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An infinitely descending chain of ground without a lower bound

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Abstract Using only uncontentious principles from the logic of ground I construct an infinitely descending chain of ground without a lower bound. I then compare the construction to the constructions due to Dixon (forthcoming) and Rabin and Rabern (J Philos Log, 2015).

Keywords Ground · Infinite descent · Logic of ground · Infinitary disjunction

1 Introduction

In this note I construct an infinitely descending chain of partial ground without a lower bound. It is of particular interest that even those philosophers who think that all truths are ultimately grounded in ungrounded truths have to accept the possibility of this chain. While this is not the first such chain that has been proposed in literature, the present construction is both novel and requires less assumptions than the previous proposals.

The main part of the paper (Sects. 2–4) is taken up by the construction of the infinitely descending chain. In the final section (Sect. 5) I compare the present construction with the constructions due to Dixon (forthcoming) and Rabin and Rabern (2015).

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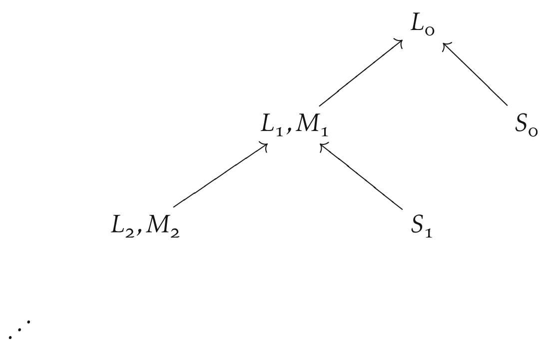
2 Terminology

Let us use "<" as a sentential operator[1] for strict full ground.[2] Let us write $\Gamma < \phi$ to mean that the truths Γ strictly fully ground the truth ϕ.[3] What we will show is that there are truths L_0, L_1, …, S_0, S_1, …, M_1, M_2, … such that for each n:

(1)$S_n < L_n$
(2)L_{n+1}, $M_{n+1} < L_n$
(3)There is no set of truths Δ such that $\Delta < L_n$, for each n

The resulting structure can be depicted as follows.



Let us call a structure like this an *unboundedly ever-descending tree of ground.*[4] Using $L_i \succ L_{i+1}$ to say that L_i is partially grounded in L_{i+1}, this tree witnesses that $L_0 \succ L_1 \succ L_2 \succ \ldots$, that is, that L_0, L_1, L_2, … is an infinitely descending chain of partial ground.

3 Constructing the tree

Let S_0, S_1, … be countably many sentences expressing distinct ungrounded truths.[5] For each natural number $n \geq 0$, let L_n be the sentence:

(L_n) Either S_n or L_{n+1} is true

It is clear that L_n is true for each n. After all, S_n is true for each n, and S_n is a disjunct of L_n.

[1] Those who like to think of ground as a relation between facts can make the appropriate adjustments.

[2] See (Fine 2012a, pp. 48–54) for a discussion of various distinct notions of ground.

[3] In the interest of readability I will not rigorously observe the use/mention distinction; the interested reader will have no problem restoring rigor.

[4] If only (1) and (2) are satisfied we just have an ever-descending tree of ground.

[5] We do not require that the S_i are ungrounded—or even that there are *any* ungrounded truths—but it is presentationally convenient to make this assumption. We dispense with this assumption in Sect. 4.

Let us write $T\ulcorner L_n \urcorner$ for the sentence "L_n is true". Since L_n is true for each n, $T\ulcorner L_n \urcorner$ is also true for each n. Since $T\ulcorner L_n \urcorner$ is true for each n, and $T\ulcorner L_{n+1} \urcorner$ is a disjunct of L_n, the following holds:

(Disjunctive-Grounding) $T\ulcorner L_{n+1} \urcorner < L_n$

This is a special case of the principle that a disjunction is grounded in its true disjuncts. While it is true that there are situations where the general principle has to be given up[6] this is not one of those situations.[7]

For each n, let M_n be the truth that L_n means that (S_n or $T\ulcorner L_{n+1} \urcorner$). The following principle is accepted by almost all writers on ground[8]:

(Truth-Grounding) $M_n, L_n < T\ulcorner L_n \urcorner$

(If the sentence S means that p, then S's being true is grounded in p's being the case together with S's meaning that p.)[9] It follows that[10]:

$$M_{n+1}, L_{n+1} < L_n$$

This finishes the construction.

A variant of the above construction establishes that there is a *dense* chain of ground. Let $\{S_r : r \in \mathbb{Q}\}$ be some sentences expressing distinct ungrounded truths. For each rational number $r \in \mathbb{Q}$, let L_r be the sentence

(L_r) S_r or for some q less than r, L_q is true

For each $q \in \mathbb{Q}$ let M_q be the truth that L_q means what it does. The same form of argument as given above establishes that ($L_r, M_r < L_q$ iff r is less than q).

It is intuitively obvious that the tree is unbounded, but it is instructive to make explicit the assumptions that are required to prove this: we only require some of the principles of the impure logic of ground presented in (Fine 2012a, pp. 58–67).

6 (Fine 2010, p. 117n15)

7 This can be rigorously established using the theory developed in (Litland 2015).

8 See, e.g., (Fine 2010, p. 106), (Schnieder 2006, pp. 35–36) and (Correia 2014, p. 24). Interestingly, the principle is also accepted by a deflationist like Horwich (2008, p. 266). For the present applications it also suffices that M_n, L_n is a weak full ground for $T\ulcorner L_n \urcorner$.

9 Such a collection of sentences is clearly *metaphysically* possible. One can also construct a sequence L_0, L_1, ... using diagonalization, though some care has to be taken. It does not suffice that L_n is (provably) equivalent to $S_n \vee T\ulcorner L_{n+1} \urcorner$. If L_n was merely (provably) equivalent to $S_n \vee T\ulcorner L_{n+1} \urcorner$ there would be no guarantee that L_n *said* that $S_n \vee T\ulcorner L_{n+1} \urcorner$; in that case the applications of (Truth-Grounding) would be unacceptable. By using Jeroslow's diagonal lemma, however, we can find $\tau_0, \tau_1, \ldots$ such that $\tau_i = \ulcorner S_i \vee T\ulcorner \tau_{i+1} \urcorner \urcorner$. We can then let L_i be the sentence $S_i \vee T\ulcorner \tau_{i+1} \urcorner$. Thanks to Tim Button for discussion on this point.

10 We here rely on the following principle from the logic of ground

(Cut) If $\Gamma < \phi$ and $\Delta, \phi < \psi$ then $\Gamma, \Delta < \psi$

4 The tree is unbounded

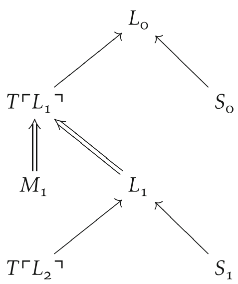
To prove that the tree is unbounded it is convenient to introduce the notion of *weak ground*, writing $\Gamma \leq \phi$ for the claim that the truths Γ weakly fully ground the truth ϕ.[11] Whereas strict ground is irreflexive, weak ground is not: in fact, for each truth ϕ we have $\phi \leq \phi$. We say that ϕ is a weak partial ground for ψ if there is some Γ such that $\phi, \Gamma \leq \psi$.

We can now dispense with the assumption that the S_i express distinct and *ungrounded* truths; it suffices for our purposes that the S_i, pairwise, do not have a common weak partial ground.[12]

We require two assumptions from the logic of ground. The first is that mediate ground is the closure of immediate ground under Cut.[13] The second assumption is that the immediate grounds for a disjunction $\phi \vee \psi$ are either (1) the truth ϕ; (2) the truth ψ; or (3) the truths ϕ, ψ taken together.

I find these assumptions from the impure logic of ground unobjectionable, but I should flag that they are not uncontroversial. Interestingly, one of the previous examples of infinitely descending chains of ground requires rejecting them; we will return to this issue in Sect. 5 below.

With the relationships of immediate ground made explicit the tree looks as follows:



In this figure the single arrows indicate relationships of strict full ground; the double arrows indicate that M_n, L_n *together* strictly fully ground $T\ulcorner L_n \urcorner$. (The truths M_i and S_j might themselves be (weakly) grounded, but we (harmlessly) omit displaying their grounds.)

We require two non-logical assumptions. The first assumption is that $M_0, M_1, \ldots$ do not together ground any L_i. This is reasonable: that the sentences L_i mean what

[11] One does not require the notion of weak ground for the following argument, but it makes the argument less cumbersome.

[12] A monist like (Schaffer 2010) would deny that this is possible: all true propositions have a common ground. But even for the monist the construction establishes that there is an ever-descending tree of ground.

[13] The distinction between mediate and immediate ground can be made clear by example: if ϕ, ψ, θ are three distinct truths, then ϕ is an immediate ground of $\phi \vee \psi$, but only a mediate ground of $(\phi \vee \psi) \vee \theta$. For more on the distinction between mediate and immediate ground, see (Fine 2012a, pp. 50–51).

they do is not enough to account for the truth of any L_j. The second assumption is that no grounds for the truths S_i (weakly) partially ground any of the truths M_j (for $j > i$). This assumption, too, is reasonable. The truth M_j is the truth that L_j means what it does: we can surely choose the truths S_i such that no grounds for S_i grounds M_j, for $j > i$.

From these assumptions it follows that the above tree is unboundedly ever-descending.

Observe first that if a truth ψ is a partial ground of L_0, then any occurrence of ψ in the above tree occurs some finite number of steps below L_0. (This follows from mediate ground's being the closure of immediate ground.)[14]

Suppose for contradiction that there are some truths Δ such that $\Delta < L_n$ for each n. Suppose first that $T\ulcorner L_n \urcorner$ is in Δ for some n. Let us write $\Delta = \Delta' \cup \{T\ulcorner L_n \urcorner\}$. Then, since $\Delta', T\ulcorner L_n \urcorner < L_n$ (by assumption) and $L_n, M_n < T\ulcorner L_n \urcorner$, a strict partial ground for $T\ulcorner L_n \urcorner$ is $T\ulcorner L_n \urcorner$ itself, contradicting the irreflexivity of partial ground. The same argument establishes that $L_n \notin \Delta$ for each n.

Suppose next that $\Delta' \leq S_n$ for no $\Delta' \subset \Delta$ and no n. Since each $\delta \in \Delta$ is such that each occurrence of δ is only finitely many steps away from L_0 it follows that each occurrence of a $\delta \in \Delta$ has to be below some M_i. Since the truths M_0, M_1, … do not ground L_0, the truths Δ do not ground L_0 either.

It follows that there is an m such that $\Delta = \Delta_m \cup \Delta'_m$ and $\Delta_m \leq S_m$. If $i \neq j$, S_i and S_j have no common (weak partial) ground. It follows that Δ_m does not weakly partially ground any S_n, $n \neq m$. By our second non-logical assumption Δ_m does not weakly partially ground any $M_n, n > k$. It follows that $\Delta_m \cup \Delta'_m = \Delta$ does not strictly ground L_n for $n > m$.

Since Δ was arbitrary this shows that there is no Δ such that $\Delta < L_n$ for each n. The tree is unbounded.[15]

A notable feature of the construction is that all the principles we have relied on should be acceptable to those who think that, necessarily, all truths are grounded in ungrounded truths. Even if it is necessary that all truths are grounded in ungrounded truths it is possible that there are infinitely descending chains of ground without a lower bound.

5 Comparison with other unbounded chains

This is not the first purported unbounded infinitely descending chain of ground in the literature. Dixon (forthcoming) gives two examples of infinitely descending chains without a lower bound. It must be admitted that in one respect his chains are more striking: they are examples of unbounded infinitely descending chains of *full* ground. Unfortunately, there are problems. One of his example does not work; and

[14] This was, in effect, pointed out by Fine (2012a, p. 51).

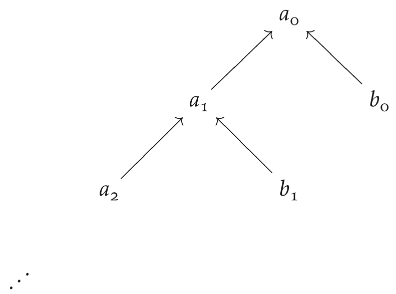
[15] In fact—assuming the transitivity rules of the Pure Logic of Ground (Fine 2012b, p. 5)—we can establish the stronger conclusion that there is no Δ that, for each n, weakly fully grounds L_n.

while the other might work, it requires infinitary resources and unusually problematic ones at that.[16]

In one of his examples Dixon considers a situation where we have countably many mereologically simple objects $o_0, o_1, \ldots$. They have the following masses. o_0 has mass exactly 1 kg; o_1 has mass exactly 1/2 kg; and so on. He then considers the following two sequences of truths $a_0, a_1, \ldots, b_0, b_1, \ldots$

a_0	$\exists z$ (z has non-zero mass at most 1 kg)
a_1	$\exists z$ (z has non-zero mass at most 1/2 kg)
a_2	$\exists z$ (z has non-zero mass at most 1/4 kg)
$\vdots$	
b_0	$\exists z$ (z has non-zero mass exactly 1 kg)
b_1	$\exists z$ (z has non-zero mass exactly 1/2 kg)
b_2	$\exists z$ (z has non-zero mass exactly 1/4 kg)
$\vdots$	

Dixon then claims that the following is an unboundedly ever-descending tree of ground.



Dixon here relies on the principle labeled (Determinables) below.

(Determinables) For any positive real numbers x and y, if (1) something with a non-zero mass of at most y kg exists and (2) $y = 0.5x$, then $\exists z\,(z \text{ has non-zero mass of at most } y) < \exists z\,(z \text{ has non-zero mass of at most } x)$

(Determinables) is false; we can see this as follows. Since a_0 is an existentially quantified truth, it follows by the elimination rules for $\exists$ (Fine 2012a, pp. 65–66) that if a_1 strictly fully grounds a_0, then a_1 weakly fully grounds that some o_i has

[16] I should stress that these unbounded infinitely descending chains play only a limited role in Dixon and Rabin and Rabern's important papers. Their main concern is to characterize different "foundationalist" theses and determine their relative strengths. For these purposes mathematical models suffice; there is no need for the mathematical models to correspond to real metaphysical possibilities.

non-zero mass at most $1/2^i$ kg.[17] Since a_1 is existential it will be grounded in its instances. In particular, for each $j > i$ it will be grounded in the truth that o_j has non-zero mass at most $1/2^j$. But since o_i, o_j are distinct mereologically simple objects the truths about the mass of one does not ground the truths about the mass of the other.

It is worth noting that while (Determinables) is false there is a correct principle in its vicinity, namely:

(Determinables′) If $y = .5x$ then the truth that z has non-zero mass at most y grounds the truth that z has non-zero mass at most x

Dixon's second example—an example also given by Rabin and Rabern (2015)—relies on infinitary disjunction. Let $p_0, p_1, \ldots$ be countably many distinct fundamental truths. Consider now the infinite disjunction:

$$p_0 \vee (p_1 \vee (p_2 \vee \ldots)))$$

They claim that we now get the following infinitely descending chain of ground:

$$p_0 \vee (p_1 \vee (p_2 \vee \ldots)))$$
$$\vee$$
$$p_1 \vee (p_2 \vee (p_3 \vee \ldots)))$$
$$\vee$$
$$p_2 \vee (p_3 \vee (p_4 \vee \ldots)))$$
$$\vdots$$

One advantage of the construction given in this paper is that it requires no infinitary resources at all. But even if one has no problem with infinitary disjunction as such—and I do not—Dixon and Rabin and Rabern have to make some problematic assumptions about how infinitary disjunctions work.[18]

Let us first set aside a superficial worry. From the notation it might appear that the infinite disjunction is formed by infinitely many applications of the binary operation of disjunction. Dixon, however, is clear that he does not understand infinite disjunction in this way; rather, he thinks that we form infinite disjunctions by applying the disjunction operator once to a collection of infinitely many sentences $\{p_0, p_1, \ldots\}$, in a single operation forming the complex sentence: $\bigvee\{p_0, p_1, \ldots\}$.

I have no objection to being able to form infinite disjunctions in this way. What I am worried about is the claim that $\bigvee\{p_0, p_1, \ldots\}$ is strictly fully grounded in

[17] This is a slight simplification. What follows is that, for some I, a_1 weakly distributively fully grounds the truths $\{o_i \text{ has non-zero mass at most } 1/2^i : i \in I\}$. But the argument below can be modified to work even when this complication is taken into account. (There is a further complication: the elimination rule makes use of the existence predicate "E" or the "totality"-predicate "T"; suffice it to say that the argument can be modified to deal with this too.)

[18] I am very grateful to an anonymous referee for pressing me to get clearer on the issues discussed in the remainder of this section.

$\bigvee\{p_1, p_2, p_3 \ldots\}$. Let us use $\approx$ as an operator for mutual weak full ground (or factual equivalence in the sense of (Correia 2010)). The crux of the issue is whether one accepts the following associativity thesis:

(Associativity)

$$\bigvee\{p_0, p_1, p_2, \ldots\} \approx \bigvee\{p_0, \bigvee\{p_1, p_2, \ldots\}\}$$

(Dixon explicitly accepts this principle.[19])

I do not accept (Associativity).

First, it seems we should distinguish between $\bigvee\{p_0, p_1, \ldots\}$ and $\bigvee\{p_0, \bigvee\{p_1, p_2, \ldots\}\}$. In forming the first truth we apply the disjunction operator *once*; in forming the second truth we apply it *twice*. The immediate reply is that this is merely a difference in the expression of the truth and not a difference in the truths expressed.

Secondly, however, (Associativity) is inconsistent with the assumption that the immediate full grounds for a truth $\phi \vee \psi$ are all and only ϕ, ψ, and ϕ, ψ taken together. (We here assume that ϕ, ψ are both true.) To see this let p, q, r be three sentences expressing three truths that, pairwise, have no common ground. It follows by (Associativity) that $(p \vee (q \vee r)) \approx ((p \vee q) \vee r)$. Since $(p \vee q)$ is an immediate full ground for $(p \vee q) \vee r$, if $(p \vee (q \vee r)) \approx ((p \vee q) \vee r)$ then $p \vee q$ has to be an immediate full ground for $p \vee (q \vee r)$ as well. But $p \vee q$ is identical to neither p nor $q \vee r$.

Third, and most importantly, while one might respond by giving up the claim about the immediate grounds of disjunctions, holding on to (Associativity) is subject to a serious but hitherto unnoticed difficulty.

Let S, T be two sentences expressing distinct, fundamental truths. Consider now the sentences (Good) and (Bad).

(Good) (The unique sentence with label (Bad) is true $\vee S) \vee T$
(Bad) The unique sentence with label (Bad) is true $\vee (S \vee T)$

These sentences are of the forms: $(\phi \vee \psi) \vee \theta$ and $\phi \vee (\psi \vee \theta)$, respectively. However, (Good) and (Bad) do not mutually weakly fully ground each other.[20] For (the truth expressed by) "The unique sentence with label (Bad) is true" is a ground for (the truth expressed by) (Good), but it is not a ground for (the truth expressed by) (Bad): the latter would contradict the irreflexivity of partial ground.[21]

Admittedly, this only shows that (Associativity) is not valid; it does not show that there is anything wrong with the particular application Dixon and Rabin and Rabern

[19] See note 25 to of (Dixon forthcoming).

[20] Or to be more precise: the truths expressed by (Good) and (Bad) do not mutually fully weakly ground each other.

[21] A similar example shows that the following principle about conjunction—$\phi \wedge (\psi \wedge \theta) \approx (\phi \wedge \psi) \wedge \theta$—also has to be given up. Consider the sentence

(Worse) $P \wedge (Q \wedge (R \vee$ the unique sentence with label (Worse) is true$))$

make of (Associativity). Recently, however, it has become clear that grounding is treacherous when account is taken of self-reference and impredicativity [see e.g., Fine (2010), Litland (2015), Correia (2014) and Krämer (2013)]. Until we have a general theory of ground telling us which instances of (Associativity) are safe and which are not, some skepticism about the chain due to Dixon and Rabin and Rabern is warranted.

I conclude that the herein presented chain is the most secure unbounded infinitely descending chain yet presented.

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