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MV-algebras and Partially Cyclically Ordered Groups

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Abstract
We prove that there exists a functorial correspondence between MV-algebras and partially cyclically ordered groups which are the wound-rounds of lattice-ordered groups. It follows that some results about cyclically ordered groups can be stated in terms of MV-algebras. For example, the study of groups together with a cyclic order allows to get a first-order characterization of groups of unimodular complex numbers and of finite cyclic groups. We deduce a characterization of pseudofinite MV-chains and of pseudo-simple MV-chains (i.e. which share the same first-order properties as some simple ones). We generalize these results to some non-linearly ordered MV-algebras, for example hyperarchimedean MV-algebras.

Keywords MV-algebras · MV-chains · Partially cyclically ordered abelian groups · Cyclically ordered abelian groups · Pseudofinite

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1 Introduction

This article has been written in such a way that it can be read by someone who does not have prior knowledge of cyclically ordered groups. We state the basic concepts of partially ordered groups that we require. We also list all the definitions and properties about MV-algebras and logic that we need. Whenever possible, in our proofs we try to use only properties of partially ordered groups, lattice-ordered groups or linearly ordered groups.

Unless otherwise stated all groups in this paper are abelian groups.

Every MV-algebra can be obtained in the following way. Let $(G, \leq, \wedge, u)$ be a lattice-ordered group (briefly ℓ-group) together with a distinguished strong unit $u > 0$ (i.e. for every $x \in G$ there is a positive integer n such that $x \leq nu$); such a group is called a unital ℓ-group. We set $[0, u] := \{x \in G \mid 0 \leq x \leq u\}$. For every x, y in $[0, u]$ we let $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$. We see that the restriction of the partial order $\leq$ to $[0, u]$ can be defined by $x \leq y \Leftrightarrow \exists z\ y = x \oplus z$.

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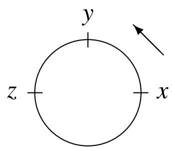
Now, the quotient group $C = G/\mathbb{Z}u$ can be equipped with a partial cyclic order. First, we explain what is a cyclic order. On a circle C, there is no canonical linear order, but there exists a canonical cyclic order. Assume that one traverses a circle counterclockwise. For every $x \neq y \neq z \neq x$, we declare that $R(x, y, z)$ holds if one can find x, y, z in this order starting from some point of the circle. Now, starting from another point one can find them in the order y, z, x or z, x, y. So in turn $R(y, z, x)$ and $R(z, x, y)$ hold. We say that R is cyclic. Furthermore, for any x in C the relation $y <_x z \Leftrightarrow R(x, y, z)$ is a linear order relation on the set $C\backslash\{x\}$.



These rules give the definition of a cyclic order. We generalize this definition to a (strict) partial cyclic order by assuming that $<_x$ is a partial order relation which needs not be a linear order.

Turning to the cyclic order $R(\cdot, \cdot, \cdot)$ on the quotient group $C = G/\mathbb{Z}u$ it is defined by setting $R(x_1+\mathbb{Z}u, x_2+\mathbb{Z}u, x_3+\mathbb{Z}u)$ if there exists n_2 and n_3 in $\mathbb{Z}$ such that $x_1 < x_2+n_2u < x_3+n_3u < x_1+u$ (see Proposition 3.5). One can prove that (C, R) satisfies for every x, y, z, v in C:

- $R(x, y, z) \Rightarrow x \neq y \neq z \neq x$ (R is strict)
- $R(x, y, z) \Rightarrow R(y, z, x)$ (R is cyclic)
- by setting $y \leq_x z$ if either $R(x, y, z)$ or $x \neq y = z$ or $x = y \neq z$ or $x = y = z$, then $\leq_x$ is a partial order relation on C
- $R(x, y, z) \Rightarrow R(x + v, y + v, z + v)$ (R is compatible)
- $R(x, y, z) \Rightarrow R(-z, -y, -x)$.

Any group equipped with a ternary relation which satisfies those properties is called a partially cyclically ordered group. If all the orders $\leq_x$ are linear orders, then (C, R) is called a cyclically ordered group. In the case where $C = G/\mathbb{Z}u$, where G is a partially ordered group, we say that C is the wound-round of a partially ordered group, or a wound-round.

If A is an MV-algebra, then there is a unital ℓ-group (G_A, u_A) (uniquely determined up to isomorphism) such that A is isomorphic to the MV algebra $[0, u_A]$. The group G_A is called the Chang ℓ-group of A. We show that the MV-algebra A is definable in the partially cyclically ordered group $(G_A/\mathbb{Z}u_A, R)$. It follows that there is a functorial correspondence between cyclically ordered groups and MV-chains (i.e. linearly ordered MV-algebras). So some properties of MV chains can be deduced from analogous properties of cyclically ordered groups.

In [8], D. Glusschankov constructed a functor between the category of projectable MV-algebras and the category of projectable lattice-ordered groups. The approach of the present paper is different, and it does not need to restrict to a subclass of MV-algebras.

In Section 2 we list basic properties of MV-algebras and of their Chang ℓ-groups. We also give a few basic notions of logic which we need in this paper. Section 3 starts with definitions and basic properties of partially cyclically ordered groups. We focus on those partially cyclically ordered groups which can be seen as wound-rounds of partially ordered groups or lattice-ordered groups. In particular, we show that there is a functor from the category of unital partially ordered groups to the category of their wound-rounds. The restriction of this functor to the linearly ordered groups gives rise to a full and faithful functor. In Section 4 we show that there is a functor $\Theta\Xi$ from the category of MV-algebras to the category of partially cyclically ordered groups together with c-homomorphisms. We define a class $\mathcal{AC}$ of partially cyclically ordered groups C in which we can define an MV-algebra $A(C) \cup \{\mathbf{1}\}$ (Theorem 4.11). Then we prove that the wound-rounds of ℓ-groups belong to $\mathcal{AC}$ (Theorem 4.16). Furthermore, the subgroup generated by $A(C)$ being the wound-round of an ℓ-group is a first-order property (Theorem 4.17). Section 5 is dedicated to MV-chains. In this case the one-to-one mapping $C \mapsto A(C)$ defines a functorial correspondence between the class of MV-chains and the class of cyclically ordered groups. We describe this functor. Next we prove that if A and A' are two MV-chains, then A and A' are elementarily equivalent if, and only if, $\Theta\Xi(A)$ and $\Theta\Xi(A')$ are elementarily equivalent (Proposition 5.2). We also prove that any two MV-chains are elementarily equivalent if, and only if, their Chang ℓ-groups are elementarily equivalent (Proposition 5.4). The class of pseudo-simple MV-chains is defined to be the elementary class generated by the simple chains. We define in the same way the pseudofinite MV-chains. One can prove that a pseudo-simple MV-chain is an MV-chain which is elementarily equivalent to some MV-subchain of $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$, and a pseudofinite MV-chain is an MV-chain which is elementarily equivalent to some ultraproduct of finite MV-chains. We use the results of [12] on cyclically ordered groups to deduce characterizations of pseudo-simple and of pseudofinite MV-chains. Furthermore, we get necessary and sufficient conditions for such MV-chains being elementarily equivalent (Theorems 5.8, 5.9). In Section 6, we generalize the results of Section 5 about pseudofinite and pseudo-simple MV-chains to pseudo-finite and pseudo-hyperarchimedean MV-algebras which are cartesian products of finitely many MV-chains (Theorems 6.4, 6.5).

