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On the weak compactness of b-weakly compact operators

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Abstract We characterize Banach lattices on which the class of b-weakly compact operators coincides with that of weakly compact operators.

Keywords b-Weakly compact operator · Weakly compact operator · Order continuous norm · Reflexive Banach lattice

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1 Introduction and notation

If E is a vector lattice, we denote by $E^{\sim}$ its order dual. All the vector lattices that we consider in this paper have separating order duals. Recall from [2] that a subset A of a vector lattice E is called b-order bounded in E if it is order bounded in the order bidual $(E^{\sim})^{\sim}$. It is clear that every order bounded subset of E is b-order bounded. However, the converse is not true in general. In fact, the subset $A = \{e_n : n \in \mathbb{N}\}$ is b-order bounded in the vector lattice c_0 but A is not order bounded in c_0, where e_n is the sequence of reals numbers with all terms zero except for the nth which is 1.

An operator T from a Banach lattice E into a Banach space X is said to be b-weakly compact whenever T carries each b-order bounded subset of E into a relatively weakly compact subset of X. For different studies and characterizations for b-weakly compact

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operators, we refer the reader to [2–5,7] . Also, for the duality property of this class of operators, we refer the reader to [8].

On the other hand, it is easy to see that each weakly compact operator is b-weakly compact but the converse may be false in general. For example, the identity operator $Id_{L^1[0,1]} : L^1[0,1] \to L^1[0,1]$ is b-weakly compact but it is not weakly compact. Moreover, we observe that even the second power of a b-weakly compact operator is not necessarily weakly compact. However, if E is an AM-space, then each b-weakly compact operator $T : E \to X$ is weakly compact for every Banach space X.

The objective of this paper is to generalize the latter result by proving that each b-weakly compact operator T from a Banach lattice E into a Banach space X is weakly compact if and only if the norm of E' is order continuous or X is reflexive. As consequences, we give some characterizations of the order continuity of the dual norm and a characterization of reflexive Banach lattices. Next, we will give a necessary and sufficient condition for which the second power of a b-weakly compact operator is weakly compact.

To state our results, we need to fix some notation and recall some definitions. Let E be a vector lattice, for each $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E \ : \ x \leq z \leq y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm and dual order, is also a Banach lattice. A norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the generalized sequence (x_α) converges to 0 for the norm $\|.\|$ where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. It follows from Theorem 5.16 of Schaefer [10], that a Banach lattice E is reflexive if and only if the norms of its topological dual E' and of its topological bidual E'' are order continuous. A Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. It is an AL-space if its topological dual E' is an AM-space.

An operator $T : E \longrightarrow F$ between two Banach lattices is a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. Note that each positive linear mapping on a Banach lattice is continuous.

For terminology concerning Banach lattice theory and positive operators, we refer the reader to the excellent book of Aliprantis–Burkinshaw [1].

2 Main results

To establish a necessary and sufficient condition for which each b-weakly compact operator is weakly compact, we need to give the following Lemma.

Lemma 2.1 *Let E be a Banach lattice. If E' does not have an order continuous norm, then there exists a disjoint sequence (u_n) of positive elements in E with $\|u_n\| \leq 1$ for all n and there exist some $0 \leq \phi \in E'$ and some $\varepsilon > 0$ satisfying $\phi(u_n) > \varepsilon$ for all n. Moreover, the components ϕ_n of ϕ in the carrier C_{u_n} form an order bounded disjoint sequence in $(E')^+$ such that $\phi_n(u_n) = \phi(u_n)$ for all n and $\phi_n(u_m) = 0$ if $n \neq m$.*

Proof It follows from Theorem 116.1 of Zaanen [11] that there is a norm bounded disjoint sequence (u_n) of positive elements in E which does not converge weakly to zero. Hence, we may assume that $\|u_n\| \leq 1$ for all n and also that for some $0 \leq \phi \in E'$ and some $\varepsilon > 0$ we have $\phi(u_n) > \varepsilon$ for all n (see also the proof of the implication $(iii) \Rightarrow (iv)$ of Theorem 117.2 of Zaanen [11]). The rest of the proof follows from Theorem 116.3 (i) of Zaanen [11].

A Banach space E is said to have the Schur property if every sequence in E weakly convergent to zero is norm convergent to zero. For example, the Banach space l^1 has the Schur property.

Note that if E is an AM-space, then the norm of E' is order continuous because E' is an AL-space.

Theorem 2.2 *Let E be a Banach lattice and X a Banach space. Then the following assertions are equivalent*:

1) *Each b-weakly compact operator $T : E \longrightarrow X$ is weakly compact.*
2) *One of the following conditions holds*:

 i) *the norm of E' is order continuous.*
 ii) *X is reflexive.*

Proof 2-i) $\Longrightarrow$ 1) Let $T : E \longrightarrow X$ be a b-weakly compact operator and let B be the band generated by E in its topological bidual E''. Then it follows from Proposition 2 of [4] that $T''(B) \subset X$ where T'' is the second adjoint of T.

