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On Order Convergence of Nets

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Abstract. In this paper we show that any order continuous operator between two Riesz spaces is automatically order bounded. We also investigate different types of order convergence.

1. Order Convergent Nets

First part of this paper deals with two modes of convergence of nets in Riesz paces. For terminology and elementary properties of Riesz spaces not explained in this paper the reader can consult either of the following sources [1,2,7,8], or [11].

A number of authors have attached various meanings to the statement "Net x_α order converges to the element x." In the literature on the Riesz space theory the order convergence of nets is defined in one of the following two ways.

DEFINITION 1.1. ([2,7,8,11]) A net $(x_\alpha)_{\alpha\in A}$ is **order convergent** to x, if there exists a net $(y_\alpha)_{\alpha\in A}$ such that:
1. $y_\alpha \downarrow 0$, and
2. $|x_\alpha - x| \leqslant y_\alpha$ for all $\alpha \in A$.

DEFINITION 1.2. ([1,11]) A net $(x_\alpha)_{\alpha\in A}$ is **order convergent** to x, if there exists a net $(y_\beta)_{\beta\in B}$ such that:
1. $y_\beta \downarrow 0$, and
2. for each $\beta \in B$ there exists some $\alpha_0 \in A$ satisfying $|x_\alpha - x| \leqslant y_\beta$ for all $\alpha \geqslant \alpha_0$.

In the book of Schaefer [11] the latter definition is referred to as order convergence of the section filter of the net $(x_\alpha)_{\alpha\in A}$.

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It should be noticed that the first definition does not satisfy our understanding of the word "convergence". A converging net must remain converging even if we attach additional terms at the "beginning" of the net. The following simple example demonstrates that this is not true for the convergence described by Definition 1.1.

EXAMPLE 1.3. For any positive element b in an Archimedean vector lattice E a net $(x_n)_{n\in\mathbf{N}}$ defined by $x_n = \frac{1}{n}b$ is order convergent to zero. On the other hand let us attach negative integers but place them between 1 and 2 of the original index set. Define the elements of the extended net by $x_n = |n|b$ for each negative integer n. Then x_n is not order convergent in the sense of Definition 1.1. It should be noticed that the extended net x_n is still converging to zero by Definition 1.2.

We fix Definition 1.1 by changing
2. $|x_\alpha - x| \leqslant y_\alpha$ for all $\alpha \in A$ to
2′. $|x_\alpha - x| \leqslant y_\alpha$ for all $\alpha \in A$ satisfying $\alpha \geqslant \alpha_0$ for some $\alpha_0 \in A$.
This corrected version of Definition 1.1 seems to originate from [4].

Similarly to Anderson and Mathews [3] we will call a net **1-converging** if it is order convergent in the sense of corrected Definition 1.1 and **2-converging** if it is convergent in the sense of Definition 1.2. If lattice E is Dedekind complete, then two definitions above become equivalent. However, if E is not Dedekind complete they define two different convergences.

EXAMPLE 1.4. (Fremlin [11], p. 141). Consider the one-point compactification K of an uncountable discrete space, and let $E = C(K)$. If $(x_n)_{n\in\mathbf{N}}$ denotes the characteristic functions of a sequence of distinct singletons in K, the sequence (x_n) is 2-convergent but not 1-convergent to zero.

Despite the last example, there is very intimate relation between these two convergence in any vector lattice. Namely, the following proposition holds.

PROPOSITION 1.5. *Let $(x_\alpha)_{\alpha\in A}$ be a net in a Riesz space E and $x \in E$. Denote a Dedekind completion of E by E^δ. Then the following are equivalent.*
1. *The net $(x_\alpha)_{\alpha\in A}$ 2-converges to x in E;*
2. *The net $(x_\alpha)_{\alpha\in A}$ 1-converges to x in E^δ.*

Proof. Since the lattice E^δ is Dedekind complete the implication (1) $\Rightarrow$ (2) is trivial. Let us prove the reverse implication. Let net $(x_\alpha)_{\alpha\in A}$ 1-converge to x in E^δ. Then there exists a net $(z_\alpha)_{\alpha\in A} \subset E^\delta$ and some index $\alpha_0 \in A$ such that $z_\alpha \downarrow 0$ and $|x_\alpha - x| \leqslant z_\alpha$ for all $\alpha \in A$ satisfying $\alpha \geqslant \alpha_0$.

