(T_j) \subset \mathrm{dom}(T_j)$. So by Theorem 3.4, $\tilde{\mathbf{T}}_j x, \tilde{\mathbf{T}}_j x^2 \in \mathcal{L}^1(T_1)$ but by Theorem 2.5, $0 \le \tilde{\mathbf{T}}_j x^2 - (\tilde{\mathbf{T}}_j x)^2$ making $(\tilde{\mathbf{T}}_j x)^2 \le \tilde{\mathbf{T}}_j x^2 \in \mathcal{L}^1(T_1)$. Now since $\mathcal{L}^1(T_1)$ is solid, $\tilde{\mathbf{T}}_j x \in \mathcal{L}^2(T_1)$.

Henceforth $\tilde{\mathbf{T}}_j$ will just be denoted as T_j, which is encapsulated by the following definition.

Definition 4.1 We call (f_i, T_i) an $\mathcal{L}^p(T_1)$-martingale (-sub-martingale), $p = 1, 2, \infty$ if (T_i) is a filtration on $\mathcal{L}^1(T_1)$ restricted to $\mathcal{L}^p(T_1)$ and $T_i f_j = (\geq) f_i \in R(T_i) \cap \mathcal{L}^p(T_1)$ for all $i \leq j$.

Denote by $\mathrm{Mart}^2(T_i)$ the vector space, and module over the ring $\mathcal{L}^\infty(T_1)$, of all the $\mathcal{L}^2(T_1)$-martingales (f_i, T_i).

The following theorem follows directly from the Doob–Meyer decomposition for sub-martingales in Dedekind complete Riesz spaces, see [9].

Theorem 4.2 *Let E be a Dedekind complete Riesz space with a weak order unit e, (T_i) a filtration on E with the property that e is T_1-invariant and (f_i, T_i) is a positive sub-martingale in E. Then there exist a unique martingale (m_i, T_1) and a unique increasing positive adapted sequence (F_i) such that $F_1 = 0$ and $f_i = m_i + F_i$.*

Theorem 4.3 *Let E be a Dedekind complete Riesz space with a weak order unit e and (T_i) a filtration on E with the property that e is T_1-invariant. If (f_i, T_i) is an $\mathcal{L}^2(T_1)$-martingale, then (f_i^2, T_i) is an $\mathcal{L}^1(T_1)$-sub-martingale.*

Proof Since $f_i \in \mathcal{L}^2(T_1)$ it follows that $f_i, f_i^2 \in \mathcal{L}^1(T_1)$ and, since $T_i f_i = f_i$ it follows the averaging properties of T_i (see Theorem 2.1) that

$$T_i f_i^2 = T_i(f_i f_i) = f_i T_i f_i = f_i^2.$$

Hence $f_i^2 \in R(T_i)$.

If $i \leq j$, then it follows that

$$\begin{aligned}
0 &\leq T_i(f_j - f_i)^2 \\
&= T_i f_j^2 + T_i f_i^2 - 2T_i(f_j f_i) \\
&= T_i f_j^2 + T_i(f_i f_i) - 2T_i(f_j f_i) \\
&= T_i f_j^2 + f_i T_i(f_i) - 2 f_i T_i f_j \\
&= T_i f_j^2 + f_i f_i - 2 f_i f_i \\
&= T_i f_j^2 - f_i^2,
\end{aligned}$$

which concludes the proof that (f_i^2, T_i) is a sub-martingale. □

We can now apply the Doob–Meyer decomposition of sub-martingales to (f_i^2, T_i).