On the other hand, since the norm of E' is order continuous, it follows from Theorem 2.4.14 of [9] that $B = E''$. Thus $T''\left(E''\right) \subset X$ and hence the result follows from Theorem 5.23 of [1].

2-ii) $\Longrightarrow$ 1) In this case, each operator $T : E \longrightarrow X$ is weakly compact.

1) $\Longrightarrow$ 2) Assume by way of contradiction that the norm of E' is not order continuous and X is not reflexive. To finish the proof, we have to construct a b-weakly compact operator $T : E \longrightarrow X$ which is not weakly compact.

Since the norm of E' is not order continuous, it follows from Lemma 2.1 that there exists a disjoint sequence (u_n) of positive elements in E with $\|u_n\| \leq 1$ for all n and that for some $0 \leq \phi \in E'$ and some $\varepsilon > 0$ such that $\phi(u_n) > \varepsilon$ for all n. Moreover, the components ϕ_n of ϕ in the carrier C_{u_n} form an order bounded disjoint sequence in $\left(E'\right)^+$ such that

$$\phi_n(u_n) = \phi(u_n) \quad \text{for all } n \text{ and } \phi_n(u_m) = 0 \text{ if } n \neq m. \tag{*}$$

Note that $0 \leq \phi_n \leq \phi$ holds for all n. Define the operator $T_1 : E \to l^1$ by

$$T_1(x) = \left(\frac{\phi_n(x)}{\phi(u_n)}\right)_{n=1}^{\infty} \quad \text{for all } x \in E.$$

Since $\sum_{n=1}^{\infty} \left|\frac{\phi_n(x)}{\phi(u_n)}\right| \leq \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \phi_n(|x|) \leq \frac{1}{\varepsilon}\phi(|x|)$ holds for each $x \in E$, the operator T_1 is well defined.

On the other hand, since X is not reflexive, then the closed unit ball B_X of X is not weakly compact. Thus, by Theorem 3.40 (Eberlein-Šmulian Theorem) of [1], there exists a sequence $(y_n) \subset B_X$ without any weakly convergent subsequence. Now, consider the operator $T_2 : l^1 \to X$ defined by

$$T_2\left((\lambda_n)\right) = \sum_{n=1}^{\infty} \lambda_n y_n \quad \text{for all } (\lambda_n) \in l^1.$$

Note that in view of $\sum_{n=1}^{\infty} \|\lambda_n y_n\| \leq \sum_{n=1}^{\infty} |\lambda_n| < \infty$, the series defining T_2 is norm convergent for each $(\lambda_n) \in l^1$.

Next, we consider the composed operator $T = T_2 \circ T_1 : E \to l^1 \to X$ defined by

$$T(x) = \sum_{n=1}^{\infty} \frac{\phi_n(x)}{\phi(u_n)} . y_n \quad \text{for all } x \in E.$$

Since l^1 has the Schur property, the first operator $T_1 : E \to l^1$ is b-weakly compact and therefore T is b-weakly compact. But T is not weakly compact. To see this, note that from (*) we have $T(u_n) = y_n$ for all n. Thus, since (y_n) has no weakly convergent subsequence, we conclude that T is not weakly compact, and this ends the proof of the Theorem.

If we assume X is a Banach lattice in Theorem 2.2, we obtain

Theorem 2.3 *Let E and F be two Banach lattices. Then the following assertions are equivalent*:

1) *Each b-weakly compact operator $T : E \longrightarrow F$ is weakly compact.*
2) *Each positive b-weakly compact operator $T : E \longrightarrow F$ is weakly compact.*
3) *One of the following conditions holds*:

   i) *the norm of E' is order continuous.*
   ii) *F is reflexive.*

Proof 1) $\Longrightarrow$ 2) Obvious.

3) $\Longrightarrow$ 1) It is just the implication 2) $\Longrightarrow$ 1) of Theorem 2.2.

2) $\Longrightarrow$ 3) The same proof as the implication 1) $\Longrightarrow$ 2) of Theorem 2.2 if we note that if F is not reflexive, then there exists a positive sequence (y_n) in the closed unit ball B_F of F without any weakly convergent subsequence.

As a consequence, we obtain a characterization of the order continuity of the dual norm.

Corollary 2.4 *Let E be a Banach lattice. Then the following assertions are equivalent*:

1) *Each b-weakly compact operator $T : E \longrightarrow E$ is weakly compact.*
2) *Each positive b-weakly compact operator $T : E \longrightarrow E$ is weakly compact.*
3) *The norm of E' is order continuous.*

A Banach lattice E is said to be a KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent.

Note that if F is a KB-space (in particular, an AL-space) then each operator $T : E \longrightarrow F$ is b-weakly compact (see Corollary of [4, p. 577]).