Consider a set B defined by $B = \{\beta \in E : \exists\alpha \geqslant \alpha_0$ such that $\beta \geqslant z_\alpha\}$ and ordered by $\beta_1 \geqslant \beta_2$ in $B \Leftrightarrow \beta_1 \leqslant \beta_2$ in E. Since $z_\alpha \downarrow$ the set B is directed. So, we consider a net on B defined by $y_\beta = \beta$. Since E is order dense in E^δ it follows that $y_\beta \downarrow 0$ in E. Moreover, for each $\beta \in B$ there exists some $\alpha_1 \in A$ such that $z_{\alpha_1} \leqslant \beta = y_\beta$. Therefore, for all $\alpha \geqslant \alpha_1$ we have

$$|x_\alpha - x| \leqslant z_\alpha \leqslant z_{\alpha_1} \leqslant y_\beta.$$

Thus, the net $(x_\alpha)_{\alpha\in A}$ 2-converges to x in E. □

Since the theory of vector spaces is first of all theory of linear operators, it is important to compare two convergences from the point of view of operators. Namely, are the operators preserving 1-convergence different from operators preserving 2-convergence? Although these operators are different, there are many similarities. It should be noticed that all results about order-continuous operators from [2,7,8,11] which are proved for Definition 1.1 remain valid for the operators preserving 1-convergence. Keeping this in mind let us prove the following connection.

PROPOSITION 1.6. *Let $T : E \to F$ be an operator between two Riesz spaces. If T preserves 1-convergence, then T preserves 2-convergence.*

Proof. Let $T : E \to F$ be an operator that preserves 1-convergence. It follows from our Theorem 2.1 that T is order bounded. Consider T as an operator acting from E into a Dedekind completion F^δ. Then operator $|T|{:}E \to F^\delta$ exists and preserves 1-convergence (see for instance Theorem 4.3 in [2]). By an extension theorem which is due to A. I. Veksler and can be found, for instance, in [2] (Theorem 4.12) there is a unique extension $|T|^\delta$ of operator $|T|$ which acts between two Dedekind completions $|T|^\delta{:}E^\delta \to F^\delta$ and preserves 1-convergence.

If net x_α 2-converges to x in E, then, by Proposition 1.5, x_α 1-converges to x in E^δ. Therefore, $|T||x_\alpha - x|$ 1-converges to 0 in F^δ since $|T|$ preserves 1-convergence. So, since $|Tx_\alpha - Tx| \leqslant |T||x_\alpha - x|$ we obtain 1-convergence $Tx_\alpha \to Tx$ in F^δ and, therefore, 2-convergence $Tx_\alpha \to Tx$ in F again by Proposition 1.5. □

For positive operators the picture is even better.

THEOREM 1.7. *Let $T{:}E \to F$ be a positive operator between two Riesz spaces, then the following are equivalent.*

1. *Operator T preserves 1-convergence;*
2. *Operator T preserves 2-convergence.*

Proof. Consider a positive operator $T{:}E \to F$ between two Riesz spaces. The implication (1) ⇒ (2) follows from Proposition 1.6. For the reverse implication, assume that T preserves 2-convergence and consider a 1-convergent net $x_\alpha \to 0$. Then there is a net $y_\alpha \downarrow 0$ and α_0 such that $y_\alpha \geqslant |x_\alpha|$ for all $\alpha \geqslant \alpha_0$. Since T is positive the net Ty_α is directed down and consists of positive elements. Since y_α is 2-convergent to zero, Ty_α is also 2-convergent to zero and, therefore, $Ty_\alpha \downarrow 0$. It remains to notice that due to positiveness of T we have $Ty_\alpha \geqslant |Tx_\alpha|$ for all $\alpha \geqslant \alpha_0$. Hence, Tx_α 1-converges to zero. □

Let us use Example 1.4 to show that the implication (2) $\Rightarrow$ (1) of Theorem 1.7 does not work in general.

EXAMPLE 1.8. Let E be the vector lattice of sequences of real numbers that are constant except for finitely many terms and let $F = C(K)$ be from Example 1.4. Consider vectors $u_n \in E$ which have first $n-1$ zero terms and all the other terms equal one. Notice that vectors u_n form an algebraic basis of E. Let $v_n = \chi_{\{t_n\}} \in F$ be the characteristic functions from Example 1.4 above. Then operator $T{:}E \to F$ which maps $T(u_n) = v_n$ preserves 2-convergence but not 1-convergence.

Proof. The fact that T does not preserve 1-convergence follows from the fact that the net u_n 1-converges to zero while $v_n = Tu_n$ does not.

