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Gelfand-Hille type theorems in ordered Banach algebras

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Abstract. We consider the Gelfand-Hille Theorems, specifically conditions under which an element in an ordered Banach algebra (A, C) with spectrum $\{1\}$ is the identity of the algebra. In particular we show that for $x, x^{-1} \in C$, where C is a closed normal algebra cone, if $\sigma(x) = \{1\}$ and x is doubly Abel bounded then $x = \mathbf{1}$. Furthermore in the case where $\sigma(x) = \{1\}$ and C is a closed proper algebra cone, then $x = \mathbf{1}$ if and only if x^L is Abel bounded and $x^N \geq 1$ for some $L, N \in \mathbb{N}$.

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1. Introduction

Throughout this paper we will use A to denote a complex unitary Banach algebra with unit $\mathbf{1}$.

The presence of an ordering within a Banach algebra allows us to weaken the sufficient conditions for an $x \in A$ with unit spectrum to be the identity. The ordering that we introduce is via an algebra cone.

An *algebra cone* $C \subseteq A$ satisfies the following

1. $C + C \subseteq C$
2. $\lambda C \subseteq C$ (for every non-negative real scalar λ)
3. $C \cdot C \subseteq C$
4. $\mathbf{1} \in C$

Any cone C of A induces an ordering $\leq$ on A in the following way:

$$a \leq b \quad \text{if and only if} \quad b - a \in C.$$

Such an $x \in C$ is referred to as *positive* and $C = \{x \in A : x \geq 0\}$.

An algebra cone C is said to be

- *proper* if $C \bigcap (-C) = \{0\}$.

- *normal* if there exists $\alpha > 0$ such that
$$0 \leq x \leq y \quad \Rightarrow \quad ||x|| \leq \alpha ||y||.$$
- *closed* if it is closed with respect to the norm of A.
- *inverse closed* if for all $x \in A^{-1}$
$$x \in C \quad \Rightarrow \quad x^{-1} \in C.$$

It can be shown that every normal cone is proper. In our second section, on Abel bounded elements, we show how in some cases the assumption of a normal cone can be reduced to the weaker property of a proper cone.

We use (A, C) to denote an ordered Banach algebra with algebra A and algebra cone C.

Grobler and Huijsmans [5] discussed the role played by certain boundedness conditions in providing sufficient conditions for an $x \in A$ with single spectrum to be the identity. We summarize some of these conditions briefly.

An $x \in A$ is said to be *power bounded* if there exists a $D > 0$ such that
$$||x^n|| \leq D \quad \text{for all} \quad n \in \mathbb{N}.$$
An $x \in A$ is said to be *Cesàro bounded* if there exists a $D > 0$ such that
$$||M_n(x)|| \leq D \quad \text{for all} \quad n \in \mathbb{N},$$
where
$$M_n(x) = \frac{\mathbf{1} + x + \ldots x^n}{n+1}$$
is called the n'th *Cesàro mean* of x.

An $x \in A$ is said to be *uniformly Abel bounded* if there exists a $D > 0$ such that
$$||(1-\theta)\sum_{k=0}^{n} \theta^k x^k|| \leq D \quad \text{for all} \quad \theta \in (0,1), n \in \mathbb{N}.$$
An $x \in A$ is said to be *Abel bounded* if there exists a $D > 0$ such that
$$||(1-\theta)\sum_{k=0}^{\infty} \theta^k x^k|| \leq D \quad \text{for all} \quad \theta \in (0,1).$$

The notions of Abel and uniformly Abel bounded can be generalized to *(N)-Abel bounded* and *(N)-uniformly Abel bounded* given by
$$||(1-\theta)^N \sum_{k=0}^{\infty} \theta^k x^k|| \leq D \quad \text{for a} \quad D > 0 \quad \text{and for all} \quad \theta \in (0,1),$$
and
$$||(1-\theta)^N \sum_{k=0}^{n} \theta^k x^k|| \leq D \quad \text{for a} \quad D > 0 \quad \text{and for all} \quad \theta \in (0,1), n \in \mathbb{N}$$
respectively.