Another characterization of the order continuity of the dual norm is given by,

Corollary 2.5 *Let E be a Banach lattice and F an infinite-dimensional AL-space. Then the following assertions are equivalent*:

1) *Each operator $T : E \longrightarrow F$ is weakly compact.*
2) *Each positive operator $T : E \longrightarrow F$ is weakly compact.*
3) *The norm of E' is order continuous.*

Proof Follows immediately from Theorem 2.3 and Corollary of [4, p. 577] if we note that F is not reflexive.

Also, recall that if E is a KB-space (in particular, an AL-space), then each operator $T : E \longrightarrow F$ is b-weakly compact (see Proposition 2.1 of Altin [6]).

Another consequence, of Theorem 2.3, is a characterization of reflexive Banach lattices.

Corollary 2.6 *Let E be an infinite-dimensional AL-space and F a Banach lattice. Then the following assertions are equivalent*:

1) *Each operator $T : E \longrightarrow F$ is weakly compact.*
2) *Each positive operator $T : E \longrightarrow F$ is weakly compact.*
3) *F is reflexive.*

Proof Follows immediately from Theorem 2.3 and Proposition 2.1 of Altin [6] if we note that the norm of E' is not order continuous.

Now, observe that the second power of a b-weakly compact operator $T : E \longrightarrow E$ is not necessarily weakly compact. In fact, the identity operator $Id_{L^1[0,1]} : L^1[0,1] \rightarrow L^1[0,1]$ is b-weakly compact not weakly compact. Also, its second power $\left(Id_{L^1[0,1]}\right)^2 = Id_{L^1[0,1]}$ is not weakly compact.

To give a necessary and sufficient condition for which the second power of a b-weakly compact operator is weakly compact, we need to give a Lemma.

Lemma 2.7 *Let (u_n) be a disjoint sequence of a Banach lattice E. If (u_n) converges weakly to some $u \in E$, then $u = 0$.*

Proof Let (u_n) be a disjoint sequence of E such that $u_n \rightarrow u$ for $\sigma\left(E, E'\right)$. Then the set $W = \{u, u_1, u_2, ...\}$ is weakly compact. So, by Theorem 4.34 of [1], each disjoint sequence in the solid hull of W converges weakly to zero. In particular $u_n \rightarrow 0$ for $\sigma\left(E, E'\right)$. Thus $u = 0$.

Now, we are in position to establish our Theorem.

Theorem 2.8 *Let E be a Banach lattice. Then the following assertions are equivalent*:

1) *For each positive operators S and T from E into E such that $0 \leq S \leq T$ and T is b-weakly compact, the operator S is weakly compact.*
2) *Each positive b-weakly compact operator $T : E \longrightarrow E$ is weakly compact.*
3) *For each positive b-weakly compact operator T from E into E, the second power operator T^2 is weakly compact*
4) *The norm of E' is order continuous.*

Proof 1) $\Longrightarrow$ 2) and 2) $\Longrightarrow$ 3) are obvious.

3) $\Longrightarrow$ 4) Assume by way of contradiction that the norm of E' is not order continuous. To finish the proof, we have to construct a positive b-weakly compact operator $T : E \longrightarrow E$ such that its second power operator T^2 is not weakly compact.

We proceed as in the proof of Theorem 2.2, the idea is to take $(y_n) = (u_n)$, which does not admit any weakly convergent subsequence by Lemma 2.7.

Since the norm of E' is not order continuous, it follows from Lemma 2.1 that there exists a disjoint sequence (u_n) of positive elements in E with $\|u_n\| \leq 1$ for all n and that for some $0 \leq \phi \in E'$ and some $\varepsilon > 0$ such that $\phi(u_n) > \varepsilon$ for all n. Moreover, the components ϕ_n of ϕ in the carrier C_{u_n} form an order bounded disjoint sequence in $\left(E'\right)^+$ such that

$$\phi_n(u_n) = \phi(u_n) \quad \text{for all } n \text{ and } \phi_n(u_m) = 0 \text{ if } n \neq m. \tag{*}$$

On the other hand, since (u_n) is a disjoint sequence of E such that $\phi\,(u_n) > \varepsilon$ for all n, we conclude from Lemma 2.7 that (u_n) does not have any weakly convergent subsequence.

Now, we consider the operator $T : E \rightarrow E$ defined by

$$T(x) = \sum_{n=1}^{\infty} \frac{\phi_n(x)}{\phi\,(u_n)}.u_n \quad \text{for all } x \in E.$$

It is clear that T is well defined and positive.

Since T admits a factorization through the Banach lattice l^1, it follows from the proof of Theorem 2.2 that T is b-weakly compact. But its second power T^2 is not weakly compact. To see this, note that from (*), we have $T(u_n) = u_n$ for all n, and hence $T^2(u_n) = u_n$ for all n. Thus, since (u_n) has no weakly convergent subsequence, we conclude that T^2 is not weakly compact, and the proof of 3) $\Longrightarrow$ 4) is finished.

4) $\Longrightarrow$ 1) Follows immediately from Corollary 2.9 of [2] and Corollary 2.4.

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