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2 MV-algebras

The reader can find more properties for example in [3, Chapter 1].

2.1 Definitions and Basic Properties

Definition 2.1 An *MV-algebra* is a set A equipped with a binary operation $\oplus$, a unary operation $\neg$ and a distinguished constant 0 satisfying the following equations. For every x, y and z:

MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
MV2) $x \oplus y = y \oplus x$
MV3) $x \oplus 0 = x$
MV4) $\neg\neg x = x$
MV5) $x \oplus \neg 0 = \neg 0$
MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

If this holds, then we define $1 = \neg 0$, $x \odot y = \neg(\neg x \oplus \neg y)$ and $x \leq y \Leftrightarrow \exists z,\ x \oplus z = y$. Then $\leq$ is a partial order called the *natural order* on A. This partial order satisfies: $x \leq y \Leftrightarrow \neg y \leq \neg x$, and A is a distributive lattice with smallest element 0, greatest element 1, where $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg(x \oplus \neg y) \oplus \neg y) = (x \oplus \neg y) \odot y$. The operations $\wedge$

(infimum) and $\vee$ (supremum) are compatible with $\oplus$ and $\odot$. Note that Condition MV6) can be written as $x \vee y = y \vee x$. In the case where $\leq$ is a linear order, A is called an *MV-chain*.

Following [3], if n is a positive integer and x is an element of a group, then we denote by nx the sum $x + \cdots + x$ (n times). If x belongs to an MV-algebra, then we set $n.x = x \oplus \cdots \oplus x$ (n times). Further, we will set $x^n = x \odot \cdots \odot x$.

If G is a partially ordered group and $u \in G$, u is said to be a *strong unit* if $u > 0$ and for every $x \in G$ there is $n \in \mathbb{N}$ such that $x \leq nu$. It follows that there exists $n' \in \mathbb{N}$ such that $-x \leq n'u$, hence $-n'u \leq x \leq nu$. A *unital ℓ-group* is an ℓ-group (i.e. a lattice-ordered group) with distinguished strong unit $u > 0$. More generally, a *unital* partially (resp. linearly) ordered group is a partially (resp. linearly) ordered group together with a distinguished strong unit.

Example 2.2 If (G, u) is a unital ℓ-group, then the set $[0, u] := \{x \in M \mid 0 \leq x \leq u\}$ together with the operations $x \oplus y = (x + y) \wedge u$, $\neg x = u - x$, and where 0 is the identity element of G, is an MV-algebra, whose natural partial order is the restriction of the partial order on G. It is denoted $\Gamma(G, u)$. In this case, $x \odot y = (x + y - u) \vee 0$, and it follows: $(x \oplus y) + (x \odot y) = x + y$.

A *unital* homomorphism between two unital partially ordered groups (G, u) and (G', u') is an order-preserving group homomorphism f between G and G' such that $f(u) = u'$. An ℓ-homomorphism between two ℓ-groups G and G' is a group-homomorphism such that for every x, y in G we have that $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$ (it follows that f is also an order-preserving homomorphism).

A *homomorphism* of MV-algebras is a function f from an MV-algebra A to an MV-algebra A' such that $f(0) = 0$ and for every x, y in A, $f(x \oplus y) = f(x) \oplus f(y)$ and $f(\neg x) = \neg f(x)$.

The mapping $\Gamma : (G, u) \mapsto \Gamma(G, u)$ is a full and faithful functor from the category $\mathcal{A}$ of unital ℓ-groups to the category $\mathcal{MV}$ of MV-algebras. If f is a unital ℓ-homomorphism between (G, u) and (G', u'), then $\Gamma(f)$ is the restriction of f to $[0, u]$ [3, Chapter 7].

Now, for every MV-algebra A there exists a unital ℓ-group (G_A, u_A), uniquely defined up to isomorphism, such that A is isomorphic to $[0, u_A]$ together with above operations; (G_A, u_A) is called the *Chang ℓ-group* of A (see [3, Chapter 2]). We will sometimes let G_A stand for (G_A, u_A). For further purposes we describe this Chang ℓ-group in Section 2.2. The mapping Ξ: $A \mapsto (G_A, u_A)$ is a full and faithful functor from the category $\mathcal{MV}$ to the category $\mathcal{A}$. We do not describe here the unital ℓ-homomorphism $\Xi(f)$, where f is a homomorphism of MV-algebras. The composite functors $\Gamma\Xi$ and $\Xi\Gamma$ are naturally equivalent to the identities of the respective categories (see [3, Theorems 7.1.2 and 7.1.7]).

The language of MV-algebras is $L_{MV} = (0, \oplus, \neg)$. However, since $\leq$, $\wedge$ and $\vee$ are definable in L_{MV}, we will assume that they belong to the language. We will denote by $L_o = (0, +, -, \leq)$ the language of (partially) ordered groups, and by $L_{lo} = (0, +, -, \wedge, \vee)$ the language of ℓ-groups. When dealing with unital ℓ-groups, we will add a constant symbol to the language, which will then be denoted by $L_{lou} = L_{lo} \cup \{u\}$, and $L_{loZu} = L_{lo} \cup \{Zu\}$ will denote the language of ℓ-groups together with a unary predicate for the subgroup generated by u.

The Chang ℓ-group G_A of A is an L_{lou}-structure where u is a constant predicate interpreted by the distinguished strong unit of G_A. Now, G_A is also an L_{loZu}-structure where Zu is a unary predicate interpreted by: $Zu(x)$ if, and only if, x belongs to the subgroup generated by the distinguished strong unit; in this case we denote G_A by $(G_A, \mathbb{Z}u_A)$.