If x is invertible, and both x and x^{-1} are bounded of one of the above forms of boundedness, then x is referred to as *doubly* bounded of that form; for instance doubly power bounded means that there exists a $D > 0$ such that

$$||x^{\pm n}|| \leq D \quad \text{for all} \quad n \in \mathbb{N}.$$

As illustrated by Grobler and Huijsmans [5] we have the following relationships between these different forms of boundedness

Remark 1.1.

$$\begin{array}{ccccc} \textit{Power bounded} & & & & \\ \Downarrow & & & & \\ \textit{Cesàro bounded} & \Leftrightarrow & \textit{Uniformly Abel bounded} & \Rightarrow & \textit{Abel bounded} \\ & & \Downarrow & & \Downarrow \\ & & \textit{(N)-Uniformly Abel bounded} & \Rightarrow & \textit{(N)-Abel bounded} \end{array}$$

It was shown in [5, Theorem 2] that in a Banach algebra a Cesàro bounded element is uniformly Abel bounded. Drissi and Zemánek [3, remarks preceding (10)] raised the question whether the converse is true? That question was answered by Montes-Rodríguez, Sánchez-Álvarez and Zemánek [8, Theorem 3.1] who recently showed that the notions of Cesàro bounded and uniformly Abel bounded are the same.

Gelfand [4] showed that if $\sigma(x) = \{1\}$ and x is doubly power bounded then $x = \mathbf{1}$. Hille [6] later elaborated on this result for an x that is doubly power bounded of some order. Allan and Ransford [2, Theorem 1.1] subsequently proved the same result of Gelfand more elegantly, using holomorphic functional calculus. Mbekhta and Zemánek [7, Theorem 2] showed that if $\sigma(x) = \{1\}$ and x is doubly Cesàro bounded then $x = \mathbf{1}$.

In section two of this article we consider the notion of Abel boundedness; we give a condition on the cone of an ordered Banach algebra under which the notions of Abel boundedness and Cesàro boundedness are equivalent; we investigate the relationship between the Abel boundedness of x and natural powers of x; and finally we discuss how Abel boundedness of an $x \in A$ with $\sigma(x) = \{1\}$ assists us in concluding that x is the identity. In the third section we mention some results relating to the nilpotency of $x - \mathbf{1}$. Finally, in the fourth section, we consider the role played by inverse closed algebra cones in our consideration.

2. Abel bounded elements

If we consider the complex series $\sum_{k=0}^{\infty} a_k$, then the Abel sum of the series is defined to be

$$\lim_{\theta \to 1^-} \sum_{k=0}^{\infty} a_k \theta^k$$

if the series converges for all $\theta \in (0, 1)$ and the limit exists. Abel's Theorem states that if the series $\sum_{k=0}^{\infty} a_k$ is convergent then the Abel sum exists. However, the converse is not necessarily true. In a similar way, when working in a Banach algebra we have the Abel boundedness condition, namely that an element $x \in A$ is said to be Abel bounded if there exists a $D > 0$ such that for all $\theta \in (0,1)$, $||(1-\theta)\sum_{k=0}^{\infty}\theta^k x^k|| \leq D$.

If an $x \in A$ is Cesàro bounded, then it is Abel bounded as illustrated by (1.1). The converse is in general not true, however, using an argument similar to the one given by Grobler and Huijsmans [5, Theorem 3] for Banach lattices we have the following result.