Theorem 4.4 *Let E be a Dedekind complete Riesz space with a weak order unit e and (T_i) a filtration on E with the property that e is T_1-invariant. If (f_i, T_i) is an $\mathcal{L}^2(T_1)$-martingale, there exist a unique $\mathcal{L}^1(T_1)$-martingale (m_i, T_i) and a unique increasing positive adapted sequence (F_i) in $\mathcal{L}^1(T_1)$ such that $F_1 = 0$ and*

$$f_i^2 = m_i + F_i.$$

Moreover,

$$F_{n+1} - F_n = T_n((f_{n+1} - f_n)^2) \quad \textit{for all } n \in \mathbb{N}.$$

Proof Applying the Doob–Meyer decomposition, see [9], to the sub-martingale (f_i^2, T_i) in the Dedekind complete Riesz space $\mathcal{L}^1(T_1)$ with weak order unit e, we obtain the decomposition

$$f_i^2 = m_i + F_i,$$

where (m_i, T_i) is a martingale, (F_i) is an increasing positive adapted sequence $\mathcal{L}^1(T_1)$ with $F_1 = 0$, and

$$F_j = \sum_{i=1}^{j-1} T_i(f_{i+1}^2 - f_i^2),$$
$$m_j = f_j^2 - F_j,$$

for $j \in \mathbb{N}$.

By the Conditional Variance Identity, Theorem 2.5, we have that

$$F_{n+1} - F_n = T_n(f_{n+1}^2 - f_n^2) = T_n(f_{n+1}^2) - f_n^2 = T_n(f_{n+1} - f_n)^2.$$

□

We are now in a position to define a vector valued inner product and norm like structures on the sequence space $\mathrm{Mart}^2(T_i)$.

Definition 4.5 Let E be a Dedekind complete Riesz space with filtration (T_i) and weak order unit $e = T_i e$ for all $i \in \mathbb{N}$. With each $(f_i, T_i) \in \mathrm{Mart}^2(T_i)$ we associate the $\mathcal{L}^1(T_1)$-martingale (m_i, T_i) and the increasing positive adapted sequence (F_i) in $\mathcal{L}^1(T_1)$ such that $F_1 = 0$ and $f_i^2 = m_i + F_i$.

The vector valued quadratic form $\langle\cdot\rangle$ on $\mathrm{Mart}^2(T_i)$ is defined by

$$\langle (f_i, T_i)\rangle = (F_i).$$

We define the vector-valued symmetric bilinear form $\langle , \rangle$ on $\mathrm{Mart}^2(T_i) \times \mathrm{Mart}^2(T_i)$ by

$$\langle (f_i, T_i), (g_i, T_i)\rangle = \frac{1}{4}[\langle (f_i + g_i, T_i)\rangle - \langle (f_i - g_i, T_i)\rangle].$$

Let

$$J_n := \langle (f_i, T_i), (g_i, T_i)\rangle_n, \quad n \in \mathbb{N},$$

then $J_1 = 0$ and

$$J_{n+1} - J_n = \frac{1}{4} T_n \left[[(f_{n+1} + g_{n+1})^2 - (f_n + g_n)^2] - [(f_{n+1} - g_{n+1})^2 - (f_n - g_n)^2] \right]$$

giving

$$J_{n+1} - J_n = T_n \left[f_{n+1} g_{n+1} - f_n g_n \right], \tag{4.1}$$

since $|f_n g_n| \leq \frac{1}{2}(f_n^2 + g_n^2) \in \mathcal{L}^1(T_1)$.

It follows directly from the definition of $\langle \cdot \rangle$ that $\langle \cdot, \cdot \rangle$ is an adapted sequence (F_i) in $\mathcal{L}^1(T_1)$ and thus

$$\langle (f_i, T_i), (g_i, T_i) \rangle_j \in \mathcal{L}^1(T_j), \text{ for all } j \in \mathbb{N}.$$

Lemma 4.6 *The bilinear form $\langle \cdot, \cdot \rangle$ defined in Definition 4.5 is a vector-valued semi-definite inner product in that it has the following properties:*

(a) $\langle \cdot, \cdot \rangle$ *is bilinear on* $\mathrm{Mart}^2(T_i)$ *with respect to multiplication by elements of* $\mathcal{L}^\infty(T_1) \cap R(T_1)$*;*
(b) $\langle \cdot, \cdot \rangle$ *is symmetric;*
(c) $\langle (f_i, T_i), (f_i, T_i) \rangle = \langle (f_i, T_i) \rangle \geq 0$ *for all* $(f_i, T_i) \in \mathrm{Mart}^2(T_i)$.