In the MV-algebra A, recall that $x \odot y$ stands for $\neg(\neg x \oplus \neg y)$. In $[0, u_A] \subset G_A$ we have that $x \odot y = (x + y - u_A) \vee 0$. If A is an MV-chain, then the formula $x > 0$ and $0 = x \odot x = \cdots = x^n$ is equivalent to: $0 < x < 2.x < \cdots < n.x \leq u_A$.

We conclude this subsection by stating a lemma which we will need in Section 4, in particular in the proof of Corollary 4.3.

Notations 2.3 In the following, if $x < y$ are elements of a partially ordered structure G, then we will set $[x, y] := \{z \in G \mid x \leq z \leq y\}$, $[x, y[:= [x, y]\backslash\{y\}$, $]x, y] := [x, y]\backslash\{x\}$ and $]x, y[:= [x, y]\backslash\{x, y\}$.

Lemma 2.4 *Let G be an ℓ-group and $0 < u \in G$.*

(1) *Either $[0, u] = \{0, u\}$,*
or there exists $x \in]0, u[$ such that $[0, u] = \{0, u, x\}$,
or there exists $x \in]0, u[$ such that $[0, u] = \{0, u, x, u - x\}$,
or for every $x \in]0, u[$ there exists $y \in]0, u[$ such that $x < y$ or $y < x$.
(2) *If $]0, u[$ contains x, y such that $x < y$, then for every $z \in]0, u[$ there exists $z' \in]0, u[$ such that $z < z'$ or $z' < z$.*

Proof As noted a referee, this lemma follows from a property of distributive lattices, which we prove for completeness. So we assume that G is a distributive lattice.

(1) If $[0, u]$ contains two or three elements, the property is trivial. If it contains four elements $0, x, y, u$, then either $[0, u]$ is linearly ordered, or $x \wedge y = 0$ and $x \vee y = u$. Now we assume that $[0, u]$ contains at least five elements, and we let $x \in]0, u[$. By hypotheses, there exists $x' \neq x''$ in $]0, u[\backslash\{x\}$. If $x \wedge x' = x$ or $x \vee x' = x$, then we let $y = x'$. If $x \wedge x'' = x$ or $x \vee x'' = x$, then we let $y = x''$. Otherwise, since every distributive lattice is cancellative (i.e. $(x \vee z' = x \vee z''\ \&\ x \wedge z' = x \wedge z'') \Rightarrow z' = z''$), we have $x \vee x' \neq x \vee x''$ or $x \wedge x' \neq x \wedge x''$. If $x \vee x' \neq x \vee x''$, then $x \vee x' \neq u$ or $x \vee x'' \neq u$. We set $y = x \vee x'$ or $y = x \vee x''$. The case $x \wedge x' \neq x \wedge x''$ is similar.
(2) This follows from (1). □

Corollary 2.5 *Let A be an MV-algebra. Then:*
either $A = \{0, 1\}$,
or there exists $x \in A\backslash\{0, 1\}$ such that $A = \{0, 1, x\}$,
or there exists $x \in A\backslash\{0, 1\}$ such that $A = \{0, 1, x, \neg x\}$,
or for every $x \in A\backslash\{0, 1\}$ there exists $y \in A\backslash\{0, 1\}$ such that $x < y$ or $y < x$.
If $A\backslash\{0, 1\}$ contains x, y such that $x < y$, then for every $z \in A\backslash\{0, 1\}$ there exists $z' \in A\backslash\{0, 1\}$ such that $z < z'$ or $z' < z$.

2.2 Construction of the Chang ℓ-group

The correspondence between MV-algebras and partially cyclically ordered groups relies on the so-called good sequences defined for the construction of the Chang ℓ-groups (see [13], [3, Chapter 2]). We describe this construction and we also define an analogue of the good sequences in an ℓ-group. We start with some properties of partially ordered groups.

Remark 2.6 We know that every cancellative abelian monoid M embeds canonically in a group G generated by the image of M following the construction of $\mathbb{Z}$ from $\mathbb{N}$. Now, one can deduce from properties of partially ordered groups (see for example [1], Propositions 1.1.2, 1.1.3, and also 1.2.5) that:

(1) M is the positive cone of a compatible partial order on G if, and only if, for every x, y in M, $x + y = 0 \Rightarrow x = y = 0$, and this partial order is given by $x \leq y \Leftrightarrow \exists z \in M,\ y = x + z$,
(2) G is an ℓ-group if, and only if, for every x, y in M, $x \wedge y$ exists.

The following lemmas show that every element of the positive cone of a unital ℓ-group can be associated to a unique sequence of elements of $[0, u]$. So G is determined by its restriction to $[0, u]$. This property will give rise to the construction of the Chang ℓ-group.

Lemma 2.7 *Assume that* (G, u) *is a unital* ℓ*-group. Let* $0 < x \in G$ *and* m *be a positive integer such that* $x \leq mu$. *Then, there exists a unique sequence* $x_1, \dots, x_n$ *of elements of* $[0, u]$ *such that* $x = x_1 + \cdots + x_n$ *and, for* $1 \leq i < n - 1$, $(u - x_i) \wedge (x_{i+1} + \cdots + x_n) = 0$, *and* $n \leq m$.

Proof For every $y \in G$, we have that $(u - y) \wedge (x - y) = 0 \Leftrightarrow (u \wedge x) - y = 0 \Leftrightarrow y = u \wedge x$. Set $x_1 = x \wedge u$. Then x_1 is the unique element of G such that $(u - x_1) \wedge (x - x_1) = 0$. Since $0 < x$, we have that $0 \leq x_1 \leq u$, and $0 \leq x - x_1 = x - (u \wedge x) = x + ((-u) \vee (-x)) = (x - u) \vee 0 \leq (mu - u) \vee 0 = (m - 1)u$. By taking $x - x_1$ in place of x we get $x_2 \in [0, u]$ such that $(u - x_2) \wedge (x - x_1 - x_2) = 0$, and we have that $x - x_1 - x_2 \in [0, (m - 2)u]$, and so on. Hence there exists a unique sequence $x_1, \dots, x_n$ of elements of $[0, u]$ such that $x = x_1 + \cdots + x_n$ and, for $1 \leq i < n - 1$, $(u - x_i) \wedge (x_{i+1} + \cdots + x_n) = 0$. □

