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On a New Kato Class and Singular Solutions of a Nonlinear Elliptic Equation in Bounded Domains of $\mathbb{R}^{n\star}$

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Abstract. Using a new form of the $3G$-Theorem for the Green function of a bounded domain Ω in $\mathbb{R}^n$, we introduce a new Kato class $K(\Omega)$ which contains properly the classical Kato class $K_n(\Omega)$. Next, we exploit the properties of this new class, to extend some results about the existence of positive singular solutions of nonlinear differential equations.

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1. Introduction

Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n (n \geqslant 3)$, and $G := G_\Omega$, be the Green function of the Laplacian in Ω. In [13], Zhao have established interesting inequalities for the Green function G. In particular, he proved the existence of a positive constant C, such that for each x, y, z in Ω

$$\frac{\delta(y)}{\delta(x)} G(x, y) \leqslant \frac{C}{|x-y|^{n-2}}, \tag{1.1}$$

$$\frac{1}{C} H(x, y) \leqslant G(x, y) \leqslant C H(x, y), \tag{1.2}$$

$$\frac{G(x, z) G(z, y)}{G(x, y)} \leqslant \subseteq \frac{|x-y|^{n-2}}{|x-z|^{n-2}|y-z|^{n-2}}, \tag{1.3}$$

where

$$H(x, y) := \frac{1}{|x-y|^{n-2}} \min\left(1, \frac{\delta(x)\delta(y)}{|x-y|^2}\right)$$

$\star$ This paper has not been submitted elsewhere in identical or similar form, nor will it be during the first three months after its submission to Positivity.

and $\delta(x)$ denotes the Euclidean distance between x and $\partial\Omega$.

The inequality (1.3), called 3G-Theorem is often used in this form

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leqslant C\left(\frac{1}{|x-z|^{n-2}} + \frac{1}{|y-z|^{n-2}}\right). \tag{1.4}$$

This 3G-Theorem is useful for the study of functions belonging to the Kato class $K_n(\Omega)$ (see Definition 1 below), which is widely used in the study of some nonlinear differential equations (see for example [1], [10] and [12]). More properties pertaining to this class can be found in [1] and [3].

DEFINITION 1 (See [l] or [3]). A Borel measurable function φ in Ω belongs to the Kato class $K_n(\Omega)$ if φ satisfies the following condition

$$\lim_{\alpha\to 0}\left(\sup_{x\in\Omega}\int_{\Omega\cap B(x,\alpha)} \frac{|\varphi(y)|}{|x-y|^{n-2}}\mathrm{d}y\right) = 0. \tag{1.5}$$

In [6], Kalton and Verbitsky improve (1.4), in the following form

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leqslant C_0\left[\frac{\delta(z)}{\delta(x)}G(x,z) + \frac{\delta(z)}{\delta(y)}G(y,z)\right]. \tag{1.6}$$

More precisely, they denoted by $N(x,y) = \frac{G(x,y)}{\delta(x)\delta(y)}$, the Naïm kernel and they proved in [6] (Lemma 7.1) that $\rho(x,y) = N(x,y)^{-1}$ is a quasi-metric on Ω. Thus (1.6) holds.

This new form of the 3G-Theorem allows us to introduce a new class of functions denoted by $K(\Omega)$ (see Definition 2 below), which contains properly the classical Kato class $K_n(\Omega)$ and which permits to generalize some results of [7], [10] and [12].

DEFINITION 2. A Borel measurable function φ in Ω belongs to the Kato class $K(\Omega)$ if φ satisfies the following condition

$$\lim_{\alpha\to 0}\left(\sup_{x\in\Omega}\int_{\Omega\cap B(x,\alpha)} \frac{\delta(y)}{\delta(x)}G(x,y)|\varphi(y)|\mathrm{d}y\right) = 0. \tag{1.7}$$

The first purpose of this paper is to study the properties of functions belonging to $K(\Omega)$, which we will doing in Section 2. In particular, we show for $1 \leqslant \lambda < 2$ that the function $x \to q(x) = \frac{1}{(\delta(x))^{\lambda}}$ is in $K(\Omega)$ but not in $K_n(\Omega)$.

In Section 3, we suppose that Ω contains 0 and we prove the existence of infinitely many singular positive solutions for the following nonlinear elliptic problem

$$(P)\begin{cases}\Delta u + f(\cdot, u) = 0, & \text{in } \Omega\backslash\{0\}\text{(in the sense of distributions)}\\ u|_{\partial\Omega} = 0, & \\ u(x) \sim \frac{c}{|x|^{n-2}}, \text{ near } x = 0, & \text{for any sufficiently small } c > 0.\end{cases}$$

Under some conditions on the function f which will be specified later, these solutions are continuous except at $x = 0$.

The existence of infinitely many singular positive solutions for the problem (P) has been established, by Zhang and Zhao in [12], for the special nonlinearity

$$f(x, t) = p(x)t^{\mu}, \quad \mu > 1,$$

where the function p satisfies

$$(H_0)\ x \to \frac{p(x)}{|x|^{(n-2)(\mu-1)}} \in K_n(\Omega).$$

Here we generalize the result of Zhang and Zhao [12] to the class $K(\Omega)$.

We note that the problem (P) is obviously equivalent to the following inhomogenious problem

$$\begin{cases}\Delta u + f(., u) = -\delta_0, & \text{in } \Omega\\ u|_{\partial\Omega} = 0, & \end{cases}$$

where δ_0 is the δ-function at $\{0\}$.

This latter problem has been solved in [6] with arbitrary measure data ω in place of δ_0 and the nonlinearity $f(x, t) = p(x)t^{\mu}, \mu > 1$. In fact, in [6] the authors obtained sharper results by a different method. In particular they gave a necessary and sufficient condition for the existence of positive solutions for the Dirichlet problem (P). In this paper, we require the following hypotheses:

(H_1) f is a Borel measurable function in $\Omega \times (0, \infty)$, continuous with respect to the second variable.

(H_2) $|f(x, t)| \leqslant tq(x, t)$, where q is a nonnegative Borel measurable function in $\Omega \times (0, \infty)$, nondecreasing with respect to the second variable such that $\lim_{t\to 0} q(x, t) = 0$.

