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Choice Functions on Posets

Danilov V. I.[1]

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Abstract

In the paper we study choice functions on posets satisfying the conditions of heredity and outcast. For every well-ordered sequence of elements of a poset, we define the corresponding 'elementary' choice function. Every such choice function satisfies the conditions of heredity and outcast. Inversely, every choice function satisfying the conditions of heredity and outcast can be represented as a union of several elementary choice functions. This result generalizes the Aizerman-Malishevski theorem about the structure of path-independent choice functions.

Keywords Order ideal · Filter · Well order · Path independence · Stable contract

1 Introduction

The study of choice functions is an important part of the theory of rational decision-making. The class of choice functions introduced by Plott [10] under the name "path-independent" turned out to be particularly interesting. A comprehensive description of such choice functions was given by Aizerman and Malishevsky in [1]. Later it turned out that these choice functions appear naturally in the theory of nonmonotonic logic [9] and the stable contract theory [2, 6, 7]. In the latter theory, the choice functions were used to describe preferences of agents, and choice functions proposed by Plott were the most appropriate for this situation.

In the theory of contracts, it was revealed the need for generalization and transfer of the concept of such choice functions to partially ordered sets (posets). For example, the papers [2–4, 6, 8] considered contracts with some intensity between 0 and 1. To ensure the existence of stable systems of contracts, the authors had to transfer the concept of path-independent choice functions to sets equipped with partial order. In what follows we call such choice functions *conservative*. In [6], the existence and good properties of stable systems of contracts under the assumption of conservativeness of choice functions of the agents were proved.

✉ Danilov V. I.
vdanilov43@mail.ru

1 Central Institute of Economics and Mathematics of the RAS, 47, Nakhimovskii Prospect, 117418 Moscow, Russia

However, some important questions remained open: the order of magnitude of the number of such choice functions, how to construct them, or the structure on the set of conservative choice functions. For instance, the authors of [2] limited themselves to giving two large classes of conservative choice functions; in [6] two particular examples were considered. But these examples did not exhaust the set of all conservative choice functions. In this paper, we answer these questions. Namely, for each sequence of elements of a poset P, we construct the corresponding 'elementary' conservative choice function on P. We show that an arbitrary conservative choice function is represented as the union of several elementary choice functions. This result generalizes the Aizerman-Malishevsky theorem [1] just like its infinite variant from [5].

In order to make the presentation more understandable, we first assume that the poset P is finite. The general case is considered in Section 6. We start with a reminder of some concepts and statements about posets and choice functions on them.

2 Preliminaries

Posets A *poset* is a partially ordered set, i.e. a set P equipped with an order relation $\leq$ (reflexive, transitive, and antisymmetric) on it. Since this relation will not change, the poset is simply denoted as P. A poset is called *linear* (or a *chain*) if any two of its elements are comparable ($x \leq y$ or $y \leq x$). A poset is called *trivial* (or an antichain), if any two elements are incomparable. A more general class of posets covering two previous ones is distinguished by the transitivity condition of the comparability relation. Structurally, such posets are the direct sum of chains. Posets of this form were used in [2, 4].

A subset I of P is called an (order) *ideal* (or a lower set, or a minor set), if $y \leq x \in I$ implies $y \in I$. For example, the principal ideal $I(x) = \{y \in P,\ y \leq x\}$. The ideal generated by a subset A in P is denoted as $I(A)$; $I(A) = \cup_{a \in A} I(a)$. The dual is the concept of a *filter*; a filter with each element contains larger ones. A filter generated by a subset A is denoted as $F(A)$, so that $F(A) = \{x \in P,\ x \geq a$ for some $a \in A\}$. The set of all ideals is denoted by $\mathcal{I}(P)$; it is a complete distributive sublattice of the Boolean lattice 2^P of all subsets in P.