Lemma 2.8 *The condition: for* $1 \leq i < n - 1$, $(u - x_i) \wedge (x_{i+1} + \cdots + x_n) = 0$ *is equivalent to: for* $1 \leq i < n - 1$, $(u - x_i) \wedge x_{i+1} = 0$. *If this holds, then, for* $1 \leq i < j \leq n$, $(u - x_i) \wedge x_j = 0$

Proof Assume that for $1 \leq i < n - 1$, $(u - x_i) \wedge (x_{i+1} + \cdots + x_n) = 0$. Let $i < j \leq n$. Since $0 \leq x_i$ and $0 \leq x_j \leq x_{i+1} + \cdots + x_n$, it follows that $(u - x_i) \wedge x_j = 0$. Now, let y, z, z' in $[0, u]$ such that $(u - y) \wedge z = (u - z) \wedge z' = 0$, then $0 \leq (u - y) \wedge z' = (u - y) \wedge z' \wedge u = (u - y) \wedge z' \wedge (z + u - z) \leq ((u - y) \wedge z' \wedge z) + ((u - y) \wedge z' \wedge (u - z)) = 0$. Hence by induction we can prove that the condition: for $1 \leq i < n - 1$, $(u - x_i) \wedge x_{i+1} = 0$ implies for $1 \leq i < n - 1$, $(u - x_i) \wedge (x_{i+1} + \cdots + x_n) = 0$. □

Remark 2.9 By setting, for x, y in $[0, u]$, $x \oplus y = (x + y) \wedge u$, we have that $x \odot y = 0 \vee (x + y - u)$. Hence, using the fact that for every z in G we have that $z = z \vee 0 + z \wedge 0$, we get: $x + y = (x \oplus y) + (x \odot y)$. We deduce from the proof of Lemma 2.7 that $x \oplus y$ is the unique element of $[0, u]$ such that $(u - x \oplus y) \wedge (x + y - x \oplus y) = 0$, and then $x + y - x \oplus y \in [0, u]$. It follows that $x = x \oplus y \Leftrightarrow (u - x) \wedge y = 0$. Furthermore, from the equality $x + y = (x \oplus y) + (x \odot y)$ we get $x \oplus y = x \Leftrightarrow x \odot y = y$.

Now, we come to the Chang ℓ-group.

Definition 2.10 Let A be an MV-algebra. A sequence (x_i) of elements of A indexed by the natural numbers $1,\ 2, \dots$ is said to be a *good sequence* if, for each i, $x_i \oplus x_{i+1} = x_i$, and it contains only a finite number of nonzero terms. If $x = (x_i)$ and $y = (y_i)$ are good sequences, then we define $z = x + y$ by the rules $z_1 = x_1 \oplus y_1$, $z_2 = x_2 \oplus (x_1 \odot y_1) \oplus y_2$, and more generally, for every positive integer i:

$$z_i = x_i \oplus (x_{i-1} \odot y_1) \oplus \cdots \oplus (x_1 \odot y_{i-1}) \oplus y_i.$$

We also define a partial order $\leq$ by $x \leq y \Leftrightarrow \exists z,\ y = x + z$.

We see that A embeds into the monoid M_A of good sequences by $x \mapsto (x, 0, 0, \dots)$, and one can prove that M_A is cancellative, it satisfies the properties 1) and 2) of Remark 2.6, where:
$x \wedge y = (x_i \wedge y_i)$, $x \vee y = (x_i \vee y_i)$, and
if $y = x + z$, then $z = (y_i) + (\neg x_n, \neg x_{n-1}, \dots, \neg x_1, 0, \dots)$, where x_n is the last nonzero term of (x_i) (see [3, Chapter 2]).

Consequently, M_A defines in a unique way an ℓ-group, and the image of $(1, 0, \dots)$ in this ℓ-group is a strong unit. This unital ℓ-group is the Chang ℓ-group G_A.

The good sequence defining some $x = (x_1, \dots, x_n, 0, \dots)$ of the positive cone of G_A is the same as the sequence defined in Lemmas 2.7 and 2.8, as shows the next proposition.

Proposition 2.11 *Let $x = (x_1, \dots, x_n, 0, \dots)$ in the positive cone of G_A. Then $x = (x_1, 0, \dots) + \cdots + (x_n, 0, \dots)$. The embedding $x_i \mapsto (x_i, 0, \dots)$ of A in G_A can be considered as an inclusion. Hence we can assume that $x_i \in G_A$ and write $x = x_1 + \cdots + x_n$.*

Proof Let (x_i) be a good sequence, and for $k \geq 1$ let $y = (x_1, \dots, x_k, 0, \dots) + (x_{k+1}, 0, \dots)$. By Lemma 2.8 and Remark 2.9, we have, for $1 \leq i < j \leq k+1$, $x_i \oplus x_j = x_i$ and $x_i \odot x_j = x_j$. It follows that
$y_1 = x_1 \oplus x_{k+1} = x_1$,
for $2 \leq i \leq n$, $y_i = x_i \oplus (x_{i-1} \odot x_{k+1} 1) \oplus \cdots \oplus (x_1 \odot 0) \oplus 0 = x_i \oplus x_{k+1} = x_i$,
$y_{k+1} = 0 \oplus (x_n \odot x_{k+1}) \oplus \cdots \oplus (x_1 \odot 0) \oplus 0 = 0 \oplus x_{k+1} \oplus 0 = x_{k+1}$,
and for $i > k + 1$, $y_i = 0$.

So by induction we get $(x_1, \dots, x_n, 0, \dots) = (x_1, 0 \dots) + \cdots + (x_n, 0, \dots)$. The remainder of the proof is straightforward. □

2.3 Elementary Equivalence, Interpretability

Two structures S and S' for a language L are *elementarily equivalent* if any L-sentence is true in S if, and only if, it is true in S'. We let $S \equiv S'$ stand for S and S' being elementarily equivalent. Furthermore if $S \subset S'$, then we say that S is an *elementary substructure* of S' (briefly $S \prec S'$) if every existential formula with parameters in S which is true in S' is also true in S. We will need the following properties.