Theorem 2.1. *Let (A, C) be an ordered Banach algebra, with C a closed normal algebra cone. If $x \in C$ and x is Abel bounded, then x is Cesàro bounded.*

Proof. Assume that C has normality constant α. Since $x \in C$ and C is closed,

$$0 \leq (1-\theta)\sum_{k=0}^{n}\theta^k x^k \leq (1-\theta)\sum_{k=0}^{\infty}\theta^k x^k \quad \text{for all} \quad \theta \in (0,1), n \in \mathbb{N}.$$

Moreover, since $\theta \in (0,1)$

$$0 \leq (1-\theta)\theta^n\sum_{k=0}^{n} x^k \leq (1-\theta)\sum_{k=0}^{n}\theta^k x^k \quad \text{for all} \quad \theta \in (0,1), n \in \mathbb{N}.$$

Hence

$$0 \leq (1-\theta)\theta^n\sum_{k=0}^{n} x^k \leq (1-\theta)\sum_{k=0}^{n}\theta^k x^k \leq (1-\theta)\sum_{k=0}^{\infty}\theta^k x^k$$

for all $\theta \in (0,1)$, $n \in \mathbb{N}$. Thus

$$||(1-\theta)\theta^n\sum_{k=0}^{n} x^k|| \leq \alpha||(1-\theta)\sum_{k=0}^{\infty}\theta^k x^k||$$

for all $\theta \in (0,1)$, $n \in \mathbb{N}$.

Since x is Abel bounded there exists a $D > 0$ such that

$$||(1-\theta)\theta^n\sum_{k=0}^{n} x^k|| \leq \alpha D \quad \text{for all} \quad \theta \in (0,1), n \in \mathbb{N}.$$

For a fixed $n \in \mathbb{N}$ we take $\theta = \frac{n}{n+1}$ to obtain

$$\left\|\frac{1}{n+1}\left(\frac{n}{n+1}\right)^n\sum_{k=0}^{n} x^k\right\| \leq \alpha D.$$

Thus

$$\left\|\frac{1}{n+1}\sum_{k=0}^{n} x^k\right\| \leq \alpha D\left(1-\frac{1}{n+1}\right)^{-n} = \alpha D\frac{n}{n+1}\left(1-\frac{1}{n+1}\right)^{-(n+1)}.$$

Therefore

$$||M_n(x)|| \leq \alpha D \left(1 - \frac{1}{n+1}\right)^{-(n+1)},$$

but since $\left(1 - \frac{1}{n+1}\right)^{-(n+1)} \to e$ as $n \to \infty$ the result follows. □

Returning to the boundedness heirachy (1.1), we notice that Theorem 2.1 allows us to reverse the horizontal implications when considering a closed, normal algebra cone C. If we recall the result due to Mbekhta and Zemánek [7, Theorem 2] - that every doubly Cesàro bounded element with single spectrum is the identity - we have the following corollary to Theorem 2.1.

Corollary 2.2. *Let (A, C) be an ordered Banach algebra, with C a closed normal algebra cone. If $\sigma(x) = \{1\}$, $x, x^{-1} \in C$ and x is doubly Abel bounded, then $x = \mathbf{1}$.*

Interestingly, the following theorem shows that Abel bounded elements cannot have the form $\mathbf{1} - x, (\mathbf{1} - x)^{-1}$ for a nonzero nilpotent element x.

Theorem 2.3. *If $x \neq 0$ is nilpotent then neither $\mathbf{1} - x$ nor $(\mathbf{1} - x)^{-1}$ is Abel bounded.*

Proof. Suppose that x is nilpotent of order N, that is $x^N = 0$. Note that $\sigma(x) = \{0\}$. Now,

$$\begin{aligned}
(1-\theta)\sum_{k=0}^{\infty}\theta^k(1-x)^k &= (1-\theta)\left[\mathbf{1}-\theta(\mathbf{1}-x)\right]^{-1} = (1-\theta)\left[(1-\theta)\mathbf{1}+\theta x\right]^{-1} \\
&= \left[\mathbf{1}+\frac{\theta x}{1-\theta}\right]^{-1} = \sum_{k=0}^{\infty}\left[\frac{-\theta x}{1-\theta}\right]^k = \sum_{k=0}^{N-1}\left[\frac{-\theta x}{1-\theta}\right]^k \\
&= \mathbf{1}+\frac{1}{(1-\theta)^{N-1}}\sum_{k=1}^{N-1}(-\theta x)^k(1-\theta)^{N-1-k}
\end{aligned}$$

