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The Wickstead problem on Dedekind σ-complete vector lattices

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Abstract In this paper, we present some results concerning the automatic order boundedness of band preserving operators on Dedekind σ-complete vector lattices.

Keywords f-Algebras · Band preserving · Dedekind σ-complete vector lattice

Mathematics Subject Classification (2000) Primary 06F25 · 47B65

1 Introduction

The question of whether a band preserving linear operator on archimedean vector lattices is automatically order bounded was posed by Wickstead [13]. There are several results that guarantee automatic order boundedness for band preserving operator acting in concrete classes of vector lattices, see [4,7,9,10]. The first example of an unbounded band preserving linear operator was announced by Abramovich et al. [1]. Later, they [1] and Mc Polin and Wickstead [9] showed that all band preserving operators in a universally complete vector lattice A are automatically bounded if and only if A is locally one-dimensional. Hence the Wickstead problem in the class of universally complete vector lattices was thus reduced to characterization of locally one-dimensional vector lattices. This characterization was studied in many works, see [6]. There is now a small body of literature devoted to the study of the Wickstead problem for the class of archimedean vector lattices. In fact, Bernau [4], Mc Polin and Wickstead [9]

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and De Pagter [10] proved, by using algebraic and technical tools, that if T is a band preserving linear operator on an archimedean vector lattice A and if for every positive sequence (x_n) in A which converges to zero relatively uniformly, $\inf_n\{|T(x_n)|\} = 0$, then T is order bounded.

In this paper, we focus our attention on the class of Dedekind σ-complete vector lattices. More precisely, if A is a Dedekind σ-complete vector lattice and if $T : A \to A$ is a band preserving, there are special topological circumstances that may conspire to force T to be order bounded. In fact, we prove, by using new techniques, that if the universal completion A^u of A is equipped with a Hausdorff f-compatible topology τ (see Definition 2), then any continuous band preserving $T : (A, (r.u)\ top) \to (A^u, \tau)$ is automatically order bounded, where $(r.u)$ top is the relatively uniform topology. This provides a generalization of results of Bernau [4], Mc Polin and Wickstead [9] and De Pagter [10]. Moreover, we show that all band preserving operators on a Freudenthal vector lattice are automatically order bounded. As well, we study the Wickstead problem for bilinear operators.

We point out that all proofs are purely order theoretical and algebraic in nature and furthermore do not involve any analytical means. We take it for granted that the reader is familiar with the notions of vector lattices (or Riesz spaces) and operators between them. For terminology, notations and concepts that are not explained in this paper, one can refer to the standard monographs [3,8,11].

2 Definitions and notations

In order to avoid unnecessary repetition we will suppose that all vector lattices and ℓ-algebras under consideration are **Archimedean**.

Let us recall some of the relevant notions. Let A be a vector lattice. A linear operator $T : A \to A$ is called *band preserving* (resp *ideal preserving*) if $T(x) \perp y$ whenever $x \perp y$ in A (resp $T(I) \subset I$ for every order ideal I of A). A bilinear operator $\Psi : A \times A \to A$ is called *separately band preserving* (resp *separately ideal preserving*) provided that the following mappings

$$\Psi(., x) : y \mapsto \Psi(y, x) \quad \text{and} \quad \Psi(x, .) : y \mapsto \Psi(x, y) \quad (y \in A)$$

are band preserving (resp ideal preserving) for all $x \in A$.

Let A be a vector lattice and let $0 \leq a \in A$. An element $0 \leq e \in A$ is called a *component* of a if $e \wedge (a - e) = 0$.

Definition 1 A vector lattice A is called a Freudenthal vector lattice if A satisfies the following property:
If $0 \leq x \leq e$ holds in A, then there exist positive real numbers $\alpha_1, \ldots, \alpha_n$ and components $e_1, \ldots, e_n$ of e satisfying $x = \sum_{1 \leq i \leq n} \alpha_i e_i$.

Example 1 The vector space of all real stationary sequences is an atomic Freudenthal vector lattice.