Proof A simple computation yields

$$\langle (f_i, T_i), (g_i, T_i) \rangle_j = \sum_{i=1}^{j-1} T_i (f_{i+1} g_{i+1} - f_i g_i). \tag{4.2}$$

Thus $\langle \cdot, \cdot \rangle$ is additive with respect to each of its variables and since multiplication is commutative (4.2) gives that the form is symmetric. The homogeneity of the form with respect to multiplication by $h \in \mathcal{L}^\infty(T_1) \cap R(T_1)$ follows from the observation that

$$T_i h (f_{i+1} g_{i+1} - f_i g_i) = h T_i (f_{i+1} g_{i+1} - f_i g_i).$$

It remains only to prove (c). From (4.2) with $(f_i) = (g_i)$, we obtain

$$\langle (f_i, T_i), (f_i, T_i) \rangle_j = \sum_{i=1}^{j-1} T_i (f_{i+1}^2 - f_i^2) = \langle (f_i, T_i) \rangle_j.$$

Since (f_i, T_i) is a martingale, (f_i^2, T_i) is a sub-martingale making

$$T_i (f_{i+1}^2 - f_i^2) = T_i f_{i+1}^2 - f_i^2 \geq 0,$$

thus $\langle (f_i, T_i) \rangle \geq 0$. □

Lemma 4.7 *The null space,*

$$\mathcal{N} = \{(f_i, T_i) \in \mathrm{Mart}^2(T_i) | \langle (f_i, T_i) \rangle = 0\},$$

of the quadratic form $\langle\cdot\rangle$, *is the set of constant sequences from* $\mathcal{L}^2(T_1)$, *i.e.*

$$\mathcal{N} = \{(f, T_i) | f \in \mathcal{L}^2(T_1) \cap R(T_1)\},$$

which is a closed, Dedekind complete Riesz subspace of $\mathrm{Mart}^2(T_i)$.

Proof The containment

$$\mathcal{N} \supset \{(f, T_i) | f \in \mathcal{L}^2(T_1) \cap R(T_1)\}$$

follows easily, so we progress to the reverse containment.

If $(f_i, T_i) \in \mathcal{N}$ then $(f_i, T_i) \in \mathrm{Mart}^2(T_i)$ with $T_i f_{i+1}^2 = f_i^2$ for all $i \in \mathbb{N}$. Since (f_i, T_i) is a martingale we have $T_i f_{i+1} = f_i$ which, from Theorem 2.1, gives $T_i(f_{i+1} f_i) = f_i^2$. Hence

$$0 = T_i(f_{i+1}^2 - f_i^2) = T_i f_{i+1}^2 - 2T_i(f_{i+1} f_i) + T_i f_i^2 = T_i(f_{i+1} - f_i)^2.$$

Now $(f_{i+1} - f_i)^2 \geq 0$ and T_i is strictly positive, so $f_{i+1} = f_i$ for all $i \in \mathbb{N}$. Thus $f_i = f_1$ for all $i \in \mathbb{N}$ and

$$f_1 \in \bigcap_{i \in \mathbb{N}} \mathcal{L}^2(T_i) \cap R(T_i) = \mathcal{L}^2(T_1) \cap R(T_1)$$

thus proving the remaining containment. □

The following theorem is a direct consequence of the above results, but the quotient here is only in the sense of vector spaces (not Riesz spaces) as, in general, $\mathcal{N}$ is not a band in $\mathrm{Mart}^2(T_i)$.

Theorem 4.8 *Let*

$$\{(f_i, T_i) \in \mathrm{Mart}^2(T_i) | f_1 = 0\} =: \mathcal{M},$$

then as vector spaces,

$$\mathrm{Mart}^2(T_i)/\mathcal{N} \equiv \mathcal{M},$$

and $\langle\cdot,\cdot\rangle$ *induces a vector valued inner product on* $\mathcal{M}$.