Now as $\theta \to 1^-$ we get

$$\left|\left|\mathbf{1} + \lim_{\theta \to 1^-} \frac{(-x)^{N-1}}{(1-\theta)^{N-1}}\right|\right| = \infty.$$

Hence $\mathbf{1} - x$ is not Abel bounded.

Observe that

$$(\mathbf{1} - x)^{-1} = \sum_{j=0}^{N-1} x^j = \mathbf{1} + \sum_{j=1}^{N-1} x^j.$$

Hence taking $y = -\sum_{j=1}^{N-1} x^j$ we have $(\mathbf{1} - x)^{-1} = \mathbf{1} - y$. Clearly $\sigma(y) = \{0\}$ and $y^N = 0$. Therefore we can apply exactly the same argument as the one above to $\mathbf{1} - y$ with the same result. Thus, $(\mathbf{1} - x)^{-1}$ is also not Abel bounded. □

Note, however, that from the proof of the above theorem if x is nilpotent of order N then $\mathbf{1}-x$ is doubly (N)-Abel bounded. Furthermore, it follows that the strongest case of Theorem 2.3 occurs when x is nilpotent of order 2, in which case $\mathbf{1}-x$ is doubly (2)-Abel bounded.

We now turn to some useful results that will lead us to our main consideration. In particular we show that if $\sigma(x)=\{1\}$ then x is Abel bounded if and only if x^N is Abel bounded for all $N\in\mathbb{N}$.

Theorem 2.4. *Let A be a Banach algebra and $x\in A$ such that x^N is Abel bounded for some $N\in\mathbb{N}$. Then x is Abel bounded.*

Proof. Since x^N is Abel bounded, it follows that there exists a $D>0$ such that

$$||(1-\theta)\sum_{k=0}^{\infty}\theta^k x^{Nk}||\le D \quad \text{for all} \quad \theta\in(0,1).$$

Equivalently,

$$||(1-\theta^N)\sum_{k=0}^{\infty}\theta^{Nk} x^{Nk}||\le D \quad \text{for all} \quad \theta^N\in(0,1).$$

Now,

$$\begin{aligned}(1-\theta^N)\sum_{k=0}^{\infty}\theta^k x^k &= (1-\theta^N)\sum_{k=0}^{\infty}(\theta x)^{Nk}+\cdots+(1-\theta^N)\sum_{k=0}^{\infty}(\theta x)^{Nk+(N-1)}\\ &= \left[\mathbf{1}+\theta x+\cdots+(\theta x)^{N-1}\right](1-\theta^N)\sum_{k=0}^{\infty}(\theta x)^{Nk}.\end{aligned}$$

For some $K>0$ and for all $\theta\in(0,1)$,

$$||\mathbf{1}+\theta x+\cdots+(\theta x)^{N-1}||\le\sum_{i=0}^{N-1}(\theta||x||)^i\le\sum_{i=0}^{N-1}(||x||)^i\le K;$$

and so

$$||(1-\theta)\sum_{k=0}^{\infty}\theta^k x^k||\le||(1-\theta^N)\sum_{k=0}^{\infty}\theta^k x^k||\le K\cdot||(1-\theta^N)\sum_{k=0}^{\infty}(\theta x)^{Nk}||\le K\cdot D.$$

Hence x is Abel bounded. □

If we make use of an argument similar to the one above, we have the following analogous result for uniformly Abel bounded elements.

