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Positivity

Increasing functions and inverse Santaló inequality for unconditional functions

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Abstract. Let $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function and $\mathcal{L}\phi$ be its Legendre tranform. It is proved that if ϕ is invariant by changes of signs, then $\int e^{-\phi} \int e^{-\mathcal{L}\phi} \geq 4^n$. This is a functional version of the inverse Santaló inequality for unconditional convex bodies due to J. Saint Raymond. The proof involves a general result on increasing functions on $\mathbb{R}^n \times \mathbb{R}^n$ together with a functional form of Lozanovskii's lemma. In the last section, we prove that for some $c > 0$, one has always $\int e^{-\phi} \int e^{-\mathcal{L}\phi} \geq c^n$. This generalizes a result of B. Klartag and V. Milman.

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1. Introduction

In the last decades, some functional forms of inequalities holding for subsets of $\mathbb{R}^n$ were extended to functions. The most useful is probably Prekopa-Leindler inequality, which is a functional version of Brunn-Minkowski. Let us mention also Blaschke-Santaló inequality ([3] and [16]), which states that if K is a convex body in $\mathbb{R}^n$ and

$$P(K) = \min_{z \in K} |K||K^{*z}|$$

where $K^{*z} = \{y \in \mathbb{R}^n, \langle y, x - z\rangle \leq 1 \text{ for all } x \in K\}$ is the polar body of K with respect to z, then

$$P(K) \leq P(B_2^n)$$

where B_2^n is the Euclidean ball, with equality if and only if K is an ellipsoid. If $z = 0$, we use the standard notation $K^{*0} = K^\circ$. Notice that the volume product $P(K)$ is an affine invariant.

An inverse form of the Blaschke-Santaló inequality for convex sets was conjectured by Mahler in [12]. The symmetric Mahler conjecture asks if

$$P(K) \geq P(B_1^n) \text{ for every centrally symmetric convex body } K,$$

where $B_1^n = \{x \in \mathbb{R}^n;\ \|x\|_1 = \sum |x_i| \leq 1\}$. This inequality was proved for unconditional convex bodies (*i.e.* symmetric with respect to n-orthogonal hyperplanes) by Saint Raymond [14] (see also Meyer [13] and Reisner [15]). In the general case, it is still open but an asymptotic form was given by Bourgain and Milman [6]: for some constants $\alpha > 0$ and $\beta > 0$, one has

$$P(K)^{1/n} \geq \beta P(B_1^n)^{1/n} \sim \alpha P(B_2^n)^{1/n}.$$

The functional analogue of polarity for sets is the Legendre transform: for a function $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, and $z, y \in \mathbb{R}^n$, let

$$\mathcal{L}_z\phi(y) = \sup_{x \in \mathbb{R}^n} \langle x - z, y - z\rangle - \phi(x).$$

Observe that one has then $\mathcal{L}_z(\mathcal{L}_z\phi) = \phi$ if ϕ is a convex function. For $z = 0$, we denote $\mathcal{L}_0\phi = \mathcal{L}\phi$. We define

$$P(\phi) = \inf_{z \in \mathbb{R}^n} \int e^{-\phi(x)} dx \int e^{-\mathcal{L}_z\phi(y)} dy.$$

Notice that $\phi \mapsto P(\phi)$ is affine invariant too (*i.e.* $P(\phi \circ A) = P(\phi)$ for any one-to-one affine function $A : \mathbb{R}^n \to \mathbb{R}^n$). The functional version of the Blaschke-Santaló inequality states that

$$P(\phi) \leq P\left(\frac{|\cdot|^2}{2}\right)$$

where $|\cdot|$ stands here for the Euclidean norm in $\mathbb{R}^n$. This statement was proved for even functions by K. Ball [2] and in full generality by Artstein-Klartag-Milman [1] (see also Fradelizi-Meyer [8]).

We give here the following sharp functional version of the reverse inequality: if ϕ is convex and unconditional (*i.e.* $\phi(x_1, \ldots, x_n) = \phi(|x_1|, \ldots, |x_n|)$ for every $(x_1, \ldots, x_n) \in \mathbb{R}^n$), then

$$P(\phi) = \int_{\mathbb{R}^n} e^{-\phi(x)} dx \int_{\mathbb{R}^n} e^{-\mathcal{L}\phi(y)} dy \geq 4^n = P(\|\cdot\|_1).$$

For this, we establish functional versions of some results of Saint-Raymond [14] and of Bollobás Leader and Radcliffe [5], and give a functional form of the classical Lozanovskii theorem. In a forthcoming paper [9], we shall prove among other results that for $n = 1$ and general ϕ, $P(\phi) \geq e$.