Theorem 2.12 ([9] Corollary 9.6.5 on p. 462, see also [4] Theorems 5.1, 5.2) *Let L be a first-order language.*

(1) *If I is a nonempty set and for each $i \in I$, A_i and B_i are elementarily equivalent L-structures, then $\prod_{i \in I} A_i \equiv \prod_{i \in I} B_i$ (here $\prod$ denotes the direct product).*
(2) *If I is a nonempty set and for each $i \in I$, A_i and B_i are L-structures with $A_i \prec B_i$, then $\prod_{i \in I} A_i \prec \prod_{i \in I} B_i$.*

If, for every $i \in I$, S_i is an L-structure and U is an ultrafilter on I, then the *ultraproduct* of the S_i's is the quotient set $\left(\prod_{i \in I} S_i\right) / \sim$, where $\sim$ is the equivalence relation: $(x_i) \sim (y_i) \Leftrightarrow \{i \in I \mid x_i = y_i\} \in U$. If R is a unary predicate of L, then $R((x_i))$ holds in the ultraproduct if the set $\{i \in I \mid R(x_i) \text{ holds in } S_i\}$ belongs to U. Every relation symbol and every function symbol is interpreted in the same way. An *elementary class* is a class which is closed under ultraproducts.

A structure S_1 for a language L_1 is interpretable in a structure S_2 for a language L_2 if the following holds.

- There is a one-to-one mapping φ from a subset T_1 of S_2 onto S_1,
- for every L_1-formula Φ of the form $R(\bar{x})$, $F(\bar{x}) = y$, $x = y$ or $x = c$ (where R is a relation symbol, F is a function symbol and c is a constant), there is an L_2-formula Φ' such that for every $\bar{x}$ in T_1, $S_1 \models \Phi(\varphi(\bar{x})) \Leftrightarrow S_2 \models \Phi'(\bar{x})$,

(this is a particular case of the definition of interpretability p. 58 and pp. 212-214 in [9]).

Theorem 2.13 (Reduction Theorem 5.3.2, [9]) *If S_1, S_1' (resp. S_2, S_2') are structures for the language L_1 (resp. L_2) such that S_1 is interpretable in S_2 and S_1' is interpretable in S_2' by the same rules, then $S_2 \equiv S_2' \Rightarrow S_1 \equiv S_1'$ and $S_2 \prec S_2' \Rightarrow S_1 \prec S_1'$.*

Since $A = [0, u]$, $a \oplus b = (a + b) \wedge u$, $\neg a = u - a$, it follows that the L_{MV}-structure A is interpretable in the L_{lou}-structure (G_A, u_A) and in the L_{loZu}-structure $(G_A, \mathbb{Z}u_A)$. Consequently, if A, A' are MV-algebras such that $(G_A, u_A) \equiv (G_{A'}, u_{A'})$ (resp. $(G_A, \mathbb{Z}u_A) \equiv (G_{A'}, \mathbb{Z}u_{A'})$), then $A \equiv A'$. The same holds with $\prec$ instead of $\equiv$.

3 Partially Cyclically Ordered Groups

In this section we state definitions and basic properties of partially cyclically ordered groups and cyclically ordered groups. Then we turn to the wound-round partially cyclically ordered groups, in particular we deduce a transfer theorem of elementary equivalence. Next we study the functor between partially ordered groups and partially cyclically ordered groups.

Recall that all the groups are assumed to be abelian groups.

3.1 Basic Properties

Definition 3.1 We say that a group C is *partially cyclically ordered* (briefly a *pco-group*) if it is equipped with a ternary relation R which satisfies (1), (2), (3), (4), (5) below.

(1) R is strict i.e., for every x, y, z in C: $R(x, y, z) \Rightarrow x \neq y \neq z \neq x$.
(2) R is cyclic i.e., for every x, y, z in C: $R(x, y, z) \Rightarrow R(y, z, x)$.
(3) For every x, y, z in C set $y \leq_x z$ if either $R(x, y, z)$ or $y = z$ or $y = x$. Then for every x in C, $\leq_x$ is a partial order relation on C. We set $y <_x z$ for $y \leq_x z$ and $y \neq z$. If y and z admit an infimum (resp. a supremum) in $(C, \leq_x)$, then it will be denoted by $y \wedge_x z$ (resp. $y \vee_x z$).
(4) R is compatible, i.e., for every x, y, z, v in C, $R(x, y, z) \Rightarrow R(x + v, y + v, z + v)$.
(5) For every x, y, z in C, $R(x, y, z) \Rightarrow R(-z, -y, -x)$.

If for every $x \in C$ the order $\leq_x$ is a linear order, then we say that C is a *cyclically ordered group* (briefly a *co-group*).

The language $(0, +, -, R)$ of pco-groups will be denoted by L_c.

A *c-homomorphism* is a group homomorphism f between two pco-groups (or co-groups) such that for every x, y, z, if $R(x, y, z)$ holds and $f(x) \neq f(y) \neq f(z) \neq f(x)$, then $R(f(x), f(y), f(z))$ holds.

Two basic examples of co-groups are the following.

(1) Let $\mathbb{U}$ be the multiplicative group of unimodular complex numbers. For $e^{i\theta_j}$ $(1 \leq j \leq 3)$ in $\mathbb{U}$, such that $0 \leq \theta_j < 2\pi$, we let $R(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})$ if, and only if, either $\theta_1 < \theta_2 < \theta_3$ or $\theta_2 < \theta_3 < \theta_1$ or $\theta_3 < \theta_1 < \theta_2$ (in other words, when one traverses the unit circle counterclockwise, starting from $e^{i\theta_1}$ one finds first $e^{i\theta_2}$ then $e^{i\theta_3}$). Then $\mathbb{U}$ is a co-group. One sees that the group Tor($\mathbb{U}$) of torsion elements of $\mathbb{U}$ (that is, the roots of 1 in the field of complex numbers) is a co-subgroup.

(2) Any linearly ordered group is a co-group once equipped with the ternary relation: $R(x, y, z)$ iff $x < y < z$ or $y < z < x$ or $z < x < y$. Such a cyclically ordered is called a *linearly cyclically ordered group*. In the same way, any partially ordered group is a pco-group.

Now, a pco-group is not necessarily a co-group. Let (C_1, R_1) and (C_2, R_2) be nontrivial co-groups and $C = C_1 \times C_2$ be their cartesian product. For (x_1, x_2), (y_1, y_2) and (z_1, z_2) in C, set $R((x_1, x_2), (y_1, y_2), (z_1, z_2))$ if, and only if, $R_1(x_1, y_1, z_1)$ and $R_2(x_2, y_2, z_2)$. Then (C, R) is a pco-group which is not a co-group.