Corollary 2.5. *Let A be a Banach algebra and $x\in A$ such that x^N is uniformly Abel bounded for some $N\in\mathbb{N}$. Then x is uniformly Abel bounded.*

Theorem 2.6. *If $x\in A$ is Abel bounded and $\sigma(x)\subseteq[0,\infty)$, then x^N is Abel bounded for all $N\in\mathbb{N}$.*

Proof. If $\theta \in (0,1)$, then

$$(1-\theta^N)\sum_{k=0}^{\infty}(\theta x)^{Nk} = (1-\theta^N)\left(\mathbf{1}-(\theta x)^N\right)^{-1}.$$

Since

$$\mathbf{1}-(\theta x)^N = (\mathbf{1}-\theta x)\left[1+\theta x+\cdots+(\theta x)^{N-1}\right],$$

it is clear that

$$\left(\mathbf{1}-(\theta x)^N\right)^{-1} = (\mathbf{1}-\theta x)^{-1}\left[1+\theta x+\cdots+(\theta x)^{N-1}\right]^{-1}$$

and so,

$$(1-\theta^N)\sum_{k=0}^{\infty}(\theta x)^{Nk} = (1-\theta^N)(\mathbf{1}-\theta x)^{-1}\left[1+\theta x+\cdots+(\theta x)^{N-1}\right]^{-1}.$$

Since $\sigma(x) \subseteq [0,\infty)$, the map $\theta \mapsto \left[1+\theta x+\cdots+(\theta x)^{N-1}\right]^{-1}$ is continuous from $[0,1]$ into A and hence, $\left|\left|\left[1+\theta x+\cdots+(\theta x)^{N-1}\right]^{-1}\right|\right| \leq M$ for all $\theta \in [0,1]$ and some constant $M \geq 0$. Furthermore,

$$(1-\theta^N) = (1-\theta)\left(1+\theta+\cdots+\theta^{N-1}\right) \leq N(1-\theta), \quad \theta \in (0,1).$$

This implies that

$$\left|\left|(1-\theta^N)\sum_{k=0}^{\infty}(\theta x)^{Nk}\right|\right| \leq N\cdot M(1-\theta)||(1-\theta x)^{-1}||$$

for all $\theta \in (0,1)$ and the result follows. □

We can now proceed with our main result, which allows us to weaken the assumption of normality of the cone in Corollary 2.2 to a proper cone.

Theorem 2.7. *Let (A,C) be an ordered Banach algebra, with closed proper algebra cone C. Let $x \in A$ such that $\sigma(x) = \{1\}$. Then $x = \mathbf{1}$ if and only if*

1. *x^L is Abel bounded and*
2. *$x^N \geq 1$*

for some $L, N \in \mathbb{N}$.

Proof. The forward implication is obvious.

For the reverse implication, assume that x^L is Abel bounded and $x^N \geq 1$. Since x^L is Abel bounded and $\sigma(x) = \{1\}$, from Theorem 2.4 and Theorem 2.6 it follows that x^N is Abel bounded. Thus

$$||(1-\theta)(\mathbf{1}-\theta x^N)^{-1}|| = ||(1-\theta)\sum_{k=0}^{\infty}\theta^k(x^N)^k|| \leq D$$

for some $D > 0$ and for all $\theta \in (0,1)$. Manipulation gives

$$\left|\left|\left(\frac{1}{\theta}-1\right)\left[\left(\frac{1}{\theta}-1\right)\mathbf{1}-\left(x^N-\mathbf{1}\right)\right]^{-1}\right|\right| \leq D$$

Since $\theta \in (0,1)$, taking $\lambda = \frac{1}{\theta} - 1$ we see that $\lambda > 0$. Let $y = x^N - \mathbf{1}$. Hence

$$||\lambda(\lambda\mathbf{1} - y)^{-1}|| \leq D$$

for all $\lambda > 0$.

Hence $\lambda^2(\lambda\mathbf{1} - y)^{-1} \to 0$ as $\lambda \to 0^+$. Replacing $(\lambda\mathbf{1} - y)^{-1}$ with its Laurent expansion results in

$$\lambda\sum_{k=0}^{\infty}\left(\frac{y}{\lambda}\right)^k = \lambda + y + \sum_{k=2}^{\infty}\frac{y^k}{\lambda^{k-1}} \to 0 \quad \text{as} \quad \lambda \to 0^+.$$

It follows then that

$$\lambda + \sum_{k=2}^{\infty}\frac{y^k}{\lambda^{k-1}} \to -y \quad \text{as} \quad \lambda \to 0^+.$$

Note that since $x^N \geq 1$, $y \in C$. Hence $\lambda + \sum_{k=2}^{\infty}\frac{y^k}{\lambda^{k-1}} \in C$ for all $\lambda > 0$ (C is closed). Then $\lambda + \sum_{k=2}^{\infty}\frac{y^k}{\lambda^{k-1}}$ must converge to an element in C as $\lambda \to 0^+$ (since C is closed). Therefore $-y \in C$. Hence $y \in C\bigcap(-C) = \{0\}$ (since C is proper). Thus $y = 0$ and so $x^N = \mathbf{1}$. Since $\sigma(x) = \{1\}$, $x = \mathbf{1}$. □

Note that if in the above theorem we replace the restriction $\sigma(x) = \{1\}$ with $r(x) \leq 1$ (where $r(x)$ denotes the spectral radius of x), then using the expansion $\lambda^2(\lambda\mathbf{1} - y)^{-1} = \lambda + y + \frac{y^2}{\lambda+2}\sum_{k=0}^{\infty}\left(\frac{y+2}{\lambda+2}\right)^k$ in the proof, we again obtain $y = 0$. Hence $x^N = \mathbf{1}$. Furthermore, if we have $\sigma(x) \subseteq [0,\infty)$ then $x = \mathbf{1}$.

Let (A, C) be an ordered Banach algebra with C a closed proper algebra cone. If $0 \neq x \in C$ and x is quasinilpotent, then it follows from Theorem 2.7 that $\mathbf{1} + x$ is not Abel bounded. In particular, for $1 \leq p \leq \infty$ let $V : L^p[0,1] \to L^p[0,1]$ be the Volterra operator, defined by

$$(Vf)(x) = \int_0^x f(t)dt, \quad \text{for } f \in L^p[0,1].$$

Then V is a positive operator on $L^p[0,1]$, with respect to a normal algebra cone. Therefore it follows that $I + V$ is not Abel bounded. Montes-Rodríguez, Sánchez-Álvarez and Zemánek [8, Theorem 3.3] showed that $I - V$ is Abel bounded, using the resolvent. Furthermore, I. Pederson proved that $I - V = M^{-1}(I + V)^{-1}M$, where $(Mf)(x) = e^{-x}f(x)$ [1, p. 15]. It follows then that $I - V$ is Abel bounded if and only if $(I + V)^{-1}$ is Abel bounded:

If $I - V$ is Abel bounded then there exists a $D > 0$ such that

$$\left|\left|(1-\theta)\sum_{k=0}^{\infty}\theta^k\left[M^{-1}(I+V)^{-1}M\right]^k\right|\right| = ||(1-\theta)\sum_{k=0}^{\infty}\theta^k(I-V)^k|| \leq D$$

for all $\theta \in (0,1)$. Observing that $\left[M^{-1}(I+V)^{-1}M\right]^k = M^{-1}(I+V)^{-k}M$, we have

$$\left|\left|M^{-1}\left[(1-\theta)\sum_{k=0}^{\infty}\theta^k(I+V)^{-k}\right]M\right|\right| \leq D \quad \text{for all } \theta \in (0,1),$$

but since

$$||(1-\theta)\sum_{k=0}^{\infty}\theta^k(I+V)^{-k}|| \leq ||M|| \cdot \left|\left|M^{-1}\left[(1-\theta)\sum_{k=0}^{\infty}\theta^k(I+V)^{-k}\right]M\right|\right| \cdot ||M^{-1}||,$$

$(I+V)^{-1}$ is Abel bounded. Clearly if $(1+V)^{-1}$ is Abel bounded, directly from the fact that the norm is submultiplicative, $I-V$ is Abel bounded.

To summarize, $(I+V)^{-1}$ is Abel bounded, whilst $I+V$ is not. For related results, see [8].

3. Nilpotency

If $x \in A$ has $\sigma(x) = \{1\}$, then $x - \mathbf{1}$ is quasinilpotent. In this section we are going to provide conditions under which $x - \mathbf{1}$ is nilpotent. Drissi and Zemánek [3] also provided conditions under which $x - \mathbf{1}$ is nilpotent, in particular we mention one of these conditions towards the end of this section.

Our first result can be seen as a generalization of Theorem 2.7, the proof of which follows similarly.

Theorem 3.1. *Let (A,C) be an ordered Banach algebra, with closed proper algebra cone C. Let $x \in A$ such that $\sigma(x) = \{1\}$. If $x \geq \mathbf{1}$ and if x is (N)-Abel bounded then $(x-\mathbf{1})^N = 0$.*

Proof. Assume that x is (N)-Abel bounded and $x \geq 1$. From the (N)-Abel boundedness of x it follows that

$$\begin{aligned}\left|\left|\theta^N\left(\frac{1}{\theta}-1\right)^N\frac{1}{\theta}\left[\left(\frac{1}{\theta}-1\right)\mathbf{1}-(x-\mathbf{1})\right]^{-1}\right|\right| &= ||(1-\theta)^N(\mathbf{1}-\theta x)^{-1}|| \\ &= ||(1-\theta)^N\sum_{k=0}^{\infty}\theta^k x^k|| \leq D\end{aligned}$$

for some $D > 0$ and for all $\theta \in (0,1)$. Taking $\lambda = \frac{1}{\theta} - 1$ and writing $y = x - \mathbf{1}$, we have

$$\left|\left|\left(\frac{1}{\lambda+1}\right)^{N-1}\lambda^{N-1}y(\lambda\mathbf{1}-y)^{-1}\right|\right| \leq D'$$

for some $D' > 0$ and for all $\lambda > 0$. Hence

$$\lambda\left[\left(\frac{1}{\lambda+1}\right)^{N-1}\lambda^{N-1}y(\lambda\mathbf{1}-y)^{-1}\right] \to 0 \quad \text{as} \quad \lambda \to 0^+.$$

Replacing $(\lambda\mathbf{1} - y)^{-1}$ with its Laurent expansion and simplifying yields

$$\sum_{k=N}^{\infty} \frac{y^k}{\lambda^{k-N+1}} \to -y^N \quad \text{as} \quad \lambda \to 0^+.$$

Since $x \geq 1$, $y^N \in C$. Hence $\sum_{k=N}^{\infty} \frac{y^k}{\lambda^{k-N+1}} \in C$ for all $\lambda > 0$. This series must converge to an element in C as $\lambda \to 0^+$. Therefore $-y^N \in C$. It follows that $y^N \in C \bigcap (-C) = \{0\}$. Thus $(x - \mathbf{1})^N = 0$. □

If in Theorem 2.1 we relax the condition of x being Abel bounded to x being (N)-Abel bounded, we can prove the following:

Theorem 3.2. *Let (A, C) be an ordered Banach algebra with C normal and closed. If $x \in C$ is (N)-Abel bounded, then $||M_n(x)|| = o(n^N)$ as $n \to \infty$.*

Proof. Let α denote the normality constant. Since C is a closed algebra cone

$$\begin{aligned}(1-\theta)^N \sum_{k=0}^{\infty} \theta^k x^k &\geq (1-\theta)^N \sum_{k=0}^{n} \theta^k x^k \geq (1-\theta)^N \theta^n \sum_{k=0}^{n} x^k \\ &= (1-\theta)^N \theta^n (n+1) M_n(x)\end{aligned}$$

for all $\theta \in (0, 1)$, $n \in \mathbb{N}$. From the normality of C and the (N)-Abel boundedness of x it follows that

$$(1-\theta)^N \theta^n (n+1) ||M_n(x)|| \leq D$$

for some $D > 0$ and for $\theta \in (0, 1)$, $n \in \mathbb{N}$. Now, for a fixed n let $\theta = \frac{n}{n+1}$. Then

$$\begin{aligned}||M_n(x)|| &\leq \frac{D}{n+1}(n+1)^N (1 + \frac{1}{n})^n \\ &= D(n+1)^{N-1}(1 + \frac{1}{n})^n\end{aligned}$$

Now, since $(1 + \frac{1}{n})^n \to e$ as $n \to \infty$, it follows that $||M_n(x)|| = o(n^N)$ as $n \to \infty$. □

Drissi and Zemánek [3, Theorem 2] showed that if $x \in A$ with $\sigma(x) = \{1\}$ is such that $||M_n(x)|| = o(n^p)$ as $n \to \infty$ and $||M_n(x^{-1})|| = o(n^q)$ as $n \to \infty$ for some $p, q \in \mathbb{N}$ then $(x - \mathbf{1})^s = 0$ where $s = \min(p, q)$. As an immediate consequence of this result and Theorem 3.2 we have the following corollary.

Corollary 3.3. *Let (A, C) be an ordered Banach algebra with C normal and closed. Let $x \in A$ have $\sigma(x) = \{1\}$. If $x, x^{-1} \in C$ and if x is doubly (N)-Abel bounded then $(x - \mathbf{1})^N = 0$.*

4. Inverse closed algebra cones

In this section we are going to investigate the effect on an element x with $\sigma(x) = \{1\}$, if x belongs to an inverse closed algebra cone.

Theorem 4.1. *Let (A, C) be an ordered Banach algebra with a closed proper and inverse closed algebra cone C; and $x \in A$ such that $\sigma(x) = \{1\}$. If $x^N \in C$ for some $N \in \mathbb{N}$ then $x = \mathbf{1}$.*

Proof. Since $\sigma(x) = \{1\}$, for $|\lambda| > 1$

$$(\lambda\mathbf{1} - x^N)^{-1} = \sum_{k=0}^{\infty} \frac{x^{Nk}}{\lambda^{k+1}}.$$

If $\lambda > 1$, since $x^N \geq 0$, it follows that $(\lambda\mathbf{1} - x^N)^{-1} \in C$ (since C is closed). C is also inverse closed, and so we have $\lambda\mathbf{1} - x^N \in C$ for all $\lambda > 1$. If $\lambda \to 1^+$, since C is closed it follows that

$$\mathbf{1} - x^N \in C.$$

$\sigma(x) = \{1\}$ and $x^N \in C$ also imply that $x^{-N} \in C$ (C is inverse closed). From a similar argument applied to x^{-N} we can conclude that $\mathbf{1} - x^{-N} \in C$. It follows that

$$x^N(\mathbf{1} - x^{-N}) = x^N - \mathbf{1} \in C.$$

Hence, since C is proper we must have that $x^N = \mathbf{1}$. Now the factorization

$$x^N - \mathbf{1} = (x - \mathbf{1})(x^{N-1} + \ldots + \mathbf{1}) = 0,$$

together with $\sigma(x) = \{1\}$ and the Spectral Mapping Theorem, implies that $x = \mathbf{1}$. □

It follows from the proof of Theorem 4.1 that we can prove the following weaker result.

Proposition 4.2. *Let (A, C) be an ordered Banach algebra with C closed and inverse closed. If $x \in C$ then $0 \leq x \leq r(x)\mathbf{1}$.*

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