Observe that $\langle(f_i, T_i), (g_i, T_i)\rangle_1 = 0$ and that

$$(f_j g_j - \langle(f_i, T_i), (g_i, T_i)\rangle_j, T_j)$$

is a martingale.

5 Discrete stochastic integrals

In order to define the discrete Itô integral on a Riesz space, we need the following sequence spaces.

Definition 5.1 Let $(A_i) \subset \mathcal{L}^1(T_1)$ be an increasing sequence adapted to the filtration (T_i) with $A_1 = 0$. Define

$$\mathcal{L}^2_{loc}((A_n), (T_n)) := \{(\alpha_i) | \alpha_n \in R(T_n), \alpha_n^2(A_{n+1} - A_n), \alpha_n \in \mathcal{L}^1(T_1), n \in \mathbb{N}\}.$$

Definition 5.2 Let $f \in \text{Mart}^2(T_i)$ and $\alpha \in \mathcal{L}^2_{loc}(\langle f \rangle, (T_i))$, then (I_i, T_i) where

$$I_n := \sum_{k=1}^{n-1} \alpha_k (f_{k+1} - f_k), \quad n \in \mathbb{N},$$

is called the discrete Itô integral of α with respect to f, denoted

$$\int \alpha \, df = g.$$

The following lemma shows that the Itô integral returns a martingale.

Lemma 5.3 *For $f \in \text{Mart}^2(T_i)$ and $\alpha \in \mathcal{L}^2_{loc}(\langle f \angle, (T_i))$, the discrete Itô integral of α with respect to f, $\int \alpha \, df = g$, is in $\mathcal{M}$.*

Proof Let

$$(I_i, T_i)_i = \int \alpha \, df,$$

then

$$I_n := \sum_{k=1}^{n-1} \alpha_k (f_{k+1} - f_k), \quad n \in \mathbb{N}.$$

Since $\alpha_n \in R(T_n)$ it follows that

$$T_n I_{n+1} := T_n \sum_{k=1}^{n} \alpha_k (f_{k+1} - f_k) = I_n + \alpha_n T_n (f_{n+1} - f_n) = I_n$$

showing that (I_i, T_i) is a martingale.

From the definition of I it follows that $I_1 = 0$.

Let $a = \alpha_n$ and $b = (f_{n+1} - f_n)^2$ then $a \in R(T_n)$ and since $f \in \text{Mart}^2(T_i)$ we have $b \in \mathcal{L}^1(T_1)$. Finally, as $\alpha \in \mathcal{L}^2_{loc}(\langle f \rangle, (T_i))$ it follows that $a^2 T_n b \in \mathcal{L}^1(T_1)$, making Lemma 3.5 applicable. Hence $a^2 b \in \mathcal{L}^1(T_1)$. Thus

$$(I_{n+1} - I_n)^2 = \alpha_n^2 (f_{n+1} - f_n)^2 = a^2 b \in \mathcal{L}^1(T_1), \quad n \in \mathbb{N}.$$

But $I_1 = 0 \in \mathcal{L}^2(T_1)$, so assuming $I_n \in \mathcal{L}^2(T_1)$ we have by induction that $I_{n+1} = (I_{n+1} - I_n) + I_n \in \mathcal{L}^2(T_1)$. □

The following theorem shows the consistency of the martingale transform constructed in Definition 5.2 and the classical definition of the Itô integral.

Theorem 5.4 *Let* $f \in \mathrm{Mart}^2(T_i)$ *and* $\alpha \in \mathcal{L}^2_{loc}(\langle f\rangle, (T_i))$, *then* $I = \int \alpha\, df$ *is the unique solution in* $\mathcal{M}$ *of*

$$\langle I, h\rangle_{n+1} - \langle I, h\rangle_n = \alpha_n(\langle f, h\rangle_{n+1} - \langle f, h\rangle_n), \tag{5.1}$$

for all $n \in \mathbb{N}$ *and* $h \in \mathrm{Mart}^2(T_i)$.