For notational convenience, if C is a pco-group and $x_1, \dots, x_n$ belong to C, we sometimes denote by $R(x_1, \dots, x_n)$ the formula: $R(x_1, x_2, x_3)$ & $R(x_1, x_3, x_4)$ & ... & $R(x_1, x_{n-1}, x_n)$. In the unit circle $\mathbb{U}$, $R(x_1, \dots, x_n)$ means that starting from x_1 one finds the elements $x_2, \dots, x_n$ in this order (this generalizes the notation $x_1 < x_2 < \cdots < x_n$ of the language of ordered groups).

Lemma 3.2 *Let C be a pco-group, $n \geq 3$ and $x_1, \dots, x_n$ in C.*

(1) $R(x_1, \dots, x_n) \Leftrightarrow \forall (i, j, k) \in [1, n] \times [1, n] \times [1, n],\ 1 \leq i < j < k \leq n \Rightarrow R(x_i, x_j, x_k)$.
(2) $\forall y \in C,\ R(x_1, \dots, x_n) \Leftrightarrow R(x_1 + y, \dots, x_n + y)$.
(3) $\forall i \in [1, n-1],\ R(x_1, \dots, x_n) \Leftrightarrow R(x_{i+1}, \dots, x_n, x_1, \dots, x_i)$.

Proof (1) $\Leftarrow$ is straightforward. Assume that $R(x_1, \dots, x_n)$ holds and let $1 \leq i < j < k \leq n$. Then $R(x_1, x_i, x_{i+1})$ and $R(x_1, x_{i+1}, x_{i+2})$ hold. Therefore, since $<_{x_1}$ is transitive, $R(x_1, x_i, x_{i+2})$ holds, and so on. Hence $R(x_1, x_i, x_j)$ holds, and in the same way $R(x_1, x_j, x_k)$ holds. It follows that $R(x_j, x_1, x_i)$ and $R(x_j, x_k, x_1)$ hold. Hence $R(x_j, x_k, x_i)$ holds which implies that $R(x_i, x_j, x_k)$ holds.

(2) For every i in $[2, n-1]$, $R(x_1, x_i, x_{i+1})$ holds. Hence $R(x_1 + y, x_i + y, x_{i+1} + y)$ holds. Consequently, $R(x_1 + y, \dots, x_n + y)$ also holds.

(3) Assume that $R(x_1, \dots, x_n)$ holds. We have that $R(x_1, x_2, x_n)$ holds and by (1) for every i in $[3, n-1]$: $R(x_2, x_i, x_{i+1})$ holds. Therefore $R(x_2, \dots, x_n, x_1)$ holds. Now, (3) follows by induction.

□

We conclude this subsection with a characterization of the order relation $<_0$. That is, if C is a pco-group, then by the definition $<_0$ is a partial order on the set C. Conversely, if $<$ is a partial order on an abelian group, one can wonder under what conditions $<$ is the order $<_0$ of some partial cyclic order on this group.

This characterization will be the main tool of the proof of Proposition 3.5.

Proposition 3.3 *Let C be a group. Then there exists a compatible partial cyclic order R on C if, and only if, there exists a partial order $<$ on the set $C\backslash\{0\}$ such that for all x and y in $C\backslash\{0\}$: $x < y \Rightarrow (y - x < -x \;\&\; -y < -x)$.*

If this holds, then we can set $R(x, y, z) \Leftrightarrow 0 \neq y - x < z - x$, and $\leq$ is the restriction to $C\backslash\{0\}$ of the relation $\leq_0$.

Proof Assume that C is a pco-group, and $x <_0 y$ in $C\backslash\{0\}$. Then $R(0, x, y)$ holds. Hence by compatibility: $R(-x, 0, y - x)$ holds so $R(0, y - x, -x)$ holds i.e. $y - x <_0 -x$. Furthermore, we have $R(-y, -x, 0)$, hence $R(0, -y, -x)$ holds, which implies $-y <_0 -x$.

Assume that $<$ is a strict partial order on $C\backslash\{0\}$ such that for all x and y in $C\backslash\{0\}$ we have that $x < y \Leftrightarrow y - x < -x$. For all x, y, z in C set $R(x, y, z)$ if, and only if, $0 \neq y - x < z - x$.

We prove that R satisfies (1), (2), (3), (4), (5) of Definition 3.1.

(1) $R(x, y, z)$ implies $y - x \neq 0$, $z - x \neq 0$ and $y - x \neq z - x$ hence $x \neq y$, $x \neq z$ and $y \neq z$.
(2) Assume that $R(x, y, z)$ holds. This implies $y - x < z - x$, hence $z - y = (z - x) - (y - x) < -(y - x) = x - y$. Therefore: $R(y, z, x)$.
(3) Let $v \in C$ and assume that $R(x, y, z)$ and $R(x, z, v)$ hold. Then $y - x < z - x$ and $z - x < v - x$, hence: $y - x < v - x$ i.e. $R(x, y, v)$, so $\leq_x$ is transitive. Now, if $R(x, y, z)$ holds, then $y - x < z - x$, hence $z - x \not< y - x$, i.e. $\neg R(x, z, y)$. It follows that $\leq_x$ is a partial order.
(4) Assume that $R(x, y, z)$ holds. Then $0 \neq y - x < z - x$ hence $0 \neq (y + v) - (x + v) < (z + v) - (x + v)$. Therefore $R(x + v, y + v, z + v)$.
(5) The relation $R(x, y, z)$ is equivalent to $y - x < z - x$, hence it implies $x - z < x - y$. This in turn is equivalent to $-z - (-x) < -y - (-x)$, that is $R(-x, -z, -y)$. It follows that $R(x, y, z)$ implies $R(-z, -y, -x)$. □

The relation $<_0$ can make the construction of pco-groups easier. For example, let $C = \mathbb{Z}/6\mathbb{Z} = \{0, 1, 2, 3, -2, -1\}$, and set $1 <_0 2$, $1 <_0 -1$, $-2 <_0 2$, and $-2 <_0 -1$. One can check that in this case, $<_0$ cannot be extended to a total order.

3.2 Wound-round pco-groups

First we look at the case of co-groups, starting with a basic example.