On the other hand, let x_α 2-converges to zero in E. Observe that if $y_\beta \downarrow 0$ in E, then $y_\beta \downarrow 0$ coordinatewise. It follows that y_β satisfying Definition 1.2 can be chosen to be a sequence y_n defined by

$$y_n = \left(\underbrace{\frac{1}{n}, \ldots, \frac{1}{n}}_{n \text{ terms}}, A, A, \ldots \right)$$

for some $A > 1$. Next we consider the index set $\mathcal{I} = \{(n, U) : n \in \mathbf{N}, U \subset K \backslash \{t_i\}_{i=1}^{\infty}, U \text{ is finite}\}$ with natural order $(n, U) \leqslant (m, V) \Leftrightarrow (n \leqslant m \text{ and } U \subset V)$ and define $z_{n,U}(t) \in F$ by

$$z_{n,U}(t) = \begin{cases} \frac{2}{n}, & \text{if } t \in U \cup \{t_i{:}1 \leqslant i \leqslant n\}. \\ 2A, & \text{otherwise.} \end{cases}$$

Then $z_{n,U} \downarrow 0$ and for every pair n, U by Definition 1.2 there is α_0 such that $|x_\alpha| \leqslant y_n$ for every $\alpha \geqslant \alpha_0$. Notice that the explicit formula for T is:

$$T(a_1, a_2, \ldots, a_k, a_k, a_k, \ldots) = a_1 v_1 + \sum_{i=2}^{k} v_i (a_i - a_{i-1})$$

where the sum is finite because the sequence of numbers (a_i) is eventually constant. Therefore, if x_α satisfies $|x_\alpha| \leqslant y_n$, then by the formula above, for some $k > n$ we have

$$|Tx_\alpha| \leqslant \sum_{i=1}^{n} \frac{2}{n} v_i + \sum_{i=n+1}^{k} 2Av_i \leqslant z_{n,U}.$$

This implies that Tx_α 2-converges to zero. □

2. Order Continuous Operators

In this part we show that every operator preserving order convergence is order bounded. The approach to the relation between two basic classes of operators between Riesz spaces, namely, order bounded and order continuous operators, differs from paper to paper. Some authors include order boundedness as a part of the definition of an order continuous operator. Others simply consider these two classes as distinct entities. So unexpected is the following theorem.

THEOREM 2.1. *Any order continuous operator between two Riesz spaces is automatically order bounded.*

Proof. Let $T{:}E \to F$ be an order continuous operator between two Riesz spaces. We should emphasize that it does not matter whether T preserves 1- or 2- convergence. Consider an arbitrary order interval $[0, b] \subset E$. Let $\mathcal{I} = \mathbf{N} \times [0, b]$ be an index set with the lexicographical order. Namely, $(n, x) > (m, y)$ if and only if either one of the following holds true.

1. $n > m$.
2. $n = m$ and $x > y$.

It is easy to check that $\mathcal{I}$ is a directed set, so we may consider a net indexed by $\mathcal{I}$. Let us set $x_{(n,y)} = \frac{1}{n} y$. Then we have $0 \leqslant x_{(n,y)} \leqslant r_{(n,y)} = \frac{b}{n}$. It follows that $x_{(n,y)}$ 1-converges (hence, 2-converges) to zero.

If T preserves 2-convergence, then there exists a net $(z_\alpha)_{\alpha \in A}$ such that $z_\alpha \downarrow 0$ and for every $\alpha \in A$ there exists (n, y) satisfying

$$|Tx_{(m,u)}| \leqslant z_\alpha \quad \text{for all } (m, u) \geqslant (n, y).$$

If T preserves 1-convergence, then there exists a net $(z_{\alpha=(n,y)})$ such that $z_\alpha \downarrow 0$ and for every α there exists $(n, y) = \alpha$ satisfying

$$|Tx_{(m,u)}| \leqslant z_{(m,u)} \leqslant z_\alpha \quad \text{for all } (m, u) \geqslant (n, y) = \alpha.$$

Let us pick any z_α and find corresponding index (n, y). Then, in particular, $|Tx_{(n+1,u)}| \leqslant z_\alpha$ for all $u \in [0, b]$. It follows that $|Tu| \leqslant (n+1)z_\alpha$ for every $u \in [0, b]$. Thus, operator T is order bounded. □

REMARK 2.2. Theorem 2.1 is not valid for operators preserving order convergence of sequences. The “Fourier coefficients” operator is σ−order continuous but not order bounded operator. This example is due to G. Ya. Lozanovskii and can be found in [6] or in [2] on page 281.

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