Proof Uniqueness

Suppose that $g, q \in \mathcal{M}$ and

$$\langle g, h\rangle_{n+1} - \langle g, h\rangle_n = \alpha_n(\langle f, h\rangle_{n+1} - \langle f, h\rangle_n) = \langle q, h\langle_{n+1} - \langle q, h\rangle_n,$$

for all $n \in \mathbb{N}$, $h \in \mathrm{Mart}^2(T_i)$. Then

$$\langle g - q, h\rangle_{n+1} = \langle g - q, h\rangle_n \quad \text{for all } n \in \mathbb{N}, \quad h \in \mathrm{Mart}^2(T_i).$$

In particular, setting $h = g - q$ gives

$$\langle g - q\rangle_{n+1} = \langle g - q\rangle_n, \quad \text{for all } n \in \mathbb{N}.$$

By definition $\langle g - q\rangle_1 = 0$, so inductively we have that

$$\sum_{i=1}^{n-1} T_i[(g_{i+1} - q_{i+1})^2 - (g_i - q_i)^2] = 0, \quad \text{for all } n \in \mathbb{N},$$

which inductively yields

$$T_n[(g_{n+1} - q_{n+1})^2 - (g_n - q_n)^2] = 0, \quad \text{for all } n \in \mathbb{N}.$$

Here $(g_n - q_n)^2 \in \mathcal{R}(T_n)$ and thus

$$T_n(g_{n+1} - q_{n+1})^2 = (g_n - q_n)^2, \quad \text{for all } n \in \mathbb{N}. \tag{5.2}$$

Since $g, q \in \mathcal{M}$ it follows that $g_1 = 0 = q_1$. If $g_n = q_n$ then from (5.2) $T_n(g_{n+1} - q_{n+1})^2 = 0$, which, with the strict positivity of T_n, gives $g_{n+1} = q_{n+1}$. Hence, by induction $g = q$, proving the uniqueness of the solution to (5.1) from $\mathcal{M}$.

Solution

It remains to verify that (I_n, T_n) is a solution of (5.1). Note that (5.1) is equivalent to

$$T_n(I_{n+1}h_{n+1} - I_n h_n) = \alpha_n T_n(f_{n+1}h_{n+1} - f_n h_n), \quad \text{for all} \quad n \in \mathbb{N}, h \in \mathrm{Mart}^2(T_i).$$

Now, $I_n, I_{n+1}, h_n, h_{n+1} \in \mathcal{L}^2(T_1)$, $T_n I_{n+1} = I_n$ and $T_n h_{n+1} = h_n$ so $T_n I_n h_{n+1} = I_n h_n = T_n I_{n+1} h_n$. Thus

$$\begin{aligned} T_n(I_{n+1}h_{n+1} - I_n h_n) &= T_n[(I_{n+1} - I_n)h_{n+1}] \\ &= T_n[\alpha_n(f_{n+1} - f_n)h_{n+1}], \end{aligned}$$

where, by Lemma 5.3, $\alpha_n(f_{n+1}-f_n)h_{n+1} \in \mathcal{L}^1(T_1)$. Setting $a = \alpha_n \in R(T_n)\cap\mathcal{L}^1(T_1)$ and $b = (f_{n+1} - f_n)h_{n+1} \in \mathcal{L}^1(T_1)$ we have that $ab \in \mathcal{L}^1(T_1)$, and as $\mathcal{L}^1(T_1) \subset \mathcal{L}^1(T_n)$ by Theorem 2.1 we get

$$T_n[\alpha_n(f_{n+1} - f_n)h_{n+1}] = T_n(ab) = aT_n(b) = \alpha_n T_n[(f_{n+1} - f_n)h_{n+1}].$$

Hence

$$\begin{aligned} T_n(I_{n+1}h_{n+1} - I_n h_n) &= \alpha_n T_n[(f_{n+1} - f_n)h_{n+1}] \\ &= \alpha_n[T_n(f_{n+1}h_{n+1}) - f_n h_n] \end{aligned}$$

for all $n \in \mathbb{N}, h \in \text{Mart}^2(T_i)$, thus showing that (I_n, T_n) is the solution of (5.1) from $\mathcal{M}$. □

Remark From the definition of the discrete stochastic integral and the above theorem, it follows that the stochastic integral, $\int f\, d\alpha$, is linear in both f and α. In addition it is order continuous with respect to both f and α as multiplication and addition are order continuous operations. Here, as in [11], by

$$(f_i, T_i) \leq (g_i, T_i), \quad \text{we mean} \quad f_i \leq g_i, \quad \text{for all } i \in \mathbb{N}.$$

Remark We recall that, in the classical setting of $L^1(\Omega, \mathcal{A}, P)$, a Brownian motion is a stochastic process (f_n) for which successive increments are independent, the mean increment is zero and the quadratic variation of $f_n - f_m$ is $|n - m|$. Such a process, if considered with respect to the filtration $(\mathcal{F}_n)$, where $\mathcal{F}_n$ is the minimal σ-algebra which makes $f_1, \ldots, f_n$ measurable, is a martingale.

In probability theory, the concept of independence relies on both the presence of a probability measure and the multiplicative properties of $\mathbb{R}^+$. In the Riesz space setting we proceed as follows.

Definition 5.5 Let E be a Dedekind complete Riesz space with conditional expectation T and weak order unit $e = Te$. Let P and Q be band projections on E, we say that P and Q are independent with respect to T if

$$TPTQe = TPQe = TQTPe.$$

We can define independence, with respect to the conditional expectation T, of a family of Dedekind complete Riesz subspaces of the Riesz space E as follows.

Definition 5.6 Let E be a Dedekind complete Riesz space with conditional expectation T and weak order unit $e = Te$. Let E_λ, $\lambda \in \Lambda$, be a family of Dedekind complete Riesz subspaces of E having $e \in E_\lambda$ for all $\lambda \in \Lambda$. We say that the family is independent with respect to T if, for each pair of disjoint sets $\Lambda_1, \Lambda_2 \subset \Lambda$, we have that P_1 and P_2 are independent with respect to T, where P_1 and P_2 are band projections with

$$P_j e \in \left\langle \bigcup_{\lambda \in \Lambda_j} E_\lambda \right\rangle, \quad j = 1, 2,$$

where $\langle S \rangle$ denotes the smallest Dedekind complete Riesz subspace of E containing the set S.

Definition 5.6 leads naturally to the definition of independence for sequences in E, given below.

Definition 5.7 Let E be a Dedekind complete Riesz space with conditional expectation T and weak order unit $e = Te$. We say that the sequence (f_n) in E is independent with respect to T if the family

$$\{\langle\{f_n, e\}\rangle \, | n \in \mathbb{N}\}$$

of Dedekind complete Riesz spaces is independent with respect to T.

We are now in a position to define Brownian motion in the Riesz space setting.

Definition 5.8 Let E be a Riesz space with conditional expectation T and weak order unit $e = Te$. A sequence $(f_n) \subset \mathcal{L}^2(T)$ will be called a Brownian motion in E with respect to T and e if

(a) $(f_i - f_{i-1})$ is an independent sequence with respect to (T, e) (see [12]), where $f_0 := 0$,
(b) $T(f_i - f_{i-1}) = 0, i \in \mathbb{N}$,
(c) $T(f_n - f_m)^2 = |n - m|e$.

Given a Brownian motion, (f_n), as defined in Definition 5.8 we can construct a sequence of conditional expectations (T_n) with $R(T_n)$ the closed Riesz space in E generated by $\{R(T), f_1, \ldots, f_n\}$ [25]. Here $f := (f_n, T_n)$ is a martingale from $\mathbf{Mart}^2(T_n)$ [22], and stochastic integrals with respect to f can be computed.

References

1. Abramovich, Y.A., Aliprantis, C.D.: An Invitation to Operator Theory. Amer. Math. Soc. Prov. (2002)
2. Aliprantis, C.D., Burkinshaw, O.: Some remarks on orthomorphisms. Colloq. Math. **47**, 255–265 (1982)
3. de Pagter, B.: f-Algebras and orthomorphisms. Thesis, Leiden University (1981)
4. de Pagter, B., Grobler, J.J.: Operators representable as multiplication-conditional expectation operators. J. Oper. Theory **48**, 15–40 (2002)
5. Fremlin, D.H.: Inextensible Riesz spaces. Math. Proc. Camb. Phil. Soc. **77**, 71–89 (1975)

6. Gihman, I.I., Skorohod, A.V.: The Theory of Stochastic Processes, I. Springer-Verlag, Berlin (1974)
7. Gihman, I.I., Skorohod, A.V.: The Theory of Stochastic Processes, II. Springer-Verlag, Berlin (1975)
8. Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes, 2nd edn. Springer-Verlag, Berlin (2003)
9. Kuo, W.-C., Labuschagne, C.C.A., Watson, B.A.: Discrete time stochastic processes on Riesz spaces. Indag. Math. **15**, 435–451 (2004)
10. Kuo, W.-C., Labuschagne, C.C.A., Watson, B.A.: Conditional expectations on Riesz spaces. J. Math. Anal. Appl. **303**, 509–521 (2005)
11. Kuo, W.-C., Labuschagne, C.C.A., Watson, B.A.: Convergence of martingales in Riesz spaces. Indag. Math. N.S. **17**, 271–283 (2006)
12. Kuo, W.-C., Labuschagne, C.C.A., Watson, B.A.: Ergodic theory and the strong law of large numbers on Riesz spaces. J. Math. Anal. Appl. **325**, 422–437 (2007)
13. Kuo, W.-C., Labuschagne, C.C.A., Watson, B.A.: Amarts on Riesz spaces. Acta Math. Sin. (English Series) **24**, 329–342 (2008)
14. Luxemburg, W.A.J., Zaanen, A.C.: Riesz Spaces I. North Holland, Amsterdam (1971)
15. Neveu, J.: Discrete-Parameter Martingales. North Holland, Amsterdam (1975)
16. Rice, N.M.: Multiplication in vector lattices. Can. J. Math. **20**, 1136–1149 (1968)
17. Rogers, L.C.G., Williams, D.: Diffusions, Markov Processes, and Martingales, vols. 1 and 2. Cambridge University Press, Cambridge (2000)
18. Stoica, G.: On some stochastic-type operators. Analele Universitătii Bucuresti, Mathematică **39**, 58–62 (1990)
19. Stoica, G.: Vector valued quasi-martingales. Stud. Cerc. Mat. **42**, 73–79 (1990)
20. Stoica, G.: The structure of stochastic processes in normed vector lattices. Stud. Cerc. Mat. **46**, 477–486 (1994)
21. Stroock, D.W.: Lectures on Stochastic Analysis: Diffusion Theory. Cambridge University Press, Cambridge (1987)
22. Vardy, J.J., Watson, B.A.: Markov processes in Riesz spaces. (in preparation)
23. van Eldik, P., Venter, L.M.: Universally complete Riesz spaces and f-algebras. S. Afr. J. Sci. **84**, 343–346 (1987)
24. Venter, L.M.: Properties of complex lattice algebras with application to spectral theory. Thesis, Potchefstroom University (1987)
25. Watson, B.A.: An Andô-Douglas type theorem in Riesz spaces with a conditional expectation. Positivity **13**, 543–558 (2009)
26. Zaanen, A.C.: Riesz Spaces II. North Holland, Amsterdam (1983)
27. Zaanen, A.C.: Introduction to Operator Theory in Riesz Space. Springer-Verlag, Berlin (1997)