In the field $\mathbb{C}$ of complex numbers, the multiplicative group $\mathbb{U}$ of unimodular complex numbers is the image of the additive group $\mathbb{R}$ of real numbers under the epimorphism $\theta \mapsto e^{i\theta}$. It follows that $\mathbb{U}$ is isomorphic to the quotient group $\mathbb{R}/2\pi\mathbb{Z}$. Then one can define the cyclic order on $\mathbb{R}/2\pi\mathbb{Z}$ by: $R(x_1 + 2\pi\mathbb{Z}, x_2 + 2\pi\mathbb{Z}, x_3 + 2\pi\mathbb{Z})$ if, and only if, there exists x'_j in $[0, 2\pi[$ such that $x_j - x'_j \in 2\pi\mathbb{Z}$ $(1 \leq j \leq 3)$ and $x'_{\sigma(1)} < x'_{\sigma(2)} < x'_{\sigma(3)}$ for some σ in the alternating group A_3 of degree 3 (in other words, $x'_1 < x'_2 < x'_3$ or $x'_2 < x'_3 < x'_1$ or $x'_3 < x'_1 < x'_2$).

More generally, if (L, u) is a unital linearly ordered group, then the quotient group $L/\mathbb{Z}u$ can be cyclically ordered by setting $R(x_1 + \mathbb{Z}u, x_2 + \mathbb{Z}u, x_3 + \mathbb{Z}u)$ if, and only if, there exists x'_j in $[0, u[$ such that $x_j - x'_j \in \mathbb{Z}u$ $(1 \leq j \leq 3)$ and $x'_{\sigma(1)} < x'_{\sigma(2)} < x'_{\sigma(3)}$ for some σ in the alternating group A_3 of degree 3 [5, p. 63]. We say that $L/\mathbb{Z}u$ is the *wound-round* of L. Now, every co-group can be obtained in this way as shows the following theorem.

Theorem 3.4 (Rieger, [5]) *Every co-group is the wound-round of a unique (up to isomorphism) unital linearly ordered group* $(uw(C), u_C)$.

The ordered group $uw(C)$ defined above is called the *unwound* of C.

Now, we generalize this winding construction to partially ordered groups. First, note that if (G, u) is a unital linearly ordered group, then for every x in G there is a unique x' in $[0, u[$ such that $x - x' \in \mathbb{Z}u$. So, one can check that the condition "there exists x'_j in $[0, u[$ such that $x_j - x'_j \in \mathbb{Z}u$ $(1 \leq j \leq 3)$ and $x'_{\sigma(1)} < x'_{\sigma(2)} < x'_{\sigma(3)}$ for some σ in the alternating group A_3 of degree 3" is equivalent to "there exist n_2 and n_3 in $\mathbb{Z}$ such that $x_1 < x_2 + n_2u < x_3 + n_3u < x_1 + u$".

Proposition 3.5 *Let* $(G, <)$ *be a partially ordered group,* $0 < u \in G$*,* C *be the quotient group* $C = G/\mathbb{Z}u$ *and* ρ *be the canonical mapping from G onto* C*.*

(1) *For every x and y in G, there exists at most one* $n \in \mathbb{Z}$ *such that* $x < y + nu < x + u$*.*
(2) *For every* x_1, x_2, x_3 *in G, set* $R(\rho(x_1), \rho(x_2), \rho(x_3))$ *if, and only if, there exist* n_2 *and* n_3 *in* $\mathbb{Z}$ *such that* $x_1 < x_2 + n_2u < x_3 + n_3u < x_1 + u$*. Then* (C, R) *is a pco-group.*

Proof (1) Assume that n and n' are integers such that $x < y + nu < x + u$ and $x < y + n'u < x + u$. Then $-x - u < -y - n'u < -x$. Therefore, by addition, $-u < (n - n')u < u$, hence $n - n' = 0$.

(2) Assume that $x_1 < x_2 + n_2u < x_3 + n_3u < x_1 + u$, let x'_1, x'_2, x'_3 in G such that $\rho(x'_i) = \rho(x_i)$ $(i \in \{1, 2, 3\})$, and let n'_1, n'_2, n'_3 be the integers such that $x'_i = x_i + n_iu$ $(i \in \{1, 2, 3\})$. Then $x'_1 - n'_1u < x'_2 - n'_2u + n_2u < x'_3 - n'_3u + n_3u < x'_1 - n'_1u$. Hence $x'_1 < x'_2 + (n_2 + n'_1 - n'_2)u < x'_3 + (n_3 + n'_1 - n'_3)u < x'_1 + u$. So R is indeed a ternary relation on C.

We set $\rho(0) <_0 \rho(x) <_0 \rho(y) \Leftrightarrow R(\rho(0), \rho(x), \rho(y))$, and we prove that $<_0$ is a strict partial order relation such that $\rho(0) <_0 \rho(x) <_0 \rho(y) \Rightarrow \rho(y) - \rho(x) <_0 -\rho(x)$ & $-\rho(y) <_0 -\rho(x)$.

By the definition, $\rho(0) <_0 \rho(x) <_0 \rho(y)$ holds iff there exist m and n in $\mathbb{Z}$ such that $0 < x + mu < y + nu < u$. Trivially, $<_0$ is anti-reflexive and it follows from (1) that it is anti-symmetric. The transitivity is also trivial. Since $x + mu < y + nu < u$, we have that $0 < y - x + (n - m)u < -x + (1 - m)u$. Now, we have that $-u < -y - nu < -x - mu < 0$, hence $0 < -y + (1 - n)u < -x + (1 - m)u < u$. On the one hand, this proves $-\rho(y) <_0 -\rho(x)$. On the other hand, this completes the inequality: $0 < y - x + (n - m)u < -x + (1 - m)u < u$, and consequently $\rho(y) - \rho(x) <_0 -\rho(x)$. Now, by Proposition 3.3, $G/\mathbb{Z}u$ is a pco-group. □

Definition 3.6 Let C be a pco-group. We say that C is a *wound-round* if there exists a unital partially ordered group (G, u) such that $C \simeq G/\mathbb{Z}u$, partially cyclically ordered as in Proposition 3.5. If G is an ℓ-group, then we say that C is the *wound-round of an* ℓ*-group*. If (G, u) is uniquely defined (up to isomorphism), then it is called the *unwound* of C.

For example, let G_1 be the lexicographically ordered group $\mathbb{R} \overrightarrow{\times} \mathbb{R}$, and G_2 be the group $\mathbb{R} \times \mathbb{R}$ partially ordered in the following way: $(x, y) \leq (x', y') \Leftrightarrow x = x'$ and $y \leq y'$. In G_1 and G_2, let $u = (0, 1)$. Then $G_1/\mathbb{Z}u \simeq G_2/\mathbb{Z}u$. They are also c-isomorphic to pco-group $\mathbb{R} \times (\mathbb{R}/\mathbb{Z})$ equipped with the partial cyclic order R' defined by setting $R'((x_1, y_1), (x_2, y_2), (x_3, y_3))$ if, and only if, $x_1 = x_2 = x_3$ and $R(y_1, y_2, y_3)$, where R is the cyclic order of $\mathbb{R}/\mathbb{Z}$ defined in Proposition 3.5.