(H_3) The function θ defined on Ω by $\theta(x) = q(x, G(x, 0))$ belongs to the class $K(\Omega)$.

We point out that in the case where $f(x,t) = p(x)t^{\mu}$, the assumption (H_0) implies (H_3).

As usual, let $B(\Omega)$ be the set of Borel measurable functions in Ω and let $B^{+}(\Omega)$ be the set of the nonnegative ones. $C_0(\Omega)$ will denote the set of continuous functions in $\bar{\Omega}$ vanishing at $\partial\Omega$. The letter C will denote a generic positive constant which may vary from line to line. When two positive functions f and g are defined on a set S, we write $f \sim g$ if the two-sided inequality $\frac{1}{C} g \leqslant f \leqslant Cg$ holds on S.

2. The Kato Class $K(\Omega)$

We start this section by proving some inequalities for the Green function G, that we will use later.

PROPOSITION 1. *For each $x, y \in \Omega$, we have*

$$G(x, y) \sim \frac{\delta(x)\delta(y)}{|x-y|^{n-2}(|x-y|^2 + \delta(x)\delta(y))} \tag{2.1}$$

and

$$\delta(x)\delta(y) \leqslant CG(x, y). \tag{2.2}$$

Moreover, if $|x-y| \geqslant r$ then

$$G(x, y) \leqslant C\frac{\delta(x)\delta(y)}{r^n}. \tag{2.3}$$

Proof. Since for each $a, b > 0$, we have $\frac{ab}{a+b} \leqslant \min(a, b) \leqslant 2\frac{ab}{a+b}$, then from (1.2), we deduce (2.1). Inequalities (2.2) and (2.3) follow immediately from (2.1). □

In the sequel, we give some properties of functions belonging to the Kato class $K(\Omega)$.

LEMMA 1. *Let φ be a function in $K(\Omega)$. Then the function*

$$x \to \delta^2(x)\varphi(x)$$

is in $L^1(\Omega)$.

Proof. Let $\varphi \in K(\Omega)$, then by (1.7) there exists $\alpha > 0$ such that for each x in Ω

$$\int_{B(x,\alpha)\cap\Omega} \frac{\delta(y)}{\delta(x)} G(x, y)|\varphi(y)|\mathrm{d}y \leqslant 1.$$

Let $x_1, \ldots, x_m$ in Ω such that $\Omega \subset \cup_{1\leqslant i\leqslant m} B(x_i, \alpha)$. Then by (2.2), there exists $C > 0$ such that for all $i \in \{1, \ldots, m\}$ and $y \in B(x_i, \alpha) \cap \Omega$, we have

$$(\delta(y))^2 \leqslant C \frac{\delta(y)}{\delta(x_i)} G(x_i, y)$$

Hence, we have

$$\begin{aligned}\int_\Omega (\delta(y))^2|\varphi(y)|\mathrm{d}y &\leqslant C \sum_{1\leqslant i\leqslant m} \int_{B(x_i,\alpha)\cap\Omega} \frac{\delta(y)}{\delta(x_i)} G(x_i, y)|\varphi(y)\mathrm{d}y \\ &\leqslant Cm < \infty.\end{aligned}$$

This completes the proof. □

We use the notation

$$\|\varphi\|_\Omega := \sup_{x\in\Omega} \int_\Omega \frac{\delta(y)}{\delta(x)} G(x, y)|\varphi(y)|\mathrm{d}y.$$

PROPOSITION 2. *Let φ be a function in $K(\Omega)$, then $\|\varphi\|_\Omega < \infty$.*

Proof. Let $\varphi \in K(\Omega)$ and $\alpha > 0$. Then we have

$$\begin{aligned}\int_\Omega \frac{\delta(y)}{\delta(x)} G(x, y)|\varphi(y)|\mathrm{d}y \leqslant& \int_{\Omega\cap|x-y|\leqslant\alpha} \frac{\delta(y)}{\delta(x)} G(x, y)|\varphi(y)|\mathrm{d}y \\ &+ \int_{\Omega\cap|x-y|\geqslant\alpha} \frac{\delta(y)}{\delta(x)} G(x, y)|\varphi(y)|\mathrm{d}y.\end{aligned}$$

Now, since by (2.3), we have

$$\int_{\Omega\cap|x-y|\geqslant\alpha} \frac{\delta(y)}{\delta(x)} G(x, y)|\varphi(y)|\mathrm{d}y \leqslant \frac{C}{\alpha^n} \int_\Omega (\delta(y))^2|\varphi(y)|\mathrm{d}y,$$

then the result follows from (1.7) and Lemma 1. □

PROPOSITION 3. *Let* $\varphi \in K(\Omega), x_0 \in \overline{\Omega}$ *and* h *be a nonnegative superharmonic function in* Ω. *Then for all* x *in* Ω, *we have*

$$\int_\Omega G(x,y)h(y)|\varphi(y)|\mathrm{d}y \leqslant 2C_0\|\varphi\|_\Omega h(x), \tag{2.4}$$

where C_0 *is the constant given in* (1.6).

Moreover, we have

$$\lim_{\alpha\to 0}\left(\sup_{x\in\Omega}\frac{1}{h(x)}\int_{\Omega\cap B(x_0,\alpha)} G(x,y)h(y)|\varphi(y)|\mathrm{d}y\right) = 0. \tag{2.5}$$

Proof. Let h be a nonnegative superharmonic function in Ω. Then by ([11], Theorem 2.1, p. 164), there exists a sequence $(f_n)_n \subset B^+(\Omega)$ such that

$$h(y) = \sup_n \int_\Omega G(y,z)f_n(z)\mathrm{d}z.$$

Hence, it is enough to prove (2.4) and (2.5) for $h(y) = G(y,z)$ uniformly in $z \in \Omega$.