Example 2 Let A be a non-atomic Dedekind σ-complete vector lattice and let $e \in A^+$. It is well known, by using the representation theorem of Kakutani [11, Proposition 7.2

and Theorem 7.4], that the principal order ideal I_e generated by e in A can be identified with the vector lattice $C(X)$, for some basically disconnected compact Hausdorff space X. Let $F = \{f \in C(X),\ f(X)$ is a finite subset of $\mathbb{R}\}$. It is an easy task to prove that $0 \leq f \in F$ if and only if there exist $k \in \mathbb{N}$, $X_1, \ldots, X_k$ mutually disjoint clopen subsets of X and positive real numbers $\alpha_1, \ldots, \alpha_k$ such that $f = \sum_{i=1}^{k} \alpha_i \mathbf{1}_{X_i}$. Therefore F is a non-atomic Freudenthal vector lattice.

Remark 1 Any order ideal I of a Freudenthal vector lattice A is a Freudenthal vector lattice.

Let A be a vector lattice, let $0 \leq v \in A$, the sequence $\{a_n\}_{n\geq 0}$ in A is called (v) *relatively uniformly convergent* to $a \in A$ if for every real number $\varepsilon > 0$, there exists a natural number n_ε such that $|a_n - a| \leq \varepsilon v$ for all $n \geq n_\varepsilon$. This will be denoted by $a_n \to a\ (v)$. If $a_n \to a\ (v)$ for some $0 \leq v \in A$, then the sequence $\{a_n\}_{n\geq 1}$ is called (*relatively*) *uniformly convergent* to a, which is denoted by $a_n \to a(r.u)$. The notion of $(r.u)$ *relatively uniform Cauchy* sequence is defined in the obvious way. A vector lattice is called *relatively uniformly complete* if every relatively uniform Cauchy sequence in A has a unique limit. Relatively uniformly limits are unique if A is archimedean, see [8, Theorem 63.2]. The relatively uniform topology is denoted by $(r.u)$ topology.

Let A be a vector lattice. A net $(x_i)_{i\in I}$ of A is *order convergent* to $x \in A$ (denoted by $x_i \xrightarrow{o} x$) if there exist net $(y_j)_{j\in J}$ such that

(i) $y_j \downarrow 0$
(ii) for each $j \in J$ there exists some $i_0 \in I$ satisfying $|x_i - x| \leq y_j$ for all $i \geq i_0$.

A subset D of A is said to be *order closed* whenever $\{x_i\} \subset D$ and $x_i \xrightarrow{o} x$ imply $x \in D$, see [2].

In the following lines, we recall definitions and some basic facts about f-algebras. For more information about this field, one can refer to [3]. A (real) algebra A which is simultaneously a vector lattice such that the partial ordering and the multiplication in A are compatible, that is $a, b \in A^+$ implies $ab \in A^+$ is called *a lattice-ordered algebra* (briefly *ℓ-algebra*). In an ℓ-algebra A we denote the collection of all nilpotent elements of A by $N(A)$. An ℓ-algebra A is said to be *semiprime* if $N(A) = \{0\}$. An ℓ-algebra A is called an *f-algebra* if A verifies the property that $a \wedge b = 0$ and $c \geq 0$ imply $ac \wedge b = ca \wedge b = 0$. Every unital f-algebra (i.e., an f-algebra with a unit element) is semiprime.

The vector lattice A is called *Dedekind σ-complete* if for each non-void countable majorized set $B \subset A$, $\sup B$ exists in A. The vector lattice A is called *laterally complete* provided that every orthogonal system in A has a supremum in A. If A is Dedekind complete and laterally complete, then A is said to be *universally complete*. Every vector lattice A has a *universal completion* A^u, this means that there exists a unique (up to a lattice isomorphism) universally complete vector lattice A^u such that A can be identified with an order dense sublattice of A^u. For more properties about universal completion, see [8, Chap. VII, Sect. 51].

We end this section with some definitions about bilinear maps on vector lattices. Let A and B be vector lattices. A bilinear map Ψ from $A \times A$ into B is said to be *positive* whenever $(a, b) \in A^+ \times A^+$ imply $\Psi(a, b) \in B^+$. A bilinear map Ψ from $A \times A$ into B is said to be *orthosymmetric* if for all $(a, b) \in A \times A$ such that $a \wedge b = 0$ implies $\Psi(a, b) = 0$, see [5].

3 The main results

The present section considers band preserving linear operators on Dedekind σ*-complete* vector lattices. More precisely, we are mainly concerned with generalizing the results of Bernau [4], Mc Polin and Wickstead [9] and De Pagter [10]. In fact, in [4,9,10] the authors proved, by using an algebraic and technical tools, that if T is a band preserving linear operator on an archimedean vector lattice A and if for every positive sequence (x_n) in A which converges to zero relatively uniformly, $\inf_n\{|T(x_n)|\} = 0$ in A, then T is order bounded. Since $\inf_n\{|T(x_n)|\} = 0$ in A is equal to $\inf_n\{|T(x_n)|\} = 0$ in A^u. This implies that $T : (A, \ (r.u)top) \to (A^u, order\ top)$ is continuous.

The next proposition and theorem are an essential ingredient for our main results. They were proved by Toumi et al. [12]. In order to make this paper self contained we reproduce alternative and short proofs.

Proposition 1 *Let A be a Freudenthal vector lattice and let B be a vector space. Then any orthosymmetric bilinear operator* $\Psi : A \times A \to B$ *is symmetric.*

Proof Let $k, k' \in A$. It follows that $k = \sum_{i=1}^{i=n} \alpha_i e_i$ and $k' = \sum_{j=1}^{j=m} \beta_j f_j$, where e_i, f_j are components of $e = |k| + |k'|$. Then

$$\Psi(k, k') = \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} \alpha_i \beta_j \Psi(e_i, f_j). \tag{1}$$

Let e_i^d be the disjoint complement of e_i. Hence $e = e_i + e_i^d$ where $e_i \wedge e_i^d = 0$. Then

$$f_j = f_j \wedge e = f_j \wedge (e_i + e_i^d) = (f_j \wedge e_i) + (f_j \wedge e_i^d).$$

Since $(f_j \wedge e_i^d) \wedge e_i = 0$, then

$$\Psi(e_i, f_j) = \Psi(e_i, (f_j \wedge e_i) + (f_j \wedge e_i^d)) = \Psi(e_i, f_j \wedge e_i).$$

Using the same argument, we prove that

$$\Psi(e_i, f_j) = \Psi(e_i, f_j \wedge e_i) = \Psi(f_j \wedge e_i, f_j \wedge e_i) = \Psi(f_j, e_i).$$

Therefore, in view of equality (1), we have

$$\Psi(k, k') = \Psi(k', k),$$

which gives the desired result. □

Theorem 1 *Let A be a Dedekind σ-complete vector lattice equipped with the $(r.u)$ topology and let B be a Hausdorff topological vector space. Then any continuous orthosymmetric bilinear operator $\Psi : A \times A \to B$ is symmetric.*

Proof Let $a, b \in A$, let $e = |a| + |b|$ and let $F = \left\{k \in I_e;\ k = \sum_{i=1}^{i=n} \alpha_i e_i,\ \alpha_i \in \mathbb{R},\ e_i \text{ is a component of } e,\ n \in \mathbb{N}^*\right\}$. It is clear that F is a Freudenthal vector lattice. Moreover, by using the Freudenthal spectral Theorem [3, Theorem 40.2.], F is a dense vector subspace of I_e.

According to the previous proposition, the restriction of Ψ to $F \times F$, denoted also by Ψ, is symmetric. Now since $a, b \in I_e$ and since F is dense in I_e, there exist $a_n, b_n \in F$, for all $n \in \mathbb{N}$, such that $a_n \to a$ $(r.u)$ and $b_n \to b$ $(r.u)$. By the continuity of Ψ, we have

$$\Psi(a_n, b_n) \to \Psi(a, b) \quad \text{and} \quad \Psi(b_n, a_n) \to \Psi(b, a).$$

Since $\Psi(a_n, b_n) = \Psi(b_n, a_n)$, it follows that $\Psi(a, b) = \Psi(b, a)$, which gives the desired result. □

For the purpose of clarifying the aim of this paper, we make use of the following well known lemma.

Lemma 1 *Let A be a universally complete vector lattice and let e be a weak order unit of A. Then there exists a unique multiplication on A in such a way that A is an f-algebra with e as a unit element.*

Remark 2 We remark that any universally complete vector lattice can be seen as a universally complete unital f-algebra. Then in the sequel we denote its f-algebra multiplication by juxtaposition.

Remark 3 We note that the f-algebra structure on a universally complete vector lattice is not unique.

We intend to supply a topological condition in order to force a band preserving linear operator on a Dedekind σ-complete vector lattice to be order bounded. For this reason, we need the following definition.

Definition 2 Let A be a universally complete vector lattice. A linear topology τ on A is called an f-compatible topology if $x_n y_n \xrightarrow{\tau} 0$ whenever $x_n \xrightarrow{\tau} 0$ and $y_n \to 0$ $(r.u)$.

Example 3 The relatively uniform topology (abbreviated as $(r.u)$ top), the order topology and the norm topology are f-compatible topologies.

Remark 4 A universally complete vector lattice A cannot have a (lattice) norm topology unless it is finite dimensional.

We need the following lemma, which is of some independent interest in its own right.

Lemma 2 *Let A be a universally complete vector lattice, let τ be an f-compatible topology on A and let $\Psi : A \times A \to A$ be a positive separately band preserving bilinear map. Then $\Psi(x_n, y_n) \overset{\tau}{\to} 0$ whenever $x_n \overset{\tau}{\to} 0$ and $y_n \to 0$ $(r.u)$.*

Proof Let $\Psi : A \times A \to A$ be a positive separately band preserving bilinear map and let τ be an f-compatible topology on A. By the fact that A can be seen as an f-algebra with a unit element e, it is not hard to prove, by using [5, Corollary 2], that

$$\Psi(x, y) = \Psi(x, y)e = \Psi(e, y)x = \Psi(e, e)xy$$

for all $x, y \in A$. Therefore , if $x_n \overset{\tau}{\to} 0$ and $y_n \to 0$ $(r.u)$, it follows that $\Psi(e, e)y_n \to 0$ $(r.u)$. Since τ is an f-compatible topology on A, we get

$$\Psi(x_n, y_n) = \Psi(e, e)y_n x_n \overset{\tau}{\to} 0$$

as required. □

Let A be a vector lattice, let A^u be its universal completion. In the sequel, let us denote by i the natural embedding of A into A^u.

All the preparations have been made for the first central result in the paper.

Theorem 2 *Let A be a Dedekind σ-complete vector lattice, let A^u be its universal completion, let τ be an f-compatible topology on A^u and let $T : A \to A$ be a band preserving linear operator such that $i \circ T : (A,\ (r.u)\ top) \to (A^u,\ \tau)$ is continuous. Then T is order bounded.*

Proof Let $0 \leq x \leq y$ in A. Let I_y be the order ideal generated by y in A and let $B_y = \{y\}^{dd}$ be the band generated by y in A^u. Since T is band preserving, it follows that $T(x) \in B_x \subset B_y$. By the fact that B_y is a universally complete vector lattice, there exists a unique multiplication (denoted by $*$) on B_y in such a way that B_y is an f-algebra with y as a unit element, see Lemma 1. Moreover this multiplication can be extended (denoted also by $*$) to A^u in the obvious way. That is