Choice Functions In the classical situation (when the poset P is trivial), a choice function is a mapping $f : 2^P \to 2^P$ such that $f(X) \subseteq X$ for any $X \subseteq P$. In decision theory, choice functions are used to describe a behavior of agents; having access to a variety of alternatives X the decision-maker selects a subset $f(X)$ from X. A rational decision-maker chooses the best alternatives in some sense. The rationality conditions of the corresponding choice function have been intensively studied in the choice theory, see, for example, [1]. Two conditions turned out to be the most popular. These are the heredity and the outcast conditions.

The *heredity* property (known also as substitutability or the persistency property): if $A, B \subseteq P$ and $A \subseteq B$ then $f(B) \cap A \subseteq f(A)$.

In other words, if an element from a smaller set A is chosen in a larger set B then it should be chosen in the smaller one.

The *outcast* property (known also as consistency or as Irrelevance of Rejected Alternatives): if $f(B) \subseteq A \subseteq B$ then $f(A) = f(B)$.

In words – removing 'bad' (unselected) elements does not affect the choice.

All these notions are extended to posets without any changes.

Definition A *choice function* (CF) on a poset P is a mapping $f : \mathcal{I}(P) \to \mathcal{I}(P)$, such that $f(X) \subseteq X$ for any ideal X in P.

The conditions of heredity and outcast are formulated as above.

Definition A CF on a poset P is called *conservative* if it has the heredity and outcast properties.

One can formulate the conservativeness by a single condition ([8]): if $f(A) \subseteq B$ then $f(B) \cap A \subseteq f(A)$. We find more convenient to use heredity and outcast separately.

Some Examples

1) Of course, any path-independent CF on a trivial poset is conservative. In particular, the CF which selects the quota-many best options relatively some linear order on the ground set P. Note that this linear order has nothing to do with the underlying partial order on P which is trivial. It is an additional structure needed to define the choice function.
2) There are interesting examples conservative CFs for non-trivial posets. Suppose we need a pair of gloves. And if there are both right and left gloves in the menu, we choose one right glove and one left glove. But if there are no right gloves, we choose two left ones. And similarly, if there are no left ones, we choose two right ones. Here P is the direct sum of two exemplar of $\mathbb{Z}_+$.

   This example can be extended for arbitrary quota. It met in papers [2, 6].
3) 'Constant' CFs. Let I be a fixed ideal in a poset P. For an arbitrary ideal X, we put $f^I(X) = I \cap X$. It is easy to see that f^I is a conservative CF on P.

   If a poset P is linear (a chain), then the previous construction gives all conservative CFs. In fact, set $I = f(P)$. Since an arbitrary ideal X lies in P, from the heredity we have the inclusion $I \cap X \subseteq f(X)$. Due to the linearity of P, either $X \subseteq I$ or $I \subseteq X$. In the first case $I \cap X = X$, $X \subseteq f(X)$, from where the equality $f(X) = X = I \cap X$ comes. In the second case, $I \cap X = I$. We have the chain $I = f(P) \subseteq f(X) \subseteq P$, and the outcast gives $f(X) = f(P) = I = I \cap X$.
4) In section 3 we provide a series of important examples of conservative CFs.

Some properties of conservative CFs Below we list some simple properties of conservative CFs.

1. $f(f(X)) = f(X)$ for any ideal X.

   Apply the outcast to the chain of inclusions $f(X) \subseteq f(X) \subseteq X$.
2. Any conservative CF f has the so-called *path independence* property: for any ideals X and Y

$$f(X \cup Y) = f(f(X) \cup f(Y)).$$

   Indeed, $f(X) \subseteq X \cup Y$; from the heredity we obtain $f(X \cup Y) \cap X \subseteq f(X)$. Similarly, $f(X \cup Y) \cap Y \subseteq f(Y)$. Hence $f(X \cup Y) = f(X \cup Y) \cap (X \cup Y) \subseteq f(X) \cup f(Y) \subseteq X \cup Y$. Using the outcast, we get $f(X \cup Y) = f(f(X) \cup f(Y))$.

   In fact, this equality is true not only for two ideals, but for an arbitrary number of them. The proof is the same. Note also that the path independence implies the outcast property, but not the heredity, see [2].

3. The union of conservative CFs is a conservative CF.
   The union $\cup_j f_j$ of a family $(f_j, j \in J)$ of CFs is given by the natural formula:

$$(\cup_j f_j)(X) = \cup_j f_j(X)$$

   If all f_j are conservative CFs, and $f = \cup_j f_j$, we have to check the heredity and outcast properties for CF f.
   Heredity. Let $Y \subseteq X$. Then, for any j, we have inclusions $f_j(X) \cap Y \subseteq f_j(Y)$. Hence $f(X) \cap Y = (\cup_j f_j(X)) \cap Y) = \cup_j (f_j(X) \cap Y) \subseteq \cup_j f_j(Y) = f(Y)$.

Outcast. Let $f(X) \subseteq Y \subseteq X$. Since $f_j(X) \subseteq f(X)$, we have the chain $f_j(X) \subseteq Y \subseteq X$ for any j. The outcast for f_j implies the equality $f_j(X) = f_j(Y)$ for any j, from where $f(X) = f(Y)$.

The property 3 means that having several conservative CFs f_j one can form new CF $\cup_j f_j$ which also is conservative. This suggests to look for an enough big stock of 'simple' conservative CFs sufficient for construction of any conservative CF. The constant CFs are not suitable for this purpose. The thing is that the union of constant CFs is constant again. But examples 2) and 4) show that there exist non-constant CFs. Thus, we need more flexible construction of 'simple' conservative CFs. To make the idea more transparent, we will temporarily assume that poset P is finite. In Section 6 we consider the general case.

3 Elementary Choice Functions

Let $A = (a_1, ..., a_k)$ $(k \geq 0)$ be a sequence of elements of a poset P . We use this sequence to build the following 'elementary' CF f_A. But first we have to enter one notation. Suppose that X is an arbitrary ideal in P. Let $i = i(A, X)$ be the first number i, such that $a_i \in X$. In other words, $a_1, ..., a_{i-1}$ do not belong to X, but a_i does. If none a_i belongs to X, we set $i = k$. If $k = 0$ (that is if A is the empty sequence) we set $i = 0$.

Definition The *elementary CF f_A associated with the sequence $A = (a_1, ..., a_k)$* is given by the formula (where X is an ideal in P, $i = i(A, X)$, and $I(a_1, ..., a_i)$ is the ideal generated by $a_1, ..., a_i$)

$$f_A(X) = X \cap I(a_1, ..., a_i).$$

In particular, $f_\emptyset(X) = \emptyset$ for any X.

The intuitive meaning of the elementary CF f_A associated with the sequence $A = (a_1, ..., a_k)$ is the follows. We understand the sequence $(a_1, ..., a_k)$ as a hierarchy of goals of our decision-maker, and the importance of goal a_i decreases with the growth of i. The decision-maker tries to reach the most important goal a_1 first of all. If it is available, that is, if a_1 lies in the ideal X, he selects it (along with all smaller elements) and settles down (that is, completes the choice). If the goal a_1 is not available, it includes in the choice all elements of X which are less than a_1, and proceeds to achieve the next goal a_2. And so on.

This construction looks the simplest in the case when the poset P is trivial. In this case, the ideal X is just a subset of P, and a_i is the first (the most preferable) element of the sequence A that got into X. If none of $a_1, ..., a_k$ belongs to X, $f_A(X) = \emptyset$.

Proposition 1 *f_A is a conservative CF.*

Proof If the sequence A is empty, then $f_A = \emptyset$ and there is nothing to check.

Let us check the heredity. Suppose $Y \subseteq X$, and y belongs to both Y and $f_A(X)$. We need to show that $y \in f_A(Y)$. Let a_i be the first term of the sequence $A = (a_1, ..., a_k)$ that falls into X. The elements $a_1, ..., a_{i-1}$ do not belong to X, and therefore do not belong to Y. Hence $i' = i(A, Y) \leq i = i(A, X)$. By the construction $f_A(X)$, $y \in I(a_1, ..., a_i)$, that is $y \leq a_j$ for some $j = 1, ..., i$. But then $y \leq a_j$ for the same j, and therefore belongs to the ideal $I' = I(a_1, ..., a_{i'})$ and $f_A(Y) = Y \cap I'$.

Let us check the outcast. Let $f_A(X) \subseteq Y \subseteq X$. It is enough to check that $f_A(Y) \subseteq f_A(X)$. Let a_i be the first element of the sequence which lies in X. Then $a_1, ..., a_{i-1}$ do not fall in X, and more so in Y. As for a_i, it belongs to $f_A(X)$ and therefore belongs to Y. If $y \in f_A(Y)$, then y lies under some of $a_1, ..., a_i$. And since $y \in X$, then it belongs to $f_A(X)$. □

Definition A sequence $A = (a_1, ..., a_k)$ is *compatible with* a CF f if, for any i from 1 to k, $a_i \in f(P - F(a_1, ..., a_{i-1}))$. (For $i = 1$, this means $a_1 \in f(P)$. Recall that $F(a_1, ..., a_j)$ is the filter generated by $a_1, ..., a_j$.)

Proposition 2 *If a sequence A is compatible with a hereditary CF f then $f_A \subseteq f$.*

Proof Let X be an arbitrary ideal; we need to show that $f_A(X) \subseteq f(X)$. Recall how $f_A(X)$ was constructed. We find the first member a_i of the sequence $a_1, ..., a_k$ such that $a_i \in X$; then $f_A(X) = X \cap I(a_1, ..., a_i)$. Suppose that x is an element of $f_A(X)$ and $x \leq a_j$ for some $j \leq i$. Since $a_j \in f(P - F(a_1, ..., a_{j-1}))$ and $f(P - F(a_1, ..., a_{j-1}))$ is an ideal, we conclude that $x \in f(P - F(a_1, ..., a_{j-1}))$. On the other hand, the sets X and $F(a_1, ..., a_{j-1})$ do not intersect. Hence we have $x \in X \subset P - F(a_1, ..., a_{j-1})$, and the heredity of f implies that $x \in f(X)$. □

4 The Main Theorem in the Finite Case

Theorem 1 *Let P be a finite poset. Then any conservative CF on P is the union of some elementary CFs.*

Actually we shall prove stronger statement: *any conservative CF f is the union of elementary CFs associated with AC-sequences compatible with f*. Here a sequence $(a_1, ..., a_k)$ of elements of the poset is called an *antichain sequence* (AC-sequence) if all a_i are incomparable in P. In the case of trivial poset this assertion turns essentially into Aizerman-Malishevski theorem. Note that even for enough simple CFs akin the selection of two best items (see example 1 from Section 2) this decomposition may involve really a lot of elementary (linear) CFs.

Proposition 3 *Let f be a CF on finite poset P satisfying the outcast condition. Suppose that $x \in f(X)$ for some ideal X. Then there exists an AC-sequence A compatible with f such that $x \in f_A(X)$.*

Proof One can suppose that f is non-empty CF. We construct such a sequence A step by step.

Let us discuss the first step of the construction. If $x \in f(P)$, we put $a_1 = x$ and terminate the construction of the sequence. The definition of f_A shows that $x \in f_A(X)$.

So we can assume that x does not belong to $f(P)$. This is possible only if $f(P)$ is not contained in X. Indeed, otherwise $f(P) \subseteq X \subseteq P$ and from the outcast we get $f(P) = f(X)$ and x belongs to $f(P)$, contrary to the assumption. Hence, the set $f(P) - X$ is non-empty. We take a_1 to be a minimal element in the set $f(P) - X$, put $B_1 = P - F(a_1)$, and go to the second step. Note that $x \notin I(a_1)$.

At the k-th step, we have:

a) an AC-sequence $(a_1, ..., a_k)$,
b) x does not belong to the ideal $I(a_1, ..., a_k)$,
c) $X \subseteq B_k := P - F(a_1, ..., a_k)$ (where $B_0 = P$),
d) for any i from 1 to k, a_i is a minimal element of the set $f(B_{i-1}) - X$.

In particular, d) implies that the sequence $(a_1, ..., a_k)$ is compatible with f. As above, we consider two cases: when x belongs to $f(B_k)$ and when it doesn't.

If $x \in f(B_k)$, we set $a_{k+1} = x$ and terminate the construction of the sequence A. It is clear that $x \in f_A(X)$. Due to b) and c), a_{k+1} is not comparable with $a_1, ..., a_k$.

Now suppose that $x \notin f(B_k)$. If $f(B_k) \subseteq X$, then from c) and the outcast we have that $f(B_k) = f(X)$ and contains x; a contradiction. Hence $f(B_k)$ is not contained in X. Put a_{k+1} to be a minimal element of the non-empty set $f(B_k) - X$. We assert that *the extended sequence* $(a_1, ..., a_k, a_{k+1})$ *also satisfies the properties a)-d).*

Let us prove a). Since $a_{k+1} \in B_k$, $a_{k+1} \notin F(a_1, ..., a_k)$. Therefore we have to show that $a_{k+1} \notin I(a_1, ..., a_k)$. Suppose that $a_{k+1} \leq a_i$ for some $i \leq k$. Since $a_i \in f(B_{i-1})$, a_{k+1} belongs to $f(B_{i-1})$ as well and does not belong to X. Since a_i is a minimal element of $f(B_{i-1}) - X$ (due to d)), we obtain that $a_{k+1} = a_i$. But this is contrary to the fact that $a_{k+1} \in B_k$ and B_k (see c)) does not contain a_i.

b) We have to show that x does not lie under a_{k+1}. But if $x \leq a_{k+1}$, then x belongs to $f(B_k)$ due to the ideality of $f(B_k)$, which contradicts the assumption that $x \notin f(B_k)$.

Check c), that is $X \subseteq B_{k+1}$, or that X does not intersect with the filter $F(a_{k+1})$. If there is an $y \in X$ such that $y \geq a_{k+1}$, then by ideality of X the element a_{k+1} also belongs to X, which contradicts to the choice a_{k+1} outside X.

Finally, d) follows from the previous d) and the choice of a_{k+1}.

Since the poset P is finite, sooner or later the process ends, and we get an AC-sequence A such that $x \in f_A(X)$. □

Proof of Theorem 1. Let $g = \bigcup_A f_A$ where A runs AC-sequences compatible with conservative CF f. Due to Proposition 2, any $f_A \subseteq f$, hence $g \subseteq f$. Due to Proposition 3, for any $x \in f(X)$ we have $x \in f_A(X)$ for some A, where from $f \subseteq g$. □

5 Join-irreducibility of Elementary Choice Functions

We have shown that any conservative CF is represented as the union of several elementary CFs. Now we will show that these elementary blocks are 'simple' in the sense that they no longer decompose into a union of other conservative CFs. In other words, they are join-irreducible.

Lemma 1 *Let f and g be elementary CFs associated with AC-sequences $A = (a_1, ..., a_n)$ and $B = (b_1, ..., b_k)$ respectively, and $g \subseteq f$. Suppose, that $a_1 = b_1, ..., a_{i-1} = b_{i-1}$ but a_i is different from b_i. Then $b_i < a_i$ (or b_i is missing).*

Proof. We will assume that b_i is actually present, and consider the ideal $X = I(a_i, b_i)$. We state that $b_i \in g(X)$. If this is not the case, then b_i is not the first member of the sequence B, belonging to X; there is a smaller number $j < i$, such that $b_j \in X$. Since it is not it is true that $b_j \leq b_i$ (due to the incomparability of members of B), then $b_j \leq a_i$. But $b_j = a_j$, and we again get a contradiction with the incomparability of members of A.

Similarly a_i is the first member of the sequence A that falls into X. So $f(X) = I(a_1, ..., a_i)$. Therefore, $b_i \in I(a_1, ..., a_i)$. b_i cannot be less than $a_1 = b_1, ..., a_{i-1} = b_{i-1}$, because this would contradict the incomparability of members of the sequence B. So $b_i \leq a_i$. □

Proposition 4 *Let $f = f_A$ be an elementary CF associated with AC-sequence $A = (a_1, ..., a_n)$. If $f = g \cup h$, where g and h are conservative CFs, then g or h is equal to f.*

Proof Using Theorem 1, we decompose g and h into elementary CFs. As a result, we get the decomposition $f = g^1 \cup ... \cup g^l$, where each CF $g^j = f_{Bj}$ is an elementary CF associated with some AC-sequence $B^j = (b^j_1, ..., b^j_{k_j})$. We want to show, that at least one of these B^j is equal to A.

Consider one of these sequences $B = (b_1, ..., b_k)$. For some time, this sequence may coincide with the sequence A. Let $d(B)$ be the first index i for which b_i is different from a_i (for example, b_i is simply missing).

Assuming that all B^j are different from A, we get that all $d(B^j) \leq n$. Denote by d the maximum of the numbers $d(B^1), ..., d(B^l)$. We call the index j a 'leader' if $d(B^j) = d$.

Let us form the ideal $X = I^0(a_1) \cup ... \cup I^0(a_{d-1}) \cup I(a_d)$, where $I^0(a) = \{x \in P, x < a\} = I(a) - \{a\}$. It is clear that a_d is the first member of the sequence A that falls into X. Because a_i (with $i < d$) does not belong to $I^0(a_i)$, and also does not belong to other ideals $I^0(a_j)$ due to the incomparability with the rest a_j. Therefore, $f(X) = X$ and, in particular, $a_d \in f(X)$.

Let us now consider $g^j(X)$, where $j = 1, ..., l$; the corresponding sequence B^j is simply denoted as $B = (b_1, ..., b_k)$. Let i be the first number for which b_i falls into X. It can't be a number less than $d(B)$, because for such number j (less than i and so more than the smaller d) $b_j = a_j$ and does not belong to X (see Lemma 1). On the other hand, $b_{d(B)}$ is less than $a_{d(B)}$ (see Lemma 1), and therefore it belongs to X. If j is not a leader, then

$$g^j(X) = I^0(a_1) \cup ... \cup I^0(a_{d(B^j)})$$

(the last term is missing if $b_{d(B^j)}$ is missing). If j is a leader, then

$$g^j(X) = I^0(a_1) \cup ... \cup I^0(a_{d-1}) \cup I(b^j_d)$$

.

Since $b^j_d < a_d$ (by Lemma 1), we see that a_d does not belong to any ideal $g^j(X)$. In contradiction with $a_d \in f(X) = \cup_j g^j(X)$. □

6 The Main Theorem in the General Case

We assumed above that the poset P is finite. Now we will consider the general case (finite or infinite). Everything is done as in the finite case, only finite sequences need to be replaced by infinite ones, indexed by elements of well ordered sets (or ordinal numbers). Recall that a linearly ordered set $(I, \prec)$ is said to be *well ordered* if any non-empty subset in it has a minimal element. To be not confused, ideals in the set of indices I will be called *initial segments*. Note that a chain $(I, \prec)$ is well ordered if and only if any initial segment of I, other than the entire I, has the form $[\prec i] = \{j \in I, j \prec i\}$ for some $i \in I$.

A *well sequence* in P is a sequence $A = (a_i, i \in I)$ elements of the poset P, which set of indices I is well ordered. With such a sequence, one can associate the *elementary* CF f_A on the poset P, which is actually defined as in Section 3. Namely, let X be an ideal in P; denote $i = i(A, X)$ the first member of the sequence which belong to X. In other words, $a_i \in X$, and $a_j \notin X$ for $j \prec i$. Then $f_A(X)$ is equal to the intersection of X with the ideal in P generated by all a_j, $j \prec i$. That is

$$f_A(X) = X \cap I,$$

where $I = \bigcup_{j \prec i(A,X)} I(a_j)$. If there is no such number i, then the ideal I is generated by all a_j, $j \in I$.

It is easy to understand that we can confine ourselves by non-repeating sequences, when all a_i are different. In this case, one can consider A as a subset of P equipped with a well order $\prec$.

As in Proposition 1, it is checked that the CF f_A is conservative.

The definition of compatibility of a well sequence $A = (a_i, i \in I)$ with a CF f remains the same: for any $i \in I$, we have $a_i \in f(B_i)$, where $B_i = P - \cup_{j \prec i} F(a_j)$. And the same reasonings as in Proposition 2 show that $f_A \subseteq f$ if the sequence A is compatible with a hereditary CF f.

Theorem 2 *Any conservative CF f is the union of elementary CFs f_A, where A are well sequences compatible with f.*

As before, the most subtle part of the proof is the construction of well sequences compatible with f. The following reasoning resembles Zermelo's proof of that any set have a well order.

Namely, let $\mathcal{F}$ denote the set of filters F in P, for which $f(P - F) \neq \emptyset$. Let p be a 'selector' $p : \mathcal{F} \to P$, $p(F) \in f(P - F)$. By virtue of the axiom of choice, there are a lot of such selectors; fix someone.

Definition A *tunnel* is a subset U in P, equipped with a linear order $\prec_U$, which have the following property:

$(*)$ if V is an initial segment in U other than U then $V = [\prec_U x]$, where $x = p(F(V))$.

Recall that $F(V)$ denotes the filter in P generated by the set V. A tunnel U is called *through* one if $F(U)$ does not belong to $\mathcal{F}$, that is, if $f(P - F(U))$ is empty.

By virtue of $(*)$, the linear order $\prec_U$ of any tunnel U is well ordered. Therefore, a tunnel can be considered as a well sequence that is obviously compatible with CF f. It remains to show that there are quite a lot of tunnels. But first we need to say about the main property of tunnels.

Say that a tunnel U' *continues* a tunnel U, if $U \subseteq U'$ and U (as an ordered set) coincides with some initial segment U'. This define an order relation on the set of all tunnels. The basic property of tunnels is that this order is linear, i.e. any two tunnels are comparable, one of them continues the other.

Lemma 2 *Any two tunnels are comparable.*

Indeed, let U and V be two tunnels, and W be the largest initial segment in both tunnels, i.e. $W = \{x \in U \cap V,\ [\prec_U x] = [\prec_V x]$, and the restrictions $\prec_U$ and $\preceq_V$ on this set are the same$\}$.

We claim, that W is equal to U or V; this is exactly what should be proved.

Let us assume that this is not the case, and that W is different from U and V. Since W is an initial segment in U, then, by the property $(*)$, W has the form $[\prec_U x]$, where $x = p(F(W))$. Similarly, W has the form $[\prec_V y]$, where $y = p(F(W))$. So $x = y$. But then $x = y$ belongs to both U and V, hence belongs to W, despite the fact that $W = [\prec_U x]$. A contradiction. □

Corollary 1 *For any selector p there exists a unique through tunnel.*

Proof Take the union of all tunnels. □

The following assertion is a natural generalization of Proposition 3.

Proposition 5 *Let X be an ideal in P, and $x \in f(X)$. Then there is a through tunnel U such that $x \in f_U(X)$.*

Proof. For this aim we take a special selector p. Namely, assume that a filter F does not intersect the ideal X, that is $X \subseteq P - F$. Then there are two possible situations. The first one is when $f(P - F)$ is not contained in X; in this case, we choose $p(F)$ outside of X. The second one is $f(P - F) \subseteq X$; since $X \subseteq P - F$ then from the Outcast $f(P - F) = f(X)$ and contains x; in this case, we choose $p(F) = x$.

Now let U be the through tunnel for the selector p which exists due to Corollary. How does $f_U(X)$ look like? Let V be the largest initial segment of the 'sequence' U that does not intersect X. It can not be entire U, since then $F(U)$ does not intersect X, $X \subseteq P - F(U)$, and from the outcast property $f(X)$ is empty, contrary to $x \in f(X)$.

So $V = [\prec u]$ for $u = p(F(V))$ (see $(*)$). That is, u is the first element of the sequence U that belongs to X. According to the rule of p, this means that $u = x$. But in this case $x \in f_U(X)$, because x belongs to both X and the ideal $I(V \cup \{x\})$. □

The theorem 2 is now proved in the same way as the theorem 1.

Data Availability All data generated during this study are included in this article.

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