Let $\varphi \in K(\Omega)$. Then by (1.6), we have for all $x, z \in \Omega$

$$\begin{aligned}
&\int_\Omega G(x,y)G(y,z)|\varphi(y)|\mathrm{d}y \\
&\quad\leqslant C_0 G(x,z)\int_\Omega \left[\frac{\delta(y)}{\delta(x)}G(x,y) + \frac{\delta(y)}{\delta(z)}G(y,z)\right]|\varphi(y)|\mathrm{d}y \\
&\quad\leqslant 2C_0\|\varphi\|_\Omega G(x,z).
\end{aligned}$$

Then (2.4) holds. Now, we shall prove (2.5). Let $\varepsilon > 0$, then by (1.7), there exists $r > 0$ such that

$$\sup_{\zeta\in\Omega}\int_{\Omega\cap B(\zeta,r)} \frac{\delta(y)}{\delta(\zeta)}G(\zeta,y)|\varphi(y)|\mathrm{d}y \leqslant \varepsilon. \tag{2.6}$$

Let $\alpha > 0$. Then using (1.6), we have

$$\begin{aligned}
&\frac{1}{G(x,z)}\int_{\Omega\cap B(x_0,\alpha)} G(x,y)G(y,z)|\varphi(y)|\mathrm{d}y \\
&\quad\leqslant C_0\int_{\Omega\cap B(x_0,\alpha)} \left[\frac{\delta(y)}{\delta(x)}G(x,y) + \frac{\delta(y)}{\delta(z)}G(y,z)\right]|\varphi(y)|\mathrm{d}y \\
&\quad\leqslant 2C_0 \sup_{\zeta\in\Omega}\int_{\Omega\cap B(x_0,\alpha)} \frac{\delta(y)}{\delta(\zeta)}G(\zeta,y)|\varphi(y)|\mathrm{d}y.
\end{aligned}$$

On the other hand, it follows from (2.3) that

$$\begin{aligned}
&\int_{\Omega\cap B(x_0,\alpha)} \frac{\delta(y)}{\delta(x)} G(x,y)|\varphi(y)|\mathrm{d}y \\
&\quad \leqslant \int_{\Omega\cap(|x-y|\leqslant r)} \frac{\delta(y)}{\delta(x)} G(x,y)|\varphi(y)|\mathrm{d}y \\
&\qquad + \int_{\Omega\cap B(x_0,\alpha)\cap(|x-y|\geqslant r)} \frac{\delta(y)}{\delta(x)} G(x,y)|\varphi(y)|\mathrm{d}y \\
&\quad \leqslant \sup_{\zeta\in\Omega} \int_{\Omega\cap B(\zeta,r)} \frac{\delta(y)}{\delta(\zeta)} G(\zeta,y)|\varphi(y)|\mathrm{d}y + \frac{C}{r^n}\int_{\Omega\cap B(x_0,\alpha)} (\delta(y))^2|\varphi(y)|\mathrm{d}y
\end{aligned}$$

Which together with Lemma 1 and (2.6), end the proof by letting $\alpha \to 0$. □

COROLLARY 1. *Let φ be a function in $K(\Omega)$. Then we have*

(a) $$\sup_{x\in\Omega} \int_\Omega G(x,y)|\varphi(y)|\mathrm{d}y < \infty \tag{2.7}$$

(b) *The function $x \to \delta(x)\varphi(x)$ is in $L^1(\Omega)$.*

Proof. (a) Put $h \equiv 1$ in (2.4) and using Proposition 2, we get (2.7).
(b) Let $x_0 \in \Omega$, then by (2.2), it follows that

$$\delta(x_0)\int_\Omega \delta(y)|\varphi(y)|\mathrm{d}y \leqslant C\int_\Omega G(x_0,y)|\varphi(y)|\mathrm{d}y.$$

Hence the result follows from (a). □

Remark 1. As consequence of Corollary 1(b), if the function q defined in Ω by

$$q(x) = \frac{1}{(\delta(x))^\lambda}$$

belongs to $K(\Omega)$, then the function $x \to (\delta(x))^{1-\lambda} \in L^1(\Omega)$. Hence, it follows by [8, Lemma p. 726], that a necessary condition in order that $q \in K(\Omega)$ is $\lambda < 2$.

In fact, this condition is sufficient as it will be proved in the following.

PROPOSITION 4. *Let q be the function defined in Ω by*

$$q(x) = \frac{1}{(\delta(x))^{\lambda}}.$$

Then q belongs to $K(\Omega)$ if and only if $\lambda < 2$.

Proof. If $\lambda \leqslant 0$, then $q \in L^{\infty}(\Omega)$ and so by (1.1), $q \in K(\Omega)$.
Let $0 < \lambda < 2$ and $\alpha > 0$. We first remark by (2.1) that for each $x, y \in \Omega$, we have

$$\frac{1}{(\delta(y))^{\lambda}} \frac{\delta(y)}{\delta(x)} G(x, y) \sim \frac{(\delta(y))^{2-\lambda}}{|x-y|^{n-2}(|x-y|^2 + 4\delta(x)\delta(y))}.$$

and

$$\begin{aligned} |x-y|^2 + 4\delta(x)\delta(y) &\geqslant \max(|\delta(x) - \delta(y)|^2 + 4\delta(x)\delta(y), |x-y|^2) \\ &\geqslant \max((\delta(y))^2, |x-y|^2). \end{aligned}$$

Hence, there exists $C > 0$ such that

$$\begin{aligned} \frac{1}{(\delta(y))^{\lambda}} \frac{\delta(y)}{\delta(x)} G(x, y) &\leqslant C \frac{(\delta(y))^{2-\lambda}}{|x-y|^{n-2}|x-y|^{\lambda}(\delta(y))^{2-\lambda}} \\ &\leqslant \frac{C}{|x-y|^{n-2+\lambda}}. \end{aligned}$$

Then we have

$$\begin{aligned} I &= \int_{\Omega \cap B(x,\alpha)} \frac{\delta(y)}{\delta(x)} G(x, y) \frac{\mathrm{d}y}{(\delta(y))^{\lambda}} \\ &\leqslant C \int_{\Omega \cap B(x,\alpha)} \frac{\mathrm{d}y}{|x-y|^{n-2+\lambda}} \\ &\leqslant C \int_0^{\alpha} r^{1-\lambda} \mathrm{d}r \leqslant C\alpha^{2-\lambda}. \end{aligned}$$

Thus $I \leqslant C\alpha^{2-\lambda} \to 0$ as $\alpha \to 0$.
The converse has been shown in Remark I. □

Remark 2. We suppose that Ω contains 0. Let g be the function defined in Ω by

$$g(x) = \frac{1}{(\delta(x))^{\lambda}|x|^{\mu}}$$

Then g belongs to $K(\Omega)$ if and only if $\lambda < 2$ and $\mu < 2$.