If C is the wound-round of a partially ordered group G together with $0 < u \in G$, then we can assume that u is a strong unit. Indeed, one can easily check that the subset $H := \{x \in G \mid \exists (m, n) \in \mathbb{Z} \times \mathbb{Z},\ mu \leq x \leq nu\}$ is a subgroup of G, and u is a strong unit of H. Now, if there exists y such that $R(\rho(0), \rho(x), \rho(y))$ or $R(\rho(0), \rho(y), \rho(x))$ holds, then there exists $n \in \mathbb{Z}$ such that $0 < x + nu < u$. In particular, $-nu < x < (1-n)u$. Hence, $x \in H$. It follows that we can restrict ourselves to the subgroup $H/\mathbb{Z}u$, or assume that u is a strong unit of G.

In the remainder of this paper, we will always assume that u is a strong unit.

The construction in Proposition 3.5 gives rise to a transfer principle of elementary equivalence. In the linearly ordered case, [6, Theorem 4.1] proves that if (G, u) and (G', u') are unital linearly ordered groups, then we have $(G, u) \equiv (G', u') \Leftrightarrow G/\mathbb{Z}u \equiv G'/\mathbb{Z}u'$ and $(G, u) \prec (G', u') \Leftrightarrow G/\mathbb{Z}u \prec G'/\mathbb{Z}u'$.

The implication $\Rightarrow$ uses the fact that the restriction of the canonical epimorphism ρ: $G \to C = G/\mathbb{Z}u$ to $[0, u[$ is a one-to-one mapping. In the case where (G, u) is a unital partially ordered group, one can also define a subset G_u such that the restriction of ρ to G_u is one-to-one.

Lemma 3.7 *Let (G, u) be a unital partially ordered group, C be the quotient group $G/\mathbb{Z}u$, ρ be the canonical mapping from G onto C and $G_u = \{x \in G \mid x \geq 0\ \&\ x \not\geq u\}$. Then:*

- *the restriction of ρ to the subset G_u is a one-to-one mapping onto C,*
- *for every x, y, z in G_u, $\rho(x)+\rho(y) = \rho(z) \Leftrightarrow x+y-z \in \mathbb{Z}u$ and $R(\rho(x), \rho(y), \rho(z))$ holds if, and only if, either $x < y < z$ or $y < z < x$ or $z < x < y$.*

Proof Since u is a strong unit, for every $x \in G$ there exist integers m and n such that $mu \leq x < nu$, and we can assume that m is maximal. Then $x - mu \in G_u$, hence the restriction of ρ to G_u is onto. Let x and y in G_u such that $\rho(x) = \rho(y)$, then $x - y \in \mathbb{Z}u$. Hence there exists an integer m such that $y = x + mu$, and without loss of generality we can assume that $m \geq 0$. If $m \geq 1$, then $y \geq x+u \geq u$: a contradiction. Hence $m = 0$, and $y = x$. So the restriction of ρ to G_u is one-to-one. The remainder of the proof is straightforward using properties of Section 2.3. □

This lemma implies sufficient conditions for two wound-rounds being elementarily equivalent.

Theorem 3.8 *Let (G, u) be a unital partially ordered group, C be the quotient group $C = G/\mathbb{Z}u$, ρ be the canonical mapping from G onto C and $G_u = \{x \in G \mid x \geq 0\ \&\ x \not\geq u\}$. Then:*

- *the pco-group C, in the language L_c, is interpretable in $(G, \mathbb{Z}u)$ in the language L_{loZu},*
- *if (G', u') is a unital partially ordered group, then $(G, \mathbb{Z}u) \equiv (G', \mathbb{Z}u') \Rightarrow G/\mathbb{Z}u \equiv G'/\mathbb{Z}u'$ (the same holds with $\prec$ instead of $\equiv$).*

Proof Follows from Lemma 3.7 and properties of Section 2.3. □

3.3 The Wound-round Functor

We show that the wound-round mapping gives rise to a functor. Then we prove that its restriction to the co-groups is full and faithful. Next, we look at the case of ℓ-groups.

3.3.1 General Case

Proposition 3.9 *The wound-round mapping defines a functor Θ from the category of unital partially ordered groups, together with unital order-preserving group homomorphisms, to the category of pco-groups, together with c-homomorphisms.*

Proof We prove that if f is a unital order-preserving homomorphism between the unital partially ordered groups (G, u) and (G', u'), then we can define a c-homomorphism between $C := G/\mathbb{Z}u$ and $C' := G'/\mathbb{Z}u'$. Let ρ (resp. ρ') be the canonical epimorphism from G onto C (resp. from G' onto C'). Since $f(\mathbb{Z}u) = \mathbb{Z}u'$, we can define a group homomorphism $\bar{f}$ between C and C' by setting for every $x \in G$ $\bar{f}(\rho(x)) = \rho'(f(x))$. Let $x < y < z$ in G such that $\bar{f}(\rho(x)) \neq \bar{f}(\rho(y)) \neq \bar{f}(\rho(z)) \neq \bar{f}(\rho(x))$. Since f is order-preserving, we have that $f(x) \leq f(y) \leq f(z)$. Now, $f(x) \neq f(y) \neq f(z)$, so we have that $f(x) < f(y) < f(z)$. We deduce that if $R(\rho(x), \rho(y), \rho(z))$ holds and $\bar{f}(\rho(x)) \neq \bar{f}(\rho(y)) \neq \bar{f}(\rho(z)) \neq \bar{f}(\rho(x))$, then $R(\bar{f}(\rho(x)), \bar{f}(\rho(y)), \bar{f}(\rho(z))$ holds. Hence f is a c-homomorphism. Now, one can check that if $f \circ g$ is the composite of two unital order-preserving homomorphisms, then $\bar{f} \circ \bar{g} = \overline{f \circ g}$. □

3.3.2 Linearly Ordered Groups

First we show that a c-homomorphism is not necessarily the image of a unital order-preserving homomorphism as in the proof of Proposition 3.9. Then we will define the w-homomorphisms and we prove that the wound round functor between unital linearly ordered groups, together with unital order preserving homomorphisms, and the co-groups, together with the w-homomorphisms, is full and faithful.

We know that the kernels of order-preserving homomorