Physical Oceanography, Vol. 20, No. 5, January, 2011 (Ukrainian Original No. 5, September–October, 2010)

THERMOHYDRODYNAMICS OF THE OCEAN

HIERARCHY OF THE MODELS OF CLASSICAL MECHANICS OF INHOMOGENEOUS FLUIDS

Yu. D. Chashechkin

UDC 555.466.81

The methods of perturbation theory and integral representations are used to analyze the general properties of a system of equations of the mechanics of inhomogeneous fluids including the equations of momentum, mass, and temperature transfer. We also consider various submodels of this system, including the reduced systems in which some kinetic coefficients are equal to zero and degenerate systems in which the variations of density or some other variables are neglected. We analyze both regularly perturbed and singularly perturbed solutions of the system. In the case of reduction or degeneration of solutions, the order of the system decreases. In this case, regularly perturbed solutions are preserved (with certain modifications) but the number of singularly perturbed components participating in the formation of the boundary layers on contact surfaces and their analogs in the bulk of the fluid, i.e., the elongated high-gradient interlayers, decreases. The interaction between all components of the currents is nonlinear, despite the fact that their characteristic scales are different.

Together with the experimental and numerical methods, the analytic methods remain one of the basic tools in the investigation of the nature of currents in fluids. In the course of their development, the information variables capable of the reliable characterization of the physical properties of the media and the parameters of currents were selected and the fundamental equations aimed at the description of the mechanics and thermodynamics of fluids were deduced [1, 2]. However, the analysis of the behavior of the entire system and the properties of separate equations, as well as the construction of partial solutions encounter serious difficulties due to the presence of multiscale processes and the nonlinearity of equations and the corresponding boundary and initial conditions. Numerous important results in the theory of slow (as compared with the sound velocity) currents in low-viscous weakly stratified fluids were obtained by the methods of perturbation theory [1, 2].

Parallel with the fundamental equations, the researchers extensively use constitutive models (various versions of turbulence theory in the hydroaerodynamics of the environment [3] and the theories of boundary layer in the engineering hydromechanics [4]) whose symmetry differs from the symmetry of the fundamental equations [5]. The fact that the constitutive models are not closed stimulated the development of more detailed investigations of the fundamental system of equations and its subsystems. In [6], the analysis of the mechanisms of adaptation of physical fields to rapidly varying external conditions is performed under the assumption of existence of stationary dynamic states of inhomogeneous rotating fluids including the state of rest. The transient wave processes are analyzed in the linear approximation, and the effect of dissipative factors (viscosity, thermal diffusivity, and diffusion) is neglected [6].

Ishlinskii Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, Russia; e-mail: chakin@ipmnet.ru.

Translated from Morskoi Gidrofizicheskii Zhurnal, No. 5, pp. 3–10, September–October, 2010. Original article submitted February 3, 2009.

The presence of dissipation significantly increases the order of equations, leads to changes in the character of solutions, and makes their structure much more complicated. Thus, in particular, the stratified media bounded by solid surfaces of any shape (topography) do not approach the state of rest even in the absence of disturbing forces. The interruption of molecular flow on impermeable boundaries leads to the formation specific currents induced by diffusion, including boundary layers, large slow eddies, and dissipative gravitational waves (nonstationary currents induced by diffusion on a sphere are computed in [7]). The infinitesimal periodic currents coexisting with two fine-structure components of different kinds in viscous continuously stratified and rotating media become more complicated [8].

The analysis of all molecular effects leads to a subsequent increase in the order of the fundamental system of equations [1, 2] and makes the general solution of the linearized system more complicated. The comparative analysis of the general properties of infinitesimal periodic currents described by the complete system of equations of the mechanics of inhomogeneous fluids and its basic subsystems is performed in the present work for the first time. In what follows, for the sake of brevity, the effects of compressibility discussed in [9] are omitted.

For simplicity, the dependence of density of the stratified fluid ρ on temperature T and the concentration of dissolved (or suspended) particles S of various types (in the general case, their number n determines the number of additional diffusion equations for the components of admixtures S_n in the system) is specified in the linearized form

$$\rho = \rho_0(1 - \alpha(T - T_0) + \beta(S - S_{n0})), \quad \alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{p,S}, \quad \beta = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial S_n}\right)_{p,T}, \tag{1}$$

where α is the coefficient of thermal expansion of the fluid, β is the salt compression coefficient, and T_0 and S_{n0} are the reference temperature and salinity. We consider the steady nonperturbed distributions of temperature $T_0(z)$, salinity $S_0(z)$, and density $\rho_0(z)$ characterized by constant scales [10]

$$\Lambda_T = \left|\frac{1}{T_0(z)}\frac{dT_0(z)}{dz}\right|^{-1}, \quad \Lambda_S = \left|\frac{1}{S_0(z)}\frac{dS_0(z)}{dz}\right|^{-1}, \quad \text{and} \quad \Lambda_\rho = \left|\frac{1}{\rho_0(z)}\frac{d\rho_0(z)}{dz}\right|^{-1},$$

frequencies

$$N_S = \sqrt{\frac{g}{\Lambda_S}}, \quad N_T = \sqrt{\frac{g}{\Lambda_T}}, \quad \text{and} \quad N = \sqrt{\frac{g}{\Lambda_\rho}},$$

and the buoyancy period $T_b = \frac{2\pi}{N}$ (g is the gravitational acceleration and the z-axis is directed along the vertical). The transformation of scales [11] enables us to transfer the results of calculations performed for the fluid with constant buoyancy frequency to the case of an arbitrary smooth distribution of density.

The system of fundamental equations of the mechanics of inhomogeneous incompressible fluids includes the equation of state (1) and the differential equations of continuity (d'Alembert equation), of momentum transfer (Navier–Stokes equations), of temperature transfer (Fourier equation), and of mass transfer (Fick equation) (the effects of thermo- and barodiffusion are neglected) [1]

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0\,,$$

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v}\right] = -\nabla p + \rho\nu\Delta\mathbf{v} + (\rho - \rho_0)\mathbf{g}\,,$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \kappa_T\Delta T\,, \tag{2}$$

$$\frac{\partial S}{\partial t} + \nabla\cdot(S\mathbf{v}) = \kappa_S\Delta S\,,$$

where $\mathbf{v}$ is a velocity, p is pressure, ν, κ_T, and κ_S are, respectively, the coefficients of kinematic viscosity, thermal diffusivity, and diffusion, and Δ is the Laplace operator, with no-flow and impermeability boundary conditions on the solid walls and the conditions of decay of all perturbations at infinity.

The equations and boundary conditions include the space scales of geometric and dynamic nature. The macroscales Λ, Λ_T, and Λ_S characterize the initial stratification (as a rule, weak), the geometry of the problem (size of the obstacles L), and the length of internal waves $\lambda = UT_b$ (U is the flow velocity at infinity).

The microscales determine the transverse sizes of fine-structure components both of the diffusion nature

$$\left(\delta_N = \sqrt{\frac{\nu}{N}}, \quad \delta_T = \sqrt{\frac{\kappa_T}{N}}, \quad \text{and} \quad \delta_S = \sqrt{\frac{\kappa_S}{N}}\right.$$

are introduced for the fields of velocity, temperature, and salinity, respectively, as analogs of the Stokes scale $\delta_\omega = \sqrt{\frac{\nu}{\omega}}$ [1]) and of the dynamic nature

$$\left(\delta_U = \frac{\nu}{U}, \quad \delta_{U,T} = \frac{\kappa_T}{U}, \quad \text{and} \quad \delta_{U,S} = \frac{\kappa_S}{U}\right.$$

are analogs of the Prandtl and Péclet scales).

Large values of the ratios of macro- and microscales including the ordinary dimensionless complexes, such as the Reynolds number

$$\mathrm{Re} = \frac{UL}{\nu} = \frac{L}{\delta_U} >> 1$$

and the Péclet numbers in temperature and salinity,

$$\mathrm{Pe}_T = \frac{UL}{\kappa_T} = \frac{L}{\delta_{U,T}} >> 1 \quad \text{and} \quad \mathrm{Pe}_S = \frac{UL}{\kappa_S} = \frac{L}{\delta_{U,S}} >> 1\,,$$

reflect the physical properties of actual fluids, namely, weak stratification

$$C = \frac{\Lambda}{L} = \frac{\rho_0}{\delta\rho} >> 1$$

(small relative changes in density on the scale L) and low values of viscosity, thermal diffusivity, and diffusion, e.g.,

$$C_N = \frac{L}{\delta_N} = \sqrt{\frac{L^2 N}{\nu}} >> 1$$

(as well as $C_T = \dfrac{L}{\delta_T}$ and $C_S = \dfrac{L}{\delta_S}$: for the solutions of salts, we have $C_N << C_T << C_S$), and substantiate the applicability of perturbation theory.

The system of equations (2) with small coefficients of higher derivatives with respect to the space variables belongs to the class of singularly perturbed equations [12]. In order to obtain the complete solutions of these equations, it is necessary to find both the direct expansions of the analyzed quantities in a small parameter ε

$$k = k_0 + \varepsilon k_1 + \varepsilon^2 k_2 + \dots, \tag{3}$$

and the inverse expansions

$$k_z = \varepsilon^{-\gamma}\left(k_0 + \varepsilon k_1 + \varepsilon^2 k_2 + \dots\right), \quad \gamma > 0. \tag{4}$$

The values of the coefficient γ can be found as a result of the substitution of expansion (4) in the analyzed system (2) from the condition of seniority of the obtained leading term of the expansion.

In analyzing small periodic motions with fixed real frequency ω and a complex-valued wave vector $\mathbf{k} = (k_x, k_y, k_z)$, $\mathbf{k} = \mathbf{k}_1 + i\mathbf{k}_2$, taking into account the decay of waves, all variables are chosen in the form

$$\mathbf{v} = \mathbf{v}_0\tau(r,t), \quad \bar{p} = p_0\tau(r,t), \quad \bar{\rho} = \rho_0\tau(r,t), \quad \tau(r,t) = \exp\left(i(\mathbf{kr} - \omega t)\right). \tag{5}$$

The solution of the linearized system (2) in the Boussinesq approximation is sought in the form of expansions in plane waves as follows:

$$A = \sum_j \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} a_j(k_x, k_y)\exp\left[i(k_{zj}(k_x, k_y)z + k_x x + k_y y - \omega t)\right]dk_x dk_y, \tag{6}$$

where A stands for the components of the velocity, pressure, temperature, salinity, and density. The operation of summation in expansion (6) is carried out over all roots of the dispersion equation representing the condition of solvability of the linearized system (2) and guaranteeing the validity of the boundary conditions of the problem or the conditions of radiation in unbounded media (decay of all perturbations at infinity).

The dispersion relation for the linearized system (2) taking into account the action of all dissipative factors has the form

$$D_\nu(k,\omega)F(k,\omega)=0\ , \tag{7}$$

where

$$F(k,\omega)=-D_\nu(k,\omega)D_{\kappa_T}(k,\omega)D_{\kappa_S}(k,\omega)\left(k^2+i\,\frac{k_z(\Lambda_T+\Lambda_S)}{\Lambda_T\Lambda_S}\right)$$

$$+D_{\kappa_T}(k,\omega)\left(\frac{\omega k_z}{\Lambda_S}D_\nu(k,\omega)-N_S^2k_\perp^2\right)+D_{\kappa_S}(k,\omega)\left(\frac{\omega k_z}{\Lambda_T}D_\nu(k,\omega)-N_T^2k_\perp^2\right), \tag{8}$$

$$D_\nu(k,\omega)=-i\omega+\nu k^2\ ,\qquad D_{\kappa_T}(k,\omega)=-i\omega+\kappa_T k^2\ ,\qquad D_{\kappa_S}(k,\omega)=-i\omega+\kappa_S k^2\ , \tag{9}$$

$$k^2=k_x^2+k_y^2+k_z^2\ ,\qquad k_\perp^2=k_x^2+k_y^2\ .$$

If all dissipative effects are neglected, then the tenth-order dispersion equation (7) turns into a quadratic equation for the internal waves in the ideal fluid [and all other types of waves, such as inertia, surface gravitational, acoustic, and hybrid waves (with regard for rotation and compressibility) [8]]. It corresponds to two regularly perturbed solutions of the algebraic equation (8) and the system of differential equations (2) with appropriate boundary conditions, respectively, specifying a conic bundle of periodic internal waves. In what follows, the spectral components (5) in which $|\boldsymbol{k}_1|>>|\boldsymbol{k}_2|$ and the attenuation coefficient is proportional to the kinetic coefficients [here, $\gamma=i\left(\nu+\kappa_T+\kappa_S\right)k^2$] are called redics (regular disturbed components of the flow).

The remaining eight roots of Eq. (7) whose imaginary parts are inversely proportional to the kinetic coefficients and not small $(|\boldsymbol{k}_1|\sim|\boldsymbol{k}_2|)$ specify singularly perturbed solutions, i.e., a family of sidics (singular disturbed components of the flow). In the unbounded medium, four roots violating the condition of decay at infinity are omitted. The remaining solutions form two different groups.

The form of Eq. (7) containing the factor $D_\nu(k,\omega)$ shows that the currents formed in the fluid always include singularly perturbed components in the form of Stokes-type periodic currents on the oscillating surface of the viscous fluid [1]. Their transverse sizes are determined by the kinematic viscosity and the wave frequency $\delta_\omega=\sqrt{\nu/\omega}$ (or the buoyancy frequency $\delta_N=\sqrt{\nu/N}$).

At the same time, the presence of viscosity leads to the appearance of another component whose properties are determined by the second and third terms in (8). Its transverse size depends not only on the frequency, kinematic viscosity, thermal diffusivity, and diffusion but also on the slope of the emitting surface (in this case, on the ratio k_z/k). The singularly perturbed components play the role of linear predecessors of the eddies and eddy systems in currents of the fluid.

Unlike the Stokes periodic current located near the oscillating surface [1], sidics can be found both in the vicinity of contact surfaces and in the bulk of the fluid. Thus, in particular, they form a fine structure of bundles of internal waves in a continuously stratified fluid for which the numerical results [13] agree with the data of shadow visualization [14].

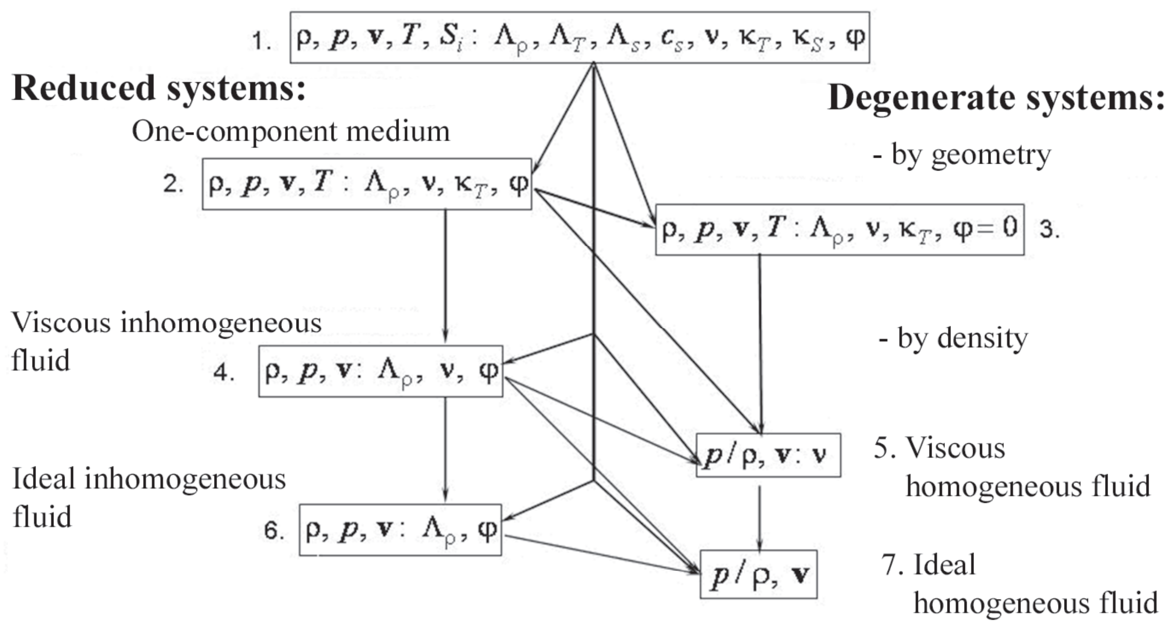

Fig. 1. Hierarchy of fundamental models of the mechanics of inhomogeneous fluids.

As the amplitude of oscillations of the source increases, the elongated high-gradient interlayers appear on the boundaries of the bundles and the eddies are formed directly in the bulk of the fluid in the regions of convergence of these interlayers [14].

It follows from the form of Eq. (7) that, in addition to two types of sidics caused by viscosity, there are two more solutions of system (2) whose properties depend on the thermal diffusivity and the diffusion coefficient. Depending on the geometry of the problem, the additional solutions can be either mixed (determined by all dissipative factors simultaneously) or split. The last case is characterized by the formation of a family of imbedded components of different scales whose location is determined by the boundary conditions of the problem.

All solutions of (2) (both regularly and singularly perturbed) form a single family described by functions of the same type (5) with different real and imaginary parts. They are simultaneously formed, transferred, and disappear, despite the difference in characteristic scales. Each component of the current causes the energy, mass, and vorticity transfer. The mechanical energy is mainly transferred by the large-scale components (redics). The dissipation of motions occurs in the fine-structure components (sidics) characterized by high values of all components of the tensor of velocity shift (including vorticity). The level of pressure in the sidics remains constant.

The general properties of solutions of the basic system and its subsystems are illustrated by the scheme presented in Fig. 1. The fundamental tenth-order system 1 describes the dynamics of four scalar fields (ρ, p, T, S) and the vector field of velocities $\mathbf{v}$ each of which is characterized by its own geometry. The parameters of the problem Λ_ρ, Λ_T, Λ_s, ν, κ_T, κ_S and the angular position of the source φ (or of the boundaries of the region filled with the fluid) determine the properties of solutions containing two (or one) regular and eight (or minimum four) singular components. System 1 is self-consistent and solvable.

If the description is simplified, e.g., if we exclude the terms with the lowest coefficients in (2) (in the actual fluids, the minimum value is taken, as a rule, by the diffusion coefficient), the order of system 2 and the order of the dispersion equation (7) are reduced. The reduced eighth-order system 2 with the parameters Λ_ρ, ν, κ_T, and φ characterizes the dynamics of six variables (ρ, p, T, and the components of the velocity $\mathbf{v}$). Its solutions contain two (or one) regularly perturbed solutions and six (or three) different singularly perturbed components. The attenuation coefficient of the regular solution also changes. The system remains solvable.

Certain geometries of the problem (special symmetry of the source and/or the vertical or horizontal locations of the boundaries: $\varphi = 0$) decrease the number of determining parameters $(\Lambda_\rho, \nu, \kappa_T)$ and the order of the reduced system 2. In this case, some singular components of the degenerate system 3 may become identical or equal to zero. The dynamics of six independent variables (ρ , p , T , and the components of the velocity $\mathbf{v}$) is determined by the behavior of two (one) regularly perturbed solutions and four (two) singularly perturbed solutions.

If we exclude the equation of state and preserve density stratification, then the complete system 1 turns into the sixth-order system 4 (with the parameters Λ_ρ, ν, and φ) whose solutions have two (one) regularly perturbed components and four (two) different singularly perturbed components. The solvability of the system is preserved.

System 5 is homogeneous in density (degenerate) and contains the d'Alembert–Navier–Stokes equation for the variables p/ϱ and $\mathbf{v}$ with a single parameter ν. This system corresponds to the following sixth-order dispersion equation:

$$k^2\left(\omega + i\nu k^2\right)^2 = 0$$

with a multiple singularly perturbed root, which reveals the identity, in the general case, of singularly perturbed components of different kinds. Hence, the problem of finding the three-dimensional fields of the variables p/ρ and $\mathbf{v}$ for $\rho = \text{const}$ and arbitrary initial conditions turns out to be ill posed. The influence of compressibility does not remove the degeneration of singularly perturbed components in which the currents are not divergent [8]. The system becomes solvable if its order is reduced (one- and two-dimensional problems and special boundary conditions).

The Euler equations for stratified media 6 with the parameters Λ_ρ and φ specify the field of internal waves (variables p , ρ , and $\mathbf{v}$) with discontinuities in the characteristics whose locations are determined by the boundary conditions. The three-dimensional Euler equations (with variables p/ρ and $\mathbf{v}$) do not contain external parameters and are not directly solvable in the proposed statement.

The solutions of the complete system 1 and reduced systems 2, 4, and 6 enable us to find the solutions of systems 3, 5, and 7 by the uniform transition to the limit as $N \to 0$ in the final expressions. Due to the reduction of the order of subsystems, the inverse transition is impossible.

The nonlinear terms in the complete system (2) characterize the direct interaction of all infinitesimal (regularly and singularly perturbed) components of the currents, which may lead to the generation of new components of currents of the same type [15] or actual eddies accompanied by new singularly perturbed components. In this case, the changes in all variables are consistent. The stationary states of stratified or rotating fluids are globally unattainable even if the effects of induced transfer (such as, e.g., thermo- and barodiffusion) are neglected.

The present work was performed under a partial financial support of the Russian Academy of Sciences (Program OE-14 OEMMPU RAN "Dynamics of Multicomponent and Inhomogeneous Fluids"), and the RFFS (Projects 08-05-00434-Ukr.).

REFERENCES

1. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics,* Pergamon Press, Oxford (1975).
2. P. Müller, *The Equations of Oceanic Motions,* Cambridge Univ. Press, Cambridge (2006).
3. A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics,* Dover, New York (2007).
4. V. Ya. Neiland, V. V. Bogolepov, G. N. Dudin, and I. I. Lipatov, *Asymptotic Theory of Supersonic Currents in Viscous Gases* [in Russian], Fizmatlit, Moscow (2004).
5. Yu. D. Chashechkin, V. G. Baydulov, and A.V. Kistovich, "Basic properties of free stratified flows," *J. Eng. Math.,* **55**, No. 1–4, 313–338 (2006).
6. A. S. Monin and A. M. Obukhov, "Weak oscillations of the atmosphere and the adaptation of meteorological fields," *Izv. Akad. Nauk SSSR, Ser. Geofiz.,* No. 11, 1360–1373 (1958).
7. V. G. Baidulov, P. V. Matyushin, and Yu. D. Chashechkin, "Evolution of currents induced by diffusion on a sphere immersed in the continuously stratified fluid," *Izv. Ros. Akad. Nauk, Mekh. Zhidk. Gaz.,* No. 2, 119–132 (2006).
8. Yu. D. Chashechkin and A.V. Kistovich, "Classification of three-dimensional periodic currents in fluids," *Dokl. Ros. Akad. Nauk,* **395**, No. 1, 55–58 (2004).
9. R. N. Bardakov, A.V. Kistovich, and Yu. D. Chashechkin, "Numerical analysis of the velocity of sound in inhomogeneous fluids," *Dokl. Ros. Akad. Nauk,* **420**, No. 3, 324–327 (2008).
10. V. V. Levitskii and Yu. D. Chashechkin, "Lateral thermoconcentration convection in weakly stratified fluids," *Izv. Ros. Akad. Nauk, Mekh. Zhidk. Gaza,* No. 3, 87–98 (2006).
11. Yu. V. Kistovich and Yu. D. Chashechkin, "Linear theory of propagation of the bundles of internal waves in arbitrarily stratified fluids," *Prikl. Mekh. Tekhn. Fiz.,* **39**, No. 5, 88–98 (1998).
12. S. A. Lomov, *Introduction to the General Theory of Singular Perturbations,* Amer. Math. Soc., Providence, RI (1992).
13. Yu. D. Chashechkin, A. Yu. Vasil'ev, and R. N. Bardakov, "Fine structure of the bundles of three-dimensional periodic internal waves," *Dokl. Ros. Akad. Nauk,* **397**, No. 3, 404–407 (2004).
14. Yu. D. Chashechkin, "Visualization of singular components of periodic motions in a continuously stratified fluid (review report)," *J. Visual.,* **10**, No. 1, 17–20 (2007).
15. Yu. V. Kistovich and Yu. D. Chashechkin, "New mechanism of nonlinear generation of internal waves," *Dokl. Ros. Akad. Nauk,* **382**, No. 6, 772–776 (2002).