Indeed, if $g \in K(\Omega)$, then by Corollary 1, we have

$$\int_{\Omega} G(0, y)g(y)\mathrm{d}y < \infty.$$

This implies by (2.1) that $\lambda < 2$ and $\mu < 2$. To prove sufficiency, let $\lambda < 2, \mu < 2$ and $\alpha > 0, r > 0$ such that $B(0, 2r) \subset \Omega$, then

$$\begin{aligned} I &= \int_{\Omega\cap B(x,\alpha)} \frac{\delta(y)}{\delta(x)} G(x, y) \frac{\mathrm{d}y}{(\delta(y))^{\lambda}|y|^{\mu}} \\ &\leqslant \int_{B(x,\alpha)\cap B(0,r)} \frac{\delta(y)}{\delta(x)} G(x, y) \frac{\mathrm{d}y}{(\delta(y))^{\lambda}|y|^{\mu}} \\ &\quad + \int_{\Omega\cap B(x,\alpha)\cap B^c(0,r)} \frac{\delta(y)}{\delta(x)} G(x, y) \frac{\mathrm{d}y}{(\delta(y))^{\lambda}|y|^{\mu}} \\ &= I_1 + I_2. \end{aligned}$$

Since $B(0, 2r) \subset \Omega$, then $\delta(y) \geqslant r$, for $y \in B(0, r)$ and using (1.1), we deduce that

$$I_1 \leqslant C \int_{B(x,\alpha)\cap B(0,r)} \frac{\mathrm{d}y}{|x-y|^{n-2}|y|^{\mu}}.$$

Hence, if we choose $\frac{n}{n-\mu} < p < \frac{n}{n-2}$, then we have by the Hölder inequality that

$$\begin{aligned} I_1 &\leqslant C \left(\int_{(|x-y|\leqslant\alpha)} \frac{\mathrm{d}y}{|x-y|^{(n-2)p}} \right)^{\frac{1}{p}} \left(\int_{B(0,r)} \frac{\mathrm{d}y}{|y|^{\frac{\mu p}{p-1}}} \right)^{\frac{p-1}{p}} \\ &\leqslant C\alpha^{\frac{n}{p}-(n-2)} r^{n\frac{p-1}{p}-\mu} \to 0 \text{ as } \quad \alpha \to 0. \end{aligned}$$

Furthermore, we have

$$I_2 \leqslant C \int_{\Omega\cap B(x,\alpha)} \frac{\delta(y)}{\delta(x)} G(x, y) \frac{\mathrm{d}y}{(\delta(y))^{\lambda}}.$$

Thus, by Proposition 4, we have $I_2 \to 0$ as $\alpha \to 0$.

Remark 3. Let $\lambda < 2$ and q be the function defined in Ω by

$$q(x) = \frac{1}{(\delta(x))^{\lambda}}.$$

Put $v(x) = \int_\Omega G(x, y)q(y)\mathrm{d}y, x \in \Omega$. Since $q \in K(\Omega)$, then by (2.2) and Corollary 1(b), we deduce that there exists $C > 0$ such that

$$\frac{1}{C}\delta(x) \leqslant v(x).$$

In fact, Mâagli [9] gives more precise estimates on the potential v of q. We recall them in the next Proposition.

PROPOSITION 5. *Let* $d = \mathrm{diam}(\Omega)$. *Then there exists a constant* $C > 0$ *such that for each* x *in* Ω *we have*

(i) $\frac{1}{C}\delta(x) \leqslant v(x) \leqslant C(\delta(x))^{2-\lambda},\ if\ 1 < \lambda < 2.$

(ii) $\frac{1}{C}\delta(x) \leqslant v(x) \leqslant C\delta(x)\log\frac{(\sqrt{5}+1)d}{2\delta(x)},\ if\ \lambda = 1,$

(iii) $\frac{1}{C}\delta(x) \leqslant v(x) \leqslant C\delta(x),\ if\ \lambda < 1.$

Remark 4. Let $1 \leqslant \lambda < 2$ and q be the function defined in Ω by

$$q(x) = \frac{1}{(\delta(x))^\lambda}.$$

Then by Proposition 4, we have $q \in K(\Omega)$.

On the other hand, $q \notin K_n(\Omega)$. Indeed, by [3] (Proposition 3.1), $K_n(\Omega) \subset L^1(\Omega)$, but using [8] (Lemma p. 726), we have for $\lambda \geqslant 1$,

$$\int_\Omega \frac{1}{\delta(x)^\lambda}\,\mathrm{d}x = \infty.$$

PROPOSITION 6. *The class* $K(\Omega)$ *properly contains* $K_n(\Omega)$.

Proof. The assertion follows from (1.1) and Remark 4. □

Remark 5. We recall (see [1]) that a radial function φ in $B(0, 1)$ is in $K_n(\Omega)$ if and only if $\int_0^1 r|\varphi(r)|\mathrm{d}r < \infty$.

Also, if $\Omega := \{x \in \mathbb{R}^n : 0 < a < |x| < b < \infty\}$, then a radial function φ in Ω is in $K_n(\Omega)$ if and only if $\int_a^b |\varphi(r)|\mathrm{d}r < \infty$.

In the two next propositions, similarly as in Remark 5, we give a characterization of the class $K(\Omega)$, in the case where Ω is invariant by rotation and φ is radial. More precisely, we prove that φ is in $K(\Omega)$ if and only if (2.7) is satisfied. For the proof, we need the next Lemma.

LEMMA 2. (i) *Let φ be a Borel radial function in $B(0,1)$. Then we have*

$$\sup_{x\in B(0,1)}\int_{B(0,1)} G_B(x,y)|\varphi(y)|\mathrm{d}y < \infty$$

if and only if

$$\int_0^1 r(1-r)|\varphi(r)|\mathrm{d}r < \infty.$$

(ii) *Let φ be a Borel radial function in $\Omega := \{x \in \mathbb{R}^n : 0 < a < |x| < b < \infty\}$. Then we have*

$$\sup_{x\in\Omega}\int_{\Omega} G_\Omega(x,y)|\varphi(y)|\mathrm{d}y < \infty$$

if and only if

$$\int_a^b (b-r)(r-a)|\varphi(r)|\mathrm{d}r < \infty.$$

Proof. We first remark the following elementary inequalities.

$$\min\left(1,\frac{\mu}{\lambda}\right)(1-t^\lambda) \leqslant 1-t^\mu \leqslant \max\left(1,\frac{\mu}{\lambda}\right)(1-t^\lambda), \tag{2.8}$$

for $t \in [0,1]$ and $\lambda, \mu \in (0,\infty)$.

(i) Since the function $x \to \int_B G_B(x,y)|\varphi(y)|\mathrm{d}y$ is radial, then by elementary calculus, we have

$$\int_{B(0,1)} G_B(x,y)|\varphi(y)|\mathrm{d}y = \frac{1}{n-2}\int_0^1 r^{n-1}\left(\frac{1}{(t\vee r)^{n-2}}-1\right)|\varphi(r)|\mathrm{d}r,$$

where $t = |x|$ and $t \vee r = \max(t,r)$.

Hence, by (2.8) we conclude that

$$\begin{aligned}\sup_{x\in B(0,1)}\int_{B(0,1)} G_B(x,y)|\varphi(y)|\mathrm{d}y &= \frac{1}{n-2}\int_0^1 r^{n-1}\left(\frac{1}{r^{n-2}}-1\right)|\varphi(r)|\mathrm{d}r\\ &\sim \int_0^1 r(1-r)|\varphi(r)|\mathrm{d}r.\end{aligned}$$

(ii) By elementary calculus, we have

$$\begin{aligned}&\int_{\Omega} G_{\Omega}(x, y)|\varphi(y)|\mathrm{d}y\\&\quad = C\int_a^b r^{n-1}((t\vee r)^{2-n} - b^{2-n})(a^{2-n} - (t\wedge r)^{2-n})|\varphi(r)|\mathrm{d}r,\end{aligned}$$

where $t = |x|$ and $t \wedge r = \min(t, r)$.
On the other hand, due to (2.8), we have

$$((t\vee r)^{2-n} - b^{2-n})(a^{2-n} - (t\wedge r)^{2-n}) \sim (b - t\vee r)(t\wedge r - a).$$

So sufficiency is clear.
To prove necessity, we take $t = \frac{a+b}{2}$, then

$$\begin{aligned}&\int_a^b (b-r)(r-a)|\varphi(r)|\mathrm{d}r\\&\quad = \int_a^{\frac{a+b}{2}} (b-r)(r-a)|\varphi(r)|\mathrm{d}r + \int_{\frac{a+b}{2}}^b (b-r)(r-a)|\varphi(r)|\mathrm{d}r\\&\quad \leqslant \int_a^{\frac{a+b}{2}} (b-a)(r-a)|\varphi(r)|\mathrm{d}r + \int_{\frac{a+b}{2}}^b (b-r)(b-a)|\varphi(r)|\mathrm{d}r\\&\quad \leqslant 2\int_a^b \left(b - \left(\frac{a+b}{2}\vee r\right)\right)\left(\left(r\wedge\frac{a+b}{2}\right) - a\right)|\varphi(r)|\mathrm{d}r < \infty.\end{aligned}$$

This completes the proof. □

PROPOSITION 7. *Let φ be a radial function in $B(0,1)$. Then the following assertions are equivalent.*

(i) $\varphi \in K(B(0,1))$.
(ii) $\int_0^1 r(1-r)|\varphi(r)|\mathrm{d}r < \infty$.

Proof. (i) $\Rightarrow$ (ii) follows from Corollary 1(a) and Lemma 2.
(ii) $\Rightarrow$ (i) Let $\alpha > 0$, then by (2.8), we have for $t = |x|$,

$$\begin{aligned}&\int_{B(0,1)\cap B(x,\alpha)} \frac{\delta(y)}{\delta(x)} G_B(x, y)|\varphi(y)|\mathrm{d}y\\&\quad \leqslant \frac{1}{n-2}\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r^{n-1}\frac{(1-r)(1-(t\vee r)^{n-2})}{(1-t)(t\vee r)^{n-2}}|\varphi(r)|\mathrm{d}r\\&\quad \leqslant C\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r(1-r)|\varphi(r)|\mathrm{d}r.\end{aligned}$$

Hence, to prove that φ is in $K(B(0,1))$, we need to show that

$$\lim_{\alpha\to 0}\left(\sup_{t\in[0,1]}\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r(1-r)|\varphi(r)|\mathrm{d}r\right)=0.$$

Let $\Phi(\zeta)=\int_0^\zeta r(1-r)|\varphi(r)|\mathrm{d}r$, for $\zeta\in[0,1]$. By hypothesis, Φ is a continuous function on $[0,1]$. Which implies that

$$\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r(1-r)|\varphi(r)|\mathrm{d}r=\Phi((t+\alpha)\wedge 1)-\Phi((t-\alpha)\vee 0)$$

converges to zero as $\alpha\to 0$ uniformly for $t\in[0,1]$. This completes the proof. □

PROPOSITION 8. *Let φ be a radial function in $\Omega:=\{x\in\mathbb{R}^n : 0<a<|x|<b<\infty\}$. Then the following assertions are equivalent.*

(i) $\varphi\in K(\Omega)$.

(ii) $\int_a^b (b-r)(r-a)|\varphi(r)|\mathrm{d}r<\infty$.

Proof. (i) ⇒ (ii) follows from Corollary 1(a) and Lemma 2.

To prove (ii) ⇒ (i), we first remark that for each $x\in\Omega$,

$$\delta(x)=\min(b-|x|,|x|-a)\sim(b-|x|)(|x|-a).$$

Now, let $\alpha>0$, then by (2.8), we have for $t=|x|$,

$$\begin{aligned}\sup_{x\in\Omega}&\int_{\Omega\cap B(x,\alpha)}\frac{\delta(y)}{\delta(x)}G_\Omega(x,y)|\varphi(y)|\mathrm{d}y\\ &\leqslant C\sup_{t\in[a,b]}\int_{(t-\alpha)\vee a}^{(t+\alpha)\wedge b} r^{n-1}\frac{(b-r)(r-a)}{(b-t)(t-a)}\\ &\quad\times((t\vee r)^{2-n}-b^{2-n})(a^{2-n}-(t\wedge r)^{2-n})|\varphi(r)|\mathrm{d}r\\ &\leqslant C\sup_{t\in[a,b]}\int_{(t-\alpha)\vee a}^{(t+\alpha)\wedge b}(b-r)(r-a)|\varphi(r)|\mathrm{d}r,\end{aligned}$$

which converges to 0 as $\alpha\to 0$. □

PROPOSITION 9. (i) *Let $p>\frac{n}{2}$. Then we have*

$$L^P(\Omega)\subset K_n(\Omega)\subset K(\Omega)\cap L^1(\Omega)\subset K(\Omega)\subset L^1(\Omega,\delta(x)\mathrm{d}x)\subset L^1_{\mathrm{loc}}(\Omega)$$

(ii) *Let $\Omega=B(0,1)$ and φ be a radial function in $K(B(0,1))\cap L^1(B(0,1))$. Then $\varphi\in K_n(B(0,1))$.*

Proof. (i) It has been shown in [1] and [3], that the classical Kato class $K_n(\Omega)$ contains $L^P(\Omega)$ for any $p > \frac{n}{2}$ and that it is contained in $L^1(\Omega)$. The rest of inclusions are clear from Proposition 6 and Corollary 1(b).

(ii) Since $\varphi \in K(B(0,1))$, then by Proposition 7, φ satisfies

$$\int_0^1 r(1-r)|\varphi(r)|\mathrm{d}r < \infty.$$

Also since $\varphi \in L^1(B(0,1))$, then

$$\int_0^1 r^{n-1}|\varphi(r)|\mathrm{d}r < \infty.$$

Now we use the fact that $1-r \sim 1-r^{n-2}$, to conclude that

$$\int_0^1 r|\varphi(r)|\mathrm{d}r < \infty.$$

Which implies by Remark 5, that $\varphi \in K_n(B(0,1))$. □

THEOREM 1. *Let φ be a function in $K(\Omega)$. Then the function $V\varphi$ defined in Ω by*

$$V\varphi(x) = \int_\Omega G(x,y)\varphi(y)\mathrm{d}y,$$

is in $C_0(\Omega)$.

Proof. Let $\varphi \in K(\Omega)$, $x_0 \in \Omega$ and $x, x' \in B(x_0,\alpha)\cap\Omega$, where $\alpha > 0$. Then we have

$$\begin{aligned}|V\varphi(x) - V\varphi(x')| &\leqslant \int_\Omega |G(x,y) - G(x',y)||\varphi(y)|\mathrm{d}y \\ &\leqslant 2\sup_{\zeta\in\Omega}\int_{\Omega\cap B(x_0,2\alpha)} G(\zeta,y)|\varphi(y)|\mathrm{d}y \\ &\quad + \int_{\Omega\cap(|x_0-y|\geqslant 2\alpha)} |G(x,y) - G(x',y)||\varphi(y)|\mathrm{d}y.\end{aligned}$$

If $|x_0 - y| \geqslant 2\alpha$, $|x - x_0| \leqslant \alpha$ and $|x' - x_0| \leqslant \alpha$, then $|x - y| \geqslant \alpha$ and $|x' - y| \geqslant \alpha$.
Hence it follows from (2.3) that

$$|G(x,y) - G(x',y)| \leqslant \frac{C}{\alpha^n}\delta(y).$$

Using the continuity of G outside the diagonal, we deduce by the dominated convergence theorem and Corollary 1(b) that

$$\int_{\Omega\cap(|x_0-y|\geqslant 2\alpha)} |G(x,y)-G(x',y)||\varphi(y)|\mathrm{d}y \to 0 \quad \text{as } |x-x'|\to 0$$

So we deduce by (2.5), with $h\equiv 1$, that

$$|V\varphi(x)-V\varphi(x')|\to 0 \quad \text{as} \quad |x-x'|\to 0.$$

Now, since for all $y\in\Omega$, $\lim_{x\to\partial\Omega} G(x,y)=0$, then by the same argument as above, we get

$$\lim_{x\to\partial\Omega} V\varphi(x)=0$$

Thus $V\varphi\in C_0(\Omega)$. □

PROPOSITION 10. *Let φ be a function in $K(\Omega)$. Then the function*

$$x\to\int_\Omega \frac{\delta(y)}{\delta(x)}G(x,y)\varphi(y)\mathrm{d}y$$

is continuous in $\bar\Omega$.

Proof. First, we remark that for $y\in\Omega$, the function $x\to\frac{G(x,y)}{\delta(x)}$ is continuous in $\bar\Omega$. So the result holds by an argument similar to that used in the proof of (2.5). □

3. Positive Singular Solutions of the Equation $\Delta u+f(\cdot,u)=0$

In this section we suppose that Ω contains 0. So, we are interested in the existence of positive singular solutions for the problem (P). We present in the next Theorem the main result of this section.

THEOREM 2. *Assume (H_1)–(H_3). Then the problem (P) has infinitely many solutions. More precisely, there exists $b_0>0$ such that for each $b\in(0,b_0]$, there exists a solution u of (P) continuous on $\Omega\backslash\{0\}$ and satisfying*

$$u(x)\sim\frac{\delta(x)}{|x|^{n-2}},\quad for\ all\ x\in\Omega\backslash\{0\}$$

and

$$\lim_{|x|\to 0} u(x)|x|^{n-2} = bc_n,$$

where $c_n = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}$.

For the proof, we put $F := \{w \in C^+(\overline{\Omega}) : ||w||_\infty \leqslant 1\}$, where $||\cdot||_\infty$ is the uniform norm. So we have the following result:

LEMMA 3. *Assume* (H_1)–(H_3). *We define the operator* T *on* F *by*

$$T\omega(x) = \frac{1}{G(x,0)} \int_\Omega G(x,y) f(y, \omega(y)G(y,0))\mathrm{d}y, \;\; x \in \Omega.$$

Then the family of functions $T(F)$ *is relatively compact in* $C(\overline{\Omega})$.

Proof. By (H_2), we have for all $\omega \in F$

$$|T\omega(x)| \leqslant \frac{1}{G(x,0)} \int_\Omega G(x,y)G(y,0)\theta(y)\mathrm{d}y.$$

Since $\theta(x) = q(x, G(x,0))$ belongs to the class $K(\Omega)$, then by (2.4), we deduce that

$$\|T\omega\|_\infty \leqslant 2C_0\|\theta\|_\Omega.$$

Hence, the family $T(F)$ is uniformly bounded. Now, we propose to prove the equicontinuity of $T(F)$ in $\overline{\Omega}$. Let $x_0 \in \overline{\Omega}$ and $\alpha > 0$. Let $x, x' \in B(x_0,\alpha) \cap \Omega$ and $\omega \in F$, then

$$\begin{aligned}
&|T\omega(x) - T\omega(x')| \\
&\quad \leqslant \int_\Omega \left| \frac{G(x,y)}{G(x,0)} - \frac{G(x',y)}{G(x',0)} \right| G(y,0)\theta(y)\mathrm{d}y \\
&\quad \leqslant 2 \sup_{x\in\Omega} \frac{1}{G(x,0)} \int_{\Omega\cap B(0,2\alpha)} G(x,y)G(y,0)\theta(y)\mathrm{d}y \\
&\qquad + 2 \sup_{x\in\Omega} \frac{1}{G(x,0)} \int_{\Omega\cap B(x_0,2\alpha)} G(x,y)G(y,0)\theta(y)\mathrm{d}y \\
&\qquad + \int_{\Omega\cap B^c(0,2\alpha)\cap B^c(x_0,2\alpha)} \left| \frac{G(x,y)}{G(x,0)} - \frac{G(x',y)}{G(x',0)} \right| G(y,0)\theta(y)\mathrm{d}y.
\end{aligned}$$

If $|x_0 - y| \geqslant 2\alpha$ and $|x - x_0| \leqslant \alpha$, then $|x - y| \geqslant \alpha$. Hence, it follows from (1.6) and (2.3), that for all $x \in B(x_0, \alpha) \cap \Omega$ and $y \in \Omega_0 := B^c(0, 2\alpha) \cap B^c(x_0, 2x) \cap \Omega$, we have

$$\frac{G(x,y)}{G(x,0)}G(y,0) \leqslant C\delta^2(y).$$

Moreover, when $y \in \Omega_0$, the function $x \to \frac{G(x,y)}{G(x,0)}$ is continuous in $B(x_0, \alpha) \cap \Omega$. Then, we deduce by Lemma 1 and the dominated convergence theorem that

$$\int_{\Omega \cap B^c(0,2\alpha) \cap B^c(x_0,2\alpha)} \left| \frac{G(x,y)}{G(x,0)} - \frac{G(x',y)}{G(x',0)} \right| G(y,0)\theta(y)\mathrm{d}y \to 0,$$

as $|x - x'| \to 0$.

By (2.5), we deduce that

$$|T\omega(x) - T\omega(x')| \to 0, \quad \text{as } |x - x'| \to 0.$$

uniformly for all $w \in F$. □

The result follows by Ascoli's Theorem.

Proof of Theorem 2. We aim to show that there exists $b_0 > 0$ such that for each $b \in (0, b_0]$, there exists a continuous function u in $\Omega \backslash \{0\}$ satisfying the following integral equation

$$u(x) = bG(x,0) + \int_\Omega G(x,y)f(y,u(y))\mathrm{d}y, \quad x \in \Omega. \tag{3.1}$$

Let $\beta \in (0,1)$. Then by Lemma 3, the function

$$T_\beta(x) = \frac{1}{G(x,0)} \int_\Omega G(x,y)G(y,0)q(y, \beta G(y,0))\mathrm{d}y$$

is continuous in $\overline{\Omega}$. Moreover, by (H_2), (H_3) and (2.4), we deduce by the dominated convergence theorem that

$$\forall x \in \overline{\Omega}, \quad \lim_{\beta \to 0} T_\beta(x) = 0.$$

Since the function $\beta \to T_\beta(x)$ is nondecreasing in (0, 1), then, by Dini Lemma, we have

$$\lim_{\beta \to 0} \left(\sup_{x \in \Omega} \frac{1}{G(x,0)} \int_\Omega G(x,y)G(y,0)q(y, \beta G(y,0))\mathrm{d}y \right) = 0.$$

Thus, there exists $\beta \in (0,1)$ such that for each $x \in \overline{\Omega}$,

$$\frac{1}{G(x,0)} \int_{\Omega} G(x,y)G(y,0)q(y,\beta G(y,0))\mathrm{d}y \leqslant \frac{1}{3}.$$

Let $b_0 = \frac{2}{3}\beta$ and $b \in (0, b_0]$. We shall use a fixed point argument. Let

$$S = \left\{ w \in C(\overline{\Omega}) \colon \frac{b}{2} \leqslant w(x) \leqslant \frac{3b}{2} \right\}.$$

Then, S is a nonempty closed bounded and convex set in $C(\overline{\Omega})$. We define the operator Γ on S by

$$\Gamma w(x) = b + \frac{1}{G(x,0)} \int_{\Omega} G(x,y) f(y, w(y)G(y,0))\mathrm{d}y, \quad x \in \Omega.$$

By Lemma 3, $\Gamma S \subset C(\overline{\Omega})$. Moreover, let $w \in S$, then for any $x \in \Omega$, we have

$$|\Gamma w(x) - b| \leqslant \frac{3b}{2} \frac{1}{G(x,0)} \int_{\Omega} G(x,y)G(y,0)q(y,\beta G(y,0))\mathrm{d}y \leqslant \frac{b}{2}.$$

It follows that $\frac{b}{2} \leqslant \Gamma w(x) \leqslant \frac{3b}{2}$ and so $\Gamma S \subset S$.

Next, we shall prove the continuity of Γ in the supermum norm. Let $(w_k)_k$ be a sequence in S which converges uniformly to $w \in S$, then since f is continuous with respect to the second variable, we deduce by the dominated convergence theorem that

$$\forall x \in \Omega, \quad \Gamma w_k(x) - \Gamma w(x) \to 0 \quad \text{as } k \to \infty.$$

Now, since ΓS is a relatively compact family in $C(\overline{\Omega})$, then

$$\|\Gamma w_k - \Gamma w\|_\infty \to 0 \quad \text{as } k \to \infty.$$

So, the Schauder fixed point theorem implies the existence of $w \in S$ such that $\Gamma w = w$.

For all $x \in \Omega$, put $u(x) = w(x)G(x,0)$. Thus, u is a continuous function in $\Omega \backslash \{0\}$ satisfying (3.1). On the other hand, by (2.1) we have

$$G(x,0) \sim \frac{\delta(x)}{|x|^{n-2}}, \quad \text{for all } x \in \Omega \backslash \{0\}.$$

Then it is clear that u is a solution of (P) such that

$$u(x) \sim \frac{\delta(x)}{|x|^{n-2}} \quad \text{for all } x \in \Omega\backslash\{0\}$$

and

$$\lim_{|x|\to 0} \frac{u(x)}{G(x,0)} = b.$$

Furthermore, since $\lim_{|x|\to 0} |x|^{n-2} G(x,0) = c_n$, then we have

$$\lim_{|x|\to 0} u(x)|x|^{n-2} = bc_n.$$

This ends the proof. □

EXAMPLE 1. Let $\Omega = B(0,1)$, and $Q(r,t) = \max_{|x|=r} q(x,t), 0 \leqslant r \leqslant 1$. If

$$\int_0^1 r(1-r) Q\left(r, \frac{1-r}{r^{n-2}}\right) \mathrm{d}r < \infty,$$

then there exists $b_0 > 0$ such that for each $b \in (0, b_0]$, the problem

$$\begin{cases} \Delta u(x) + f(x, u(x)) = 0, & \text{in } \Omega\backslash\{0\} \text{ (in the sense of distributions)} \\ u|_{\partial\Omega} = 0. \end{cases}$$

has a positive solution which is continuous in $\Omega\backslash\{0\}$ and satisfies

$$u(x) \sim \frac{1-|x|}{|x|^{n-2}}, \quad \text{for all } x \in \Omega\backslash\{0\}$$

and

$$\lim_{|x|\to 0} u(x)|x|^{n-2} = bc_n.$$

EXAMPLE 2. Let $p > 1$, $\lambda < 2$ and $\mu < 2$. Let $V \in B(\Omega)$ such that

$$\forall x \in \Omega, \quad |V(x)| \leqslant C \frac{|x|^{(n-2)(p-1)-\mu}}{(\delta(x))^{p-1+\lambda}}.$$

Then there exists $b_0 > 0$ such that for each $b \in (0, b_0]$, the problem

$$\begin{cases} \Delta u(x) + V(x) u^p(x) = 0, & \text{in } \Omega\backslash\{0\} \text{ (in the sense of distributions)} \\ u|_{\partial\Omega} = 0. \end{cases}$$

has a positive solution which is continuous in $\Omega\backslash\{0\}$ and satisfies

$$u(x) \sim \frac{\delta(x)}{|x|^{n-2}}, \quad \text{for all } x \in \Omega\backslash\{0\}$$

and

$$\lim_{|x|\to 0} u(x)|x|^{n-2} = bc_n.$$

References

1. Aizenman, M. and Simon, B.: Brownian motion and Harnack inequality for Schrödinger operators, *Comm. Pure App. Math.* **35** (1982), 209–271.
2. Chen, Z.Q. and Song, R.: Estimates on Green functions and Poisson kernels for symmetric stable processes, *Math. Ann.* **312** (1998), 465-501.
3. Chung, K.L. and Zhao, Z.: *From Brownian Motion to Schrödinger's Equation*, Springer-Verlag 1995.
4. Dautry, R., Lions, J.L., et al.: Analyse mathématique et calcul numérique pour les sciences et les techniques, coll. CEA, Vol. 2, L'opérateur de Laplace, Masson, 1987.
5. Gilbarg, D. and Trudinger, N.S.: *Elliptic Partial Differential Equations of Second order*, Springer-Verlag, 1977.
6. Kalton, N.J. and Verbitsky, I.E.: Nonlinear equations and weighted norm inequalities, *Trans. Amer. Math. Soc.* **351**(9) (1999), 3441–3497.
7. Lair, A.V. and Shaker, A.W.: Classical and weak solutions of a singular semilinear elliptic problem, *J. Math. Anal. Appl.* **211** (1997), 371–385.
8. Lazer, A.C. and Mckenna, P.J.: On a singular nonlinear elliptic boundary-value problem, *Proc. Amer. Math. Soc.* **111** (1991), 721–730.
9. Mâagli, H.: Inequalities for the Riesz potentials, *Archives. Inequalities. Appl.* **1** (2003), 285–294.
10. Mâagli, H. and Zribi, M.: Existence and estimates of solutions for singular nonlinear elliptic problems, *J. Math. Anal. Appl.* **263**(2) (2001), 522–542.
11. Port, S.C. and Stone, C.J.: *Brownian Motion and Classical Potential Theory*, Academic Press, 1978.
12. Zhang, Qi. S. and Zhao, Z.: Singular solutions of semilinear elliptic and parabolic equations, *Math. Ann.* **310** (1998), 777–794.
13. Zhao, Z.: Green function for Schrödinger operator and conditional Feynman–Kac gauge, *J. Math. Anal. Appl.* **116** (1986), 309–334.