$$a * b = a_1 * b_1$$

where a_1(resp b_1) is the band projection (in A^u) of a in B_y. It is an easy task to prove that the bilinear map associated to the multiplication $*$ is positive separately band preserving. Let $\Psi : I_y \times I_y \to A^u$ defined by $\Psi(a, b) = T(a)*b$ for all $a, b \in I_y$. The fact that τ is an f-compatible topology on A^u and $i \circ T : (A, (r.u)\ \text{top}) \to (A^u, \tau)$ is continuous coupled with Lemma 2, we deduce that $\Psi : (I_y,\ (r.u)\ top) \times (I_y,\ (r.u)\ top) \to (A^u, \tau)$ is continuous. According to Theorem 1, Ψ is symmetric. Therefore

$$\begin{aligned} T(x) &= T(x) * y \\ &= T(y) * x. \end{aligned}$$

It follows that $T(x) \in [-c, c]$, where $c = |T(y)| * y = |T(y)|$. Thus $i \circ T$ is order bounded. By Aliprantis and Burkinshaw [3, Theorem 2.40], the modulus of $i\ o\ T$ exists and $|i \circ T|(x) = |i \circ T(x)|$ for all $0 \leq x \in A$. Since $i \circ T(x) = T(x)$ for all $x \in A$, it follows that

$$|i \circ T|(x) = |T(x)| \in A$$

for all $0 \leq x \in A$. Therefore $|i \circ T|$ is a positive operator on A. Since $T = \frac{1}{2}(T + |i \circ T|) - \frac{1}{2}(|i \circ T| - T)$, it follows that T is a regular operator. Therefore T is order bounded and we are done. □

Corollary 1 *Let A be a Dedekind σ-complete vector lattice, let A^u be its universal completion, let τ be an f-compatible topology on A^u and let $T : A \to A$ be band preserving linear operator. Then the band preserving $i \circ T : (A, \ (r.u)\ top) \to (A^u, \ order\ top)$ is continuous if and only if T is order bounded.*

Proof ($\Rightarrow$) This path follows by using the fact that the order topology is an f-compatible topology and the previous theorem.
($\Leftarrow$) Since T is order bounded, then T is an orthomorphism. It follows that T is $(r.u)$ continuous. That is if $x_n \to 0$ $(r.u)$ in A then $T(x_n) \to 0(r.u)$. It follows that $\inf_n\{|T(x_n)|\} = 0$ in A. Therefore $\inf_n\{|T(x_n)|\} = 0$ in A^u, and the proof is complete. □

Similarly, we deduce the following corollary.

Corollary 2 *Let A be a Dedekind σ-complete vector lattice, let A^u be its universal completion, let τ be an f-compatible topology on A^u and let $T : A \to A$ be band preserving linear operator. Then the band preserving $i \circ T : (A, \ (r.u)\ top) \to (A^u, \ (r.u)\ top)$ is continuous if and only if T is order bounded.*

If A is a Freudenthal vector lattice, the situation improves considerably. In fact, we have the following result. Its corresponding proof is omitted, since it is similar to the proof of Theorem 2.

Theorem 3 *Let A be a Freudenthal vector lattice. Then all band preserving operators on A are automatically order bounded.*

We consider now the Wickstead bilinear problem. The findings we have mentioned can pave the way to the next theorem.

Theorem 4 *Let A be a Dedekind σ-complete vector lattice, let A^u be its universal completion, let τ be an f-compatible topology on A^u and let $\Psi : A \times A \to A$ be a separately band preserving bilinear operator. If $i \circ \Psi : (A, \ (r.u)\ top) \times (A, \ (r.u)\ top) \to (A^u, \tau)$ is continuous then Ψ is order bounded.*

Proof Let $0 \leq x_1, x_2 \leq y$ in A. Let I_y be the order ideal generated by y in A and let $B_y = \{y\}^{dd}$ be the band generated by y in A^u. Since Ψ is a separately band preserving bilinear operator, it follows that $\Psi(x_1, z) \in B_{x_1} \subset B_y$ and $\Psi(z, x_2) \in B_{x_2} \subset B_y$ for all $z \in A$. By the fact that B_y is a universally complete vector lattice, there exists a unique multiplication (denoted by $*$) on B_y in such a way that B_y is an f-algebra with y as a unit element, see Lemma 1. Moreover this multiplication can be extended (denoted also by $*$) to A^u in the obvious way. That is

$$a * b = a_1 * b_1$$

where a_1(resp b_1) is the band projection (in A^u) of a in B_y. Let $\overline{\Psi} : I_y \times I_y \times I_y \to A^u$ defined by $\overline{\Psi}(a, b, c) = \Psi(a, b) * c$ for all $a, b, c \in I_y$. $\overline{\Psi}$ is orthosymmetric in each pair of variables. Indeed, let $a, b, c \in I_y$ such that $a \wedge b = 0$, then $\Psi(a, b) = 0$ and then $\overline{\Psi}(a, b, c) = 0$. If we assume now that $a \wedge c = 0$, then, by the fact that Ψ is separately band preserving, it follows that $\Psi(a, b) \in B_a$. Moreover, since $*$ is an f-algebra multiplication, $\Psi(a, b) * c \in B_c$. Therefore $\overline{\Psi}(a, b, c) \in B_a \cap B_c = \{0\}$. Using the same argument, we deduce that $\overline{\Psi}(a, b, c) = 0$ as soon as $b \wedge c = 0$. From the fact τ is an f-compatible topology on A^u and $i \circ \Psi : (A, (r.u)\text{ top}) \times (A, (r.u)\text{ top}) \to (A^u, \tau)$ is continuous, coupled with Lemma 2, we deduce that $\overline{\Psi} : (I_y,\ (r.u)\ top) \times (I_y,\ (r.u)\ top) \times (I_y,\ (r.u)\ top) \to (A^u, \tau)$ is continuous. According to Theorem 1, $\overline{\Psi}$ is trisymmetric, that is $\overline{\Psi}(a_1, a_2, a_3) = \overline{\Psi}(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$ for any permutation σ of $\{1, 2, 3\}$. Therefore

$$\begin{aligned}
\Psi(x_1, x_2) &= \Psi(x_1, x_2) * y \\
&= \overline{\Psi}(x_1, x_2, y) \\
&= \overline{\Psi}(x_1, y, x_2) \\
&= \Psi(x_1, y) * x_2 \\
&= \Psi(x_1, y) * x_2 * y \\
&= [\Psi(x_1, y) * y] * x_2 \\
&= \overline{\Psi}(x_1, y, y) * x_2 \\
&= \overline{\Psi}(y, y, x_1) * x_2 \\
&= \Psi(y, y) * x_1 * x_2
\end{aligned}$$

It follows that $\Psi(x_1, x_2) \in [-c, c]$, where $c = |\Psi(y, y)| * y * y = |\Psi(y, y)|$. Thus $i \circ \Psi$ is order bounded. It is not hard to prove, by using [3, Theorem 2.40], that the modulus of $i \circ \Psi$ exists and $|i \circ \Psi|(x, y) = |i \circ \Psi(x, y)|$ for all $0 \leq x, y \in A$. Since $i \circ \Psi(x, y) = \Psi(x, y)$ for all $x, y \in A$, It follows that

$$|i \circ \Psi|\,(x, y) = |\Psi(x, y)| \in A$$

for all $0 \leq x, y \in A$. Therefore $|i \circ \Psi|$ is a positive bilinear operator on $A \times A$. Since $\Psi = \frac{1}{2}(\Psi + |i \circ \Psi|) - \frac{1}{2}(|i \circ \Psi| - \Psi)$, it follows that Ψ is a regular operator. Therefore Ψ is order bounded and we are done. □

An immediate consequence is the following corollary.

Corollary 3 *Let A be a Dedekind σ-complete vector lattice, let A^u be its universal completion and let $\Psi : A \times A \to A$ be a separately band preserving bilinear operator. Then the following statements are equivalent*:

(i) $i \circ \Psi : (A,\ (r.u)\ top) \times (A,\ (r.u)\ top) \to (A^u,\ order\ top)$ *is continuous*
(ii) $i \circ \Psi : (A,\ (r.u)\ top) \times (A,\ (r.u)\ top) \to (A^u,\ (r.u)\ top)$ *is continuous*
(iii) Ψ *is order bounded.*

Now, we give a new version of a theorem of Gutman et al. [7, Theorem 4.2.5] about band preserving linear operators on Dedekind σ-complete vector lattices.

Theorem 5 *Let A be a Dedekind σ-complete vector lattice and let A^u be its universal completion. Then the following are equivalent*:

(1) *All separately band preserving bilinear operators from $A \times A$ into A^u are order bounded.*
(2) *All separately band preserving bilinear operators from $A \times A$ into A^u are symmetric.*
(3) *All separately band preserving bilinear operators from $(A,\ (r.u)\ top) \times (A,\ (r.u)\ top)$ into $(A^u,\ order\ top)$ are continuous.*
(4) *All separately band preserving bilinear operators from $(A,\ (r.u)\ top) \times (A,\ (r.u)\ top)$ into $(A^u,\ (r.u)\ top)$ are continuous.*
(5) *All band preserving linear operators from A into A^u are order bounded.*
(6) *All band preserving linear operators from $(A,\ (r.u)\ top)$ into $(A^u,\ order\ top)$ are continuous.*
(7) *All band preserving linear operators from $(A,\ (r.u)\ top)$ into $(A^u,\ (r.u)\ top)$ are continuous.*

Proof (1) $\Rightarrow$ (2) This follows by Theorem 1.
(2) $\Rightarrow$ (1) This follows by using the same argument as in the proof of Theorem 4.
(1) $\Rightarrow$ (3) $\Rightarrow$ (4) Obvious.
(5) $\Rightarrow$ (6) $\Rightarrow$ (7) Obvious.
(1) $\Rightarrow$ (5) This follows by using the same argument as in the proof of Theorem 2.
(5) $\Rightarrow$ (1) Let $\Psi : A \times A \to A$ be a separately band preserving, then the mappings $\Psi(.,x) : y \mapsto \Psi(y,x)$ and $\Psi(x,.) : y \mapsto \Psi(y,x)$ are band preserving. It follows that $\Psi(.,x)$ and $\Psi(x,.)$ are order bounded. Therefore $\Psi(.,x)$ and $\Psi(x,.)$ are continuous with respect to $(r.u)$ topology. Consequently, by using Theorem 1, Ψ is symmetric and the proof is complete. □

Theorem 6 *Let A be a Dedekind σ-complete vector lattice and let A^u be its universal completion. Then the following assertions are true*:

(1) *All ideal preserving linear operators on A are order bounded.*
(2) *All ideal preserving linear operators from $(A,\ (r.u)\ top)$ into $(A,\ order\ top)$ are continuous.*
(3) *All ideal preserving linear operators from $(A,\ (r.u)\ top)$ into $(A,\ (r.u)\ top)$ are continuous.*
(4) *All separately ideal preserving bilinear operators on $A \times A$ are order bounded.*
(5) *All separately ideal preserving bilinear operators on $A \times A$ are symmetric.*
(6) *All separately ideal preserving bilinear operators from $(A,\ (r.u)\ top) \times (A,\ (r.u)\ top)$ into $(A,\ order\ top)$ are continuous.*
(7) *All separately ideal preserving bilinear operators from $(A,\ (r.u)\ top) \times (A,\ (r.u)\ top)$ into $(A,\ (r.u)\ top)$ are continuous.*

Proof (1) It follows from [13, Proposition 2.6].
(2), (3), (5), (6) and (7) Obvious.
(4) This follows by using the same argument as in the proof of Theorem 4. □

If A is a Freudenthal vector lattice, the situation improves considerably. In fact, we have the following result.

Theorem 7 *Let A be a Freudenthal vector lattice. Then all separately band preserving bilinear operators on $A \times A$ are automatically order bounded.*

Proof The proof follows by using the fact that any orthosymmetric bilinear map Ψ from $A \times A$ into B is symmetric, where B is a topological vector space. □

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