This paper is organized in the following way. In section 2, we give various inequalities about integrals of increasing functions on $\mathbb{R}^n$. In section 3, we prove a functional extension of Lozanovskii lemma and apply the results of section 2 to integrals of Legendre transforms. In section 4, we give a short proof of the Klartag-Milman's [10] functional extension of Bourgain-Milman inequality, using this inequality for sets in the most general case ([6]). Namely, we prove that there exists a constant $C > 0$ such that for every convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ with $0 < \int_{\mathbb{R}^n} e^{-\phi(x)}dx < +\infty$, one has

$$\int\limits_{\mathbb{R}^n} e^{-\phi(x)}dx \int\limits_{\mathbb{R}^n} e^{-\mathcal{L}\phi(x)}dx \geq C^n.$$

This was proved in [10] under the restrictive hypothesis that $\phi(0) = \min \phi$ (which includes of course the case of even functions).

2. Some inequalities on integrals of increasing functions

Notation. We say that a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is *unconditional* if

$$\varphi(x_1, \ldots, x_n) = \varphi(|x_1|, \ldots, |x_n|) \text{ for every } (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

We consider on $\mathbb{R}^n$ the canonical partial order:

$$x = (x_1, \ldots, x_n) \leq y = (y_1, \ldots, y_n) \quad \text{whenever } x_i \leq y_i,\ 1 \leq i \leq n.$$

For $x, y \in \mathbb{R}^n$ such that $x \leq y$, we define the order intervals

$$[x, y] = \{z \in \mathbb{R}^n; x \leq z \leq y\},$$
$$[y, +\infty) = \{z \in \mathbb{R}^n;\ z \geq y\} \quad \text{and} \quad (-\infty, x] = \{z \in \mathbb{R}^n;\ z \leq x\}.$$

We say that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is *increasing* if $\varphi(x) \leq \varphi(y)$ when $x \leq y$.

In the same way, a subset K of $\mathbb{R}^n$ is *unconditional* (respectively *increasing*) if its indicator function $\mathbf{1}_K$ is unconditional (respectively increasing). Finally, we denote by $|A|$ the Lebesgue measure of a measurable set A in $\mathbb{R}^n$.

Our first result concerns increasing functions of two variables in $\mathbb{R}^n$.

Theorem 1. *Let $F :\ \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be an increasing function. Then the following inequality holds for all $z \in \mathbb{R}^n$*

$$\int\limits_{\mathbb{R}^n} F(x, z - x)\ dx \geq \int\limits_{(-\infty, z]} \sup_{x \in \mathbb{R}^n} F(x, w - x)\ dw.$$

Proof. **1)** Assume first that $F = \mathbf{1}_K$, where $K \subset \mathbb{R}^n \times \mathbb{R}^n$ is an increasing set. We define for $z \in \mathbb{R}^n$, the sets

$$K_z = \{x \in \mathbb{R}^n;\ (x, z - x) \in K\},\ C = \{w \in \mathbb{R}^n;\ K_w \neq \emptyset\} \text{ and } C_z = C \cap (-\infty, z].$$

One has to prove that

$$\int\limits_{\mathbb{R}^n} \mathbf{1}_K(x, z - x)\ dx = |K_z| \geq \int\limits_{(-\infty, z]} \sup_{x \in \mathbb{R}^n} \mathbf{1}_K(x, w - x)\ dw = |C_z|.$$

The proof goes by induction on $n \geq 1$.

a. Suppose $n = 1$; then $K \subset \mathbb{R}^2$ and $C \subset \mathbb{R}$. Let $w \in C_z$. Then for every $x \in K_w$ and $t \in [0, z - w]$, one has

$$(x, w - x) \in K,\ x + t \geq x \quad \text{and} \quad z - (x + t) \geq w - x.$$

Since K is increasing, this implies that $(x+t, z-(x+t)) \in K$ and therefore that $x+t \in K_z$. It follows that $K_w + [0, z-w] \subset K_z$ for every $w \in C_z$. Thus $K_z \neq \emptyset$ and C is increasing. Setting $w_K = \inf C \in \mathbb{R} \cup \{-\infty\}$, we get that

$$(w_K, z] \subset C_z \subset [w_K, z]$$

and that

$$|K_z| \geq \sup_{w \in C_z} (z-w) = z - \inf C_z = z - w_K = |C_z|.$$

b. For $n \geq 2$, we proceed by induction to show that $|C_z| \leq |K_z|$ for every increasing subset K of $\mathbb{R}^n \times \mathbb{R}^n$.

Writing $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n$, one has

$$|K_z| = \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \mathbf{1}_K(x_1, x_2, z_1 - x_1, z_2 - x_2)\, dx_1 \right) dx_2.$$

Applying the 1-dimensional case to

$$M_{x_2, y_2} = \{(x_1, y_1) \in \mathbb{R}^2; (x_1, x_2, y_1, y_2) \in K\}$$

which is an increasing subset of $\mathbb{R}^2$, one has for all (x_2, y_2)

$$\begin{aligned} &|\{x_1; (x_1, x_2, z_1 - x_1, y_2) \in K\}| \\ &\quad \geq |\{w_1 \leq z_1; (t_1, x_2, w_1 - t_1, y_2) \in K \text{ for some } t_1\}|. \end{aligned}$$

It follows that

$$\begin{aligned} |K_z| &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \mathbf{1}_K(x_1, x_2, z_1 - x_1, z_2 - x_2)\, dx_1 \right) dx_2 \\ &\geq \int_{\mathbb{R}^{n-1}} \left(\int_{w_1 \leq z_1} \sup_{t_1} \mathbf{1}_K(t_1, x_2, w_1 - t_1, z_2 - x_2)\, dw_1 \right) dx_2 \\ &= \int_{w_1 \leq z_1} \left(\int_{\mathbb{R}^{n-1}} \mathbf{1}_{L_{w_1}}(x_2, z_2 - x_2)\, dx_2 \right) dw_1, \end{aligned}$$

where

$$L_{w_1} = \{(x_2, y_2) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; (t_1, x_2, w_1 - t_1, y_2) \in K \text{ for some } t_1\}.$$

We apply the induction hypothesis to this increasing subset to get

$$|\{x_2; (x_2, z_2 - x_2) \in L_{w_1}\}| \geq |\{w_2 \leq z_2; (t_2, w_2 - t_2) \in L_{w_1} \text{ for some } t_2\}|.$$

It follows that

$$
\begin{aligned}
|K_z| &\geq \int\limits_{w_1 \leq z_1} \left(\int\limits_{w_2 \leq z_2} \sup_{t_2} \mathbf{1}_{L_{w_1}}(t_2, w_2 - t_2)\ dw_2 \right)\ dw_1 \\
&= \int\limits_{w=(w_1,w_2)\leq(z_1,z_2)=z} \sup_{t_1,t_2} \mathbf{1}_K(t_1, t_2, w_1 - t_1, w_2 - t_2)\ dw = |C_z|.
\end{aligned}
$$

2) Now if $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is an increasing function, then for every $t > 0$, $\{F > t\}$ is an increasing subset of $\mathbb{R}^n \times \mathbb{R}^n$. We apply **1)** to get:

$$
\begin{aligned}
\int\limits_{\mathbb{R}^n} F(x, z - x)\ dx &= \int\limits_0^{+\infty} \left(\int\limits_{\mathbb{R}^n} \mathbf{1}_{\{F>t\}}(x, z - x)\ dx \right)\ dt \\
&\geq \int\limits_0^{+\infty} \left(\int\limits_{w \leq z} \sup_u \mathbf{1}_{\{F>t\}}(u, w - u)\ dw \right)\ dt \\
&= \int\limits_{w \leq z} \left(\int\limits_0^{+\infty} \sup_u \mathbf{1}_{\{F>t\}}(u, w - u)\ dt \right)\ dw \\
&= \int\limits_{w \leq z} \sup_u F(u, w - u)\ dw.
\end{aligned}
$$

□

Remark. It can be proved that, if $n = 1$, there is equality in Theorem 1 for every $z \in \mathbb{R}^n$ if and only if one has $F(x, y) = \inf(\alpha(x), \beta(y))$ for some increasing functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}_+$. This characterization does not hold for $n \geq 2$.

The following corollary states that the volume of a union of parallelepipeds with faces parallel to the coordinate hyperplanes is minimal if they share the same corner (for example the corner with the smallest coordinates). We use the notation introduced in the beginning of this section for order intervals in $\mathbb{R}^n$.

Corollary 2. *Let $u_1, \dots, u_N \in \mathbb{R}^n_+$. Then for every $v_1, \dots, v_N \in \mathbb{R}^n$, one has*

$$
\left| \bigcup_{i=1}^N [0, u_i] \right| \leq \left| \bigcup_{i=1}^N [v_i, u_i + v_i] \right|.
$$

Proof. Let $a_i, b_i \in \mathbb{R}^n$, $1 \leq i \leq N$. Applying Theorem 1 to the indicator function of the increasing set Q defined by

$$
Q = \bigcup_{i=1}^N\ [a_i, +\infty) \times [b_i, +\infty),
$$

we get for every $z \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \mathbf{1}_Q(x, z-x)dx = \left|\bigcup_{i=1}^N [a_i, z-b_i]\right| \geq \int_{(-\infty,z]} \sup_x \mathbf{1}_Q(x, w-x)dw$$
$$= \left|\bigcup_{i=1}^N [a_i+b_i, z]\right|.$$

The last equality follows from the fact that $\sup_x \mathbf{1}_Q(x, w-x) = 1$ if and only if for some $1 \leq i \leq N$ and some $x \in \mathbb{R}^n$, one has $a_i \leq x \leq w - b_i$, *i.e.* if and only if for some $1 \leq i \leq N$, one has $a_i \leq w - b_i$. For $1 \leq i \leq N$, let

$$u_i = z - a_i - b_i \quad \text{and} \quad v_i = b_i.$$

Since $[a_i, z-b_i] = z - [v_i, u_i+v_i]$ and $[a_i+b_i, z] = z - [0, u_i]$, one gets

$$\left|\bigcup_{i=1}^N [0, u_i]\right| \leq \left|\bigcup_{i=1}^N [v_i, u_i+v_i]\right|.$$

□

From Theorem 1 we deduce also the following functional version of an inequality due to Saint-Raymond ([14]) and of the reverse Kleitman inequality due to Bollobás, Leader and Radcliffe ([5]).

Theorem 3. *Let $f, g : \mathbb{R}^n \to \mathbb{R}_+$ be increasing functions. Define $h : \mathbb{R}^n \to \mathbb{R}_+$ by $h(z) = \sup_x f(x)g(z-x)$. Then one has:*

i) *$(f * g)(z) \geq (h * \mathbf{1}_{\mathbb{R}^n_+})(z)$ for every $z \in \mathbb{R}^n$.*

ii) *For every $\xi_1, \ldots, \xi_n > 0$,*

$$\int_{\mathbb{R}^n} e^{-\langle x,\xi\rangle} f(x)dx \int_{\mathbb{R}^n} e^{-\langle x,\xi\rangle} g(x)dx \geq \int_{\mathbb{R}^n} e^{-\langle x,\xi\rangle} h(x)dx \Big/ \left(\prod_{i=1}^n \xi_i\right).$$

Proof. To get i), we apply the preceding theorem to $F(x,y) = f(x)g(y)$. Then for every $w \in \mathbb{R}^n$, one has $\sup_x F(x, w-x) = h(w)$, so that for every $z \in \mathbb{R}^n_+$

$$(f * g)(z) = \int_{\mathbb{R}^n} F(x, z-x)dx \geq \int_{(-\infty,z]} h(w)dw = (h * \mathbf{1}_{\mathbb{R}^n_+})(z).$$

Inequality ii) follows from i) using the Laplace transform at $\xi = (\xi_1, \ldots, \xi_n)$, with $\xi_i > 0$, $1 \leq i \leq n$. □

Remarks. 1) Using the remark after Theorem 1, one can prove that, for $n = 1$, there is equality in i) or in ii) of Theorem 3 if and only if either f or g is of the form $c\mathbf{1}_{[d,+\infty)}$ for some $c > 0$ and some $d \in \mathbb{R}$.

2) The function h in Theorem 3 is known as the *Asplund sum* of f and g. This comes from the fact that for $f = \mathbf{1}_A$ and $g = \mathbf{1}_B$ then $h = \mathbf{1}_{A+B}$. If moreover A and B are increasing, Theorem 3 gives

$$\mathbf{1}_A * \mathbf{1}_B \geq \mathbf{1}_{A+B} * \mathbf{1}_{\mathbb{R}^n_+}, \ \textit{i.e.}\ |A \cap (-B)| \geq |(A+B) \cap \mathbb{R}^n_-|.$$

This inequality was proved by Saint Raymond in [14]. Bollobás, Leader and Radcliffe [5] reproved and reinterpreted it in terms of a reverse Kleitman inequality: If S is a solid subset S of $\mathbb{R}^n$ (*i.e.* $[x, y] \subset S$, for every $x, y \in S$ such that $x \leq y$), one has

$$|(S - S) \cap \mathbb{R}^n_+| \leq |S|.$$

Actually, if $S^+ = \cup_{x\in S}[x, +\infty)$ and $S^- = \cup_{x\in S}(-\infty, x]$, one has $S = S^+ \cap S^-$ and $(S - S) \cap \mathbb{R}^n_+ = (S^- - S^+) \cap \mathbb{R}^n_+$. Setting $A = S^+$ and $B = -S^-$, we conclude.

Notation. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we define $x{\cdot}y \in \mathbb{R}^n$ by

$$x \cdot y = (x_1 y_1, \dots, x_n y_n).$$

The next theorem is also a functional form of a result of Saint-Raymond ([14]).

Theorem 4. *Let $\tilde{f}, \tilde{g} : \mathbb{R}^n_+ \to \mathbb{R}_+$ be decreasing functions, and define $\tilde{h} : \mathbb{R}^n_+ \to \mathbb{R}_+$ by*

$$\tilde{h}(z) = \sup_{z=x\cdot y} \tilde{f}(x)\tilde{g}(y) \ \textit{for all}\ z \in \mathbb{R}^n_+.$$

Then for every $r_i > 0$, $1 \leq i \leq n$, one has

$$\int_{\mathbb{R}^n_+} \prod_{i=1}^n x_i^{r_i - 1} \tilde{f}(x)dx \int_{\mathbb{R}^n_+} \prod_{i=1}^n x_i^{r_i - 1} \tilde{g}(x)dx \geq \int_{\mathbb{R}^n_+} \prod_{i=1}^n x_i^{r_i - 1} \tilde{h}(x)dx \Big/ \prod_{i=1}^n r_i.$$

In particular

$$\int_{\mathbb{R}^n_+} \tilde{f}(x)dx \int_{\mathbb{R}^n_+} \tilde{g}(x)dx \geq \int_{\mathbb{R}^n_+} \tilde{h}(x)dx.$$

Proof. For $t \in \mathbb{R}^n$, we define

$f(t) = \tilde{f}(e^{-t_1}, \dots, e^{-t_n})$, $g(t) = \tilde{g}(e^{-t_1}, \dots, e^{-t_n})$ and $h(t) = \tilde{h}(e^{-t_1}, \dots, e^{-t_n})$.

Then $f, g : \mathbb{R}^n \to \mathbb{R}_+$ are increasing and for every $z \in \mathbb{R}^n$, one has

$$h(z) = \sup_{z=x+y} f(x)g(y).$$

Setting $x_i = e^{-t_i}$ for $i = 1, \dots, n$, we need to prove

$$\int_{\mathbb{R}^n} e^{-\langle t,r\rangle} f(t)dt \int_{\mathbb{R}^n} e^{-\langle t,r\rangle} g(t)dt \geq \int_{\mathbb{R}^n} e^{-\langle t,r\rangle} h(t)dt \Big/ \left(\prod_{i=1}^n r_i\right),$$

and this follows from Theorem 3. $\square$

3. Applications to the inverse Santaló functional inequality

We first establish a functional form of Lozanovskii's theorem ([11]).

Lemma 5. *Let $\phi:\ \mathbb{R}^n \to \mathbb{R}$ be a convex unconditional function such that*

$$\int_{\mathbb{R}^n} e^{-\phi(x)}dx < +\infty.$$

Then for every $z \in (0,+\infty)^n$ one has

$$\inf_{x,y\geq 0,\ x\cdot y=z} (\phi(x)+\mathcal{L}\phi(y)) = \sum_{i=1}^n z_i.$$

Proof. We follow the proof of Lozanovskii's theorem given by Saint-Raymond in [14]. Let $z\in(0,+\infty)^n$ be fixed. It is clear that for every x,y such that $x\cdot y=z$ one has

$$\phi(x)+\mathcal{L}\phi(y) \geq \langle x,y\rangle = \sum_{i=1}^n x_iy_i = \sum_{i=1}^n z_i.$$

It is thus enough show that there exist $x,y\in(0,+\infty)^n$ such that

$$x\cdot y = z \quad \text{and} \quad \phi(x)+\mathcal{L}\phi(y) = \langle x,y\rangle = \sum_{i=1}^n z_i.$$

Define $F:\ (0,+\infty)^n \to \mathbb{R}$ by

$$F(w) = \phi(w) - \sum_{i=1}^n z_i\log(w_i) \qquad \text{for } w=(w_1,\dots,w_n).$$

Then F is convex and the integrability condition $\int_{\mathbb{R}_+^n} e^{-\phi(w)}dw < +\infty$ implies easily that for some $c=(c_1,\dots,c_n)\in(0,+\infty)^n$, one has

$$\phi(w) \geq \phi(0) + \langle c,w\rangle = \phi(0) + \sum_{i=1}^n c_iw_i, \qquad \text{for every } w\in(0,+\infty)^n.$$

It follows that $F(w)\to+\infty$ when $w_1+\cdots+w_n\to+\infty$ or when for some $1\leq i\leq n$, $w_i\to 0$. Hence F reaches its global minimum at some point $x\in(0,+\infty)^n$. We define $L:\ (0,+\infty)^n\to\mathbb{R}$ by

$$L(w) = \sum_{i=1}^n z_i\log(w_i).$$

Since $F(w)\geq F(x)$ on $(0,+\infty)^n$, one has

$$\phi(w)-\phi(x) \geq L(w)-L(x) \quad \text{for every } w\in(0,+\infty)^n,$$

Since ϕ is convex and L is concave, by Hahn-Banach's theorem, we can separate the epigraph of ϕ from the subgraph of L with an affine hyperplane; thus there exists $y\in\mathbb{R}^n$ such that

$$\phi(w)-\phi(x) \geq \langle w-x,y\rangle \geq L(w)-L(x) \quad \text{for every } w\in(0,+\infty)^n.$$

The right hand side inequality implies that $y = \nabla L(x) = (\frac{z_1}{x_1}, \dots, \frac{z_n}{x_n}) \in (0, +\infty)^n$ and the left hand side inequality gives

$$\mathcal{L}\phi(y) = \sup_w \langle w, y\rangle - \phi(w) = \langle x, y\rangle - \phi(x).$$

$\square$

Remarks. 1) If A, B are subsets of $\mathbb{R}^n$, let

$$A \cdot B = \{x \cdot y;\ x \in A,\ y \in B\}\,.$$

If we apply Lemma 5 to $\phi(x) = \|x\|_K^2/2$, we get back the classical Lozanovskii theorem asserting that if K is an unconditional convex body, one has

$$K \cdot K^\circ = B_1^n := \{x;\ \sum_{i=1}^n |x_i| \le 1\}.$$

As a matter of fact, for all $z \in (0, +\infty)^n$, one has

$$\inf_{z=x\cdot y} \left(\frac{\|x\|_K^2}{2} + \frac{\|y\|_{K^\circ}^2}{2}\right) \ge \inf_{z=x\cdot y} \|x\|_K \|y\|_{K^\circ} \ge \inf_{z=x\cdot y} \langle x, y\rangle = \sum_{i=1}^n z_i.$$

From lemma 5, the left hand side is equal to the right hand side, hence

$$\inf_{z=x\cdot y} \|x\|_K \|y\|_{K^\circ} = \sum_{i=1}^n z_i = \|z\|_1,$$

which means that the gauges of $K \cdot K^\circ$ and of B_1^n are equal.
2) If we apply Theorem 4 to indicator functions, we get the following result, obtained by B. Bollobás and I. Leader, and independently by M. Meyer and A. Pajor (see [4]). Let A and B be decreasing compact subsets of $\mathbb{R}_+^n$. Then

$$|A||B| \ge |A \cdot B|.$$

Using Lozanovskii's theorem, this gives the inverse Blaschke-Santaló inequality for unconditional convex bodies due to Saint-Raymond [14]: Let K be an unconditional convex body in $\mathbb{R}^n$. Then

$$|K||K^\circ| \ge |K \cdot K^\circ| = 2^n |B_1^n| = 4^n/n!.$$

We give now our main theorem, a functional version of the inverse Santaló inequality for unconditional convex functions:

Theorem 6. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a convex unconditional function. Then for all $(r_1, \dots, r_n) \in (0, +\infty)^n$ one has*

$$\int_{\mathbb{R}_+^n} \left(\prod_{i=1}^n r_i x_i^{r_i - 1}\right) e^{-\phi(x)} dx \int_{\mathbb{R}_+^n} \left(\prod_{i=1}^n r_i x_i^{r_i - 1}\right) e^{-\mathcal{L}\phi(x)} dx \ge \prod_{i=1}^n \Gamma(r_i + 1).$$

In particular,

$$\int_{\mathbb{R}_+^n} e^{-\phi(x)} dx \int_{\mathbb{R}_+^n} e^{-\mathcal{L}\phi(x)} dx \ge 1.$$

and

$$P(\phi) = \int_{\mathbb{R}^n} e^{-\phi(x)}dx \int_{\mathbb{R}^n} e^{-\mathcal{L}\phi(x)}dx \geq 4^n = P(\|\cdot\|_1).$$

Proof. Observe that ϕ and $L\phi$ are increasing on $\mathbb{R}^n_+$. We apply Theorem 4 to $\tilde{f} = e^{-\phi}$ and $\tilde{g} = e^{-\mathcal{L}\phi}$. From Lemma 5 we get

$$\tilde{h}(z) = \sup_{z=x\cdot y} e^{-(\phi(x)+\mathcal{L}\phi(y))} = e^{-\sum_{i=1}^n z_i} \quad \text{for every } z \in (0,+\infty)^n,$$

which gives the first inequality. For the second inequality we take $r_1 = \cdots = r_n = 1$. The fact that ϕ is unconditional implies that ϕ is even and that $\mathcal{L}\phi$ is also unconditional. Hence

$$P(\phi) = \int_{\mathbb{R}^n} e^{-\phi(x)}dx \int_{\mathbb{R}^n} e^{-\mathcal{L}\phi(y)}dy = 4^n \int_{\mathbb{R}^n_+} e^{-\phi(x)}dx \int_{\mathbb{R}^n_+} e^{-\mathcal{L}\phi(y)}dy \geq 4^n.$$

□

Remarks. 1) If we apply Theorem 6 to $\phi(x) = \|x\|_K$, where K is a unconditional convex body, one gets $e^{-\mathcal{L}\phi} = \mathbf{1}_{K^\circ}$. It enables to recover inequalities of Saint Raymond [14]. For example, our last inequality gives

$$P(\|x\|_K) = n!|K||K^\circ| = n!P(K) \geq 4^n.$$

2) Using the remark after Theorem 3, it is easy to prove that, for $n = 1$, there is equality in the inequalities of Theorem 6 if and only either

$$e^{-\phi(x)} = e^{-\beta-\alpha|x|} \quad \text{and} \quad e^{-\mathcal{L}\phi(y)} = e^{\beta}\mathbf{1}_{[-\alpha,\alpha]}(y)$$

or vice-versa. The inequalities of the Theorem 6 are sharp for every n: take for instance $\phi(x) = \|x\|_1 = \sum_{i=1}^n |x_i|$. But, for $n \geq 2$, we ignore the characterization of the case of equality. For unconditional bodies, this problem was solved independantly by Meyer [13] and Reisner [15] (see also [5] for an other proof).

3) Theorem 6 can be generalized as follows. We say that a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ is *almost unconditional* if for every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1,1\}^n$,

$$(x_1, \ldots, x_n) \mapsto \phi(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$$

is increasing on $\mathbb{R}^n_+$. Using Theorem 6, it is easy to prove, as in [14] for almost unconditional bodies, that $P(\phi) \geq 4^n$ for every almost unconditional even convex function $\phi : \mathbb{R}^n \to \mathbb{R}$.

4. A general functional version of the Bourgain-Milman inequality

The following theorem was proved by Klartag and Milman [10] in the particular case when the function φ is minimal at 0. We shall prove it in full generality, using as a main tool, like in [10], the Bourgain-Milman inequality for convex sets ([6]), which says that for some $\alpha > 0$, one has

$P(K) \geq \alpha^n P(B_2^n)$ for every $n \geq 1$ and every convex body K in $\mathbb{R}^n$,

where B_2^n denotes here the Euclidean ball in $\mathbb{R}^n$.

Theorem 7. *There exists $c > 0$ such that for every convex function $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, with $0 < \int_{\mathbb{R}^n} e^{-\phi(x)} dx < +\infty$ one has,*

$$P(\phi) = \inf_z \int_{\mathbb{R}^n} e^{-\phi(x)} dx \int_{\mathbb{R}^n} e^{-\mathcal{L}_z \phi(y)} dy \geq c^n.$$

We need first a lemma giving an interesting relationship between the level sets of a convex function and of its Legendre transform.

Lemma 8. *Let $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Then for every $t > 0$ one has*

$$\{\mathcal{L}\phi \leq t + \mathcal{L}\phi(0)\} = \left\{ \frac{x}{\phi(x) - \min\phi + t} \ ; \ x \in \mathbb{R}^n \right\}^\circ \supset \ t \{\phi \leq t + \phi(0)\}^\circ .$$

and for every $s, t \in \mathbb{R}$ such that $s + t > 0$

$$\{\mathcal{L}\phi \leq t\} \subset (s + t) \{\phi \leq s\}^\circ .$$

Proof. Since for every $c \in \mathbb{R}$ one has $\mathcal{L}(\phi - c) = \mathcal{L}\phi + c$, we may assume that $\min\phi = 0$. This implies that $\mathcal{L}\phi(0) = -\min\phi = 0$. For every $t > 0$, we define

$$K_t := \left\{ \frac{x}{\phi(x) + t}; \ x \in \mathbb{R}^n \right\}.$$

Then, for $t > 0$,

$$\begin{aligned} \{\mathcal{L}\phi \leq t\} &= \{y \in \mathbb{R}^n; \ \langle x, y\rangle \leq \phi(x) + t, \ \forall x \in \mathbb{R}^n\} \\ &= \{y \in \mathbb{R}^n; \ \langle \frac{x}{\phi(x) + t}, y\rangle \leq 1, \ \forall x \in \mathbb{R}^n\} \\ &= \{y \in \mathbb{R}^n; \ \langle z, y\rangle \leq 1, \ \forall z \in K_t\} \\ &= K_t^0. \end{aligned}$$

Using the convexity of ϕ, one has for every $x \in \mathbb{R}^n$,

$$\phi\left(\frac{tx}{\phi(x) + t}\right) \leq \frac{t}{\phi(x) + t} \times \phi(x) + \frac{\phi(x)}{\phi(x) + t} \times \phi(0) \leq t + \phi(0).$$

This prove that $tK_t \subset \{\phi \leq t + \phi(0)\}$. Finally, taking the polar with respect to the origin, we obtain

$$\{\mathcal{L}\phi \leq t\} = K_t^\circ \supset t\{\phi \leq t + \phi(0)\}^\circ.$$

The second inequality follows from the fact that if $\phi(x) \geq s$ and $\mathcal{L}\phi(y) \leq t$, then

$$\langle x, y\rangle \leq \phi(x) + \mathcal{L}\phi(y) \leq s + t.$$

$\square$

Proof of Theorem 7. Let $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. For every $z \in \mathbb{R}^n$, we define $\phi_z(x) = \phi(z + x)$. As observed in [1], the function

$$z \mapsto \int_{\mathbb{R}^n} e^{-\phi(x)} dx \int_{\mathbb{R}^n} e^{-\mathcal{L}(\phi_z)(y)} dy$$

is strictly convex on $\mathbb{R}^n$ and reaches its minimum $P(\phi)$ at a unique point z_0. Using a change of variable, we may assume that $z_0 = 0$ and since $P(\phi + c) = P(\phi)$, we may assume also that $\min \phi = 0$.

One has then

$$\int_{\mathbb{R}^n} y e^{-\mathcal{L}\phi(y)} dy = 0.$$

By a change of variables, one has

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\phi(x)} dx &= \int_{\min \phi}^{+\infty} e^{-t} \big|\{\phi \le t\}\big|\, dt \ge \int_{\phi(0)}^{+\infty} e^{-t} \big|\{\phi \le t\}\big| dt \\ &= e^{-\phi(0)} \int_0^{+\infty} e^{-t} \big|\{\phi \le t + \phi(0)\}\big| dt.\end{aligned}$$

Applying this inequality to ϕ and $\mathcal{L}\phi$, using Lemma 8, Cauchy-Schwarz inequality and the Bourgain-Milman inequality [6] as recalled at the beginning of this section, we get

$$\begin{aligned}P(\phi) &= \int_{\mathbb{R}^n} e^{-\phi(x)} dx \int_{\mathbb{R}^n} e^{-\mathcal{L}\phi(y)} dy \\ &\ge e^{-\phi(0) - \mathcal{L}\phi(0)} \int_0^{+\infty} e^{-t} |\{\phi \le t + \phi(0)\}| dt \int_0^{+\infty} e^{-t} |\{\mathcal{L}\phi \le t + \mathcal{L}\phi(0)\}| dt \\ &\ge e^{-\phi(0) - \mathcal{L}\phi(0)} \int_0^{+\infty} e^{-t} |\{\phi \le t + \phi(0)\}| dt \int_0^{+\infty} e^{-t} t^n |\{\phi \le t + \phi(0)\}^\circ| dt \\ &\ge e^{-\phi(0) - \mathcal{L}\phi(0)} \left(\int_0^{+\infty} e^{-t} t^{n/2} \sqrt{|\{\phi \le t + \phi(0)\}|\ |\{\phi \le t + \phi(0)\}^\circ|} dt \right)^2 \\ &\ge e^{-\phi(0) - \mathcal{L}\phi(0)} \left(\Gamma\left(\frac{n}{2} + 1\right)\right)^2 \alpha^n |B_2^n|^2 \\ &= e^{-\phi(0) - \mathcal{L}\phi(0)} (\alpha\pi)^n.\end{aligned}$$

If we can prove that $\mathcal{L}\phi(0) + \phi(0) \le n$, we get

$$P(\phi) \ge \left(\frac{\alpha\pi}{e}\right)^n.$$

Actually, we know that $-\phi(0) = \min L\phi$, and it follows from [7] that for every convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} ye^{-\psi(y)}dy = 0$ one has

$$\psi(0) \leq \min \psi + n.$$

For the sake of completeness, we give a short proof of this fact : by Jensen's inequality and the convexity of ψ, one has

$$\psi(0) = \psi\left(\frac{\int_{\mathbb{R}^n} ye^{-\psi(y)}dy}{\int_{\mathbb{R}^n} e^{-\psi(y)}dy}\right) \leq \frac{\int_{\mathbb{R}^n} \psi(y)e^{-\psi(y)}dy}{\int_{\mathbb{R}^n} e^{-\psi}dy}.$$

Applying again the convexity of ψ, $x, y \in \mathbb{R}^n$

$$\psi(x) \geq \psi(y) + \langle x - y, (\nabla\psi)(y)\rangle.$$

Multiplying both terms by $e^{-\psi(y)}$ and integrating over $\mathbb{R}^n$, we get

$$\begin{aligned}\psi(x)\int_{\mathbb{R}^n} e^{-\psi(y)}dy &\geq \int_{\mathbb{R}^n} \psi(y)e^{-\psi(y)}dy + \int_{\mathbb{R}^n} \langle x - y, (\nabla\psi)(y)\rangle e^{-\psi(y)}dy \\ &= \int_{\mathbb{R}^n} \psi(y)e^{-\psi(y)}dy - n\int_{\mathbb{R}^n} e^{-\psi(y)}dy.\end{aligned}$$

and this gives the result. □

Remark. There exists $c > 0$ such that if $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing and $\phi : \mathbb{R}^n \to \mathbb{R}^+$ is convex and satisfies $\phi(0) = \min\phi = 0$ and $0 < \int_{\mathbb{R}^n} \rho(\phi) < +\infty$, then

$$\int_{\mathbb{R}^n} \rho(\phi) \int_{\mathbb{R}^n} \rho(\mathcal{L}\phi) \geq c^n \left(\int_{\mathbb{R}^n} \rho(|x|^2/2)dx\right)^2.$$

This follows from the same argument as in theorem 7 writing in the regular case $\rho(t) = \int_t^{+\infty} \theta(s)ds$. If moreover ρ is log-concave then it was proved in Theorem 8 of [8] that

$$\int_{\mathbb{R}^n} \rho(\phi) \int_{\mathbb{R}^n} \rho(\mathcal{L}\phi) \leq \left(\int_{\mathbb{R}^n} \rho(|x|^2/2)dx\right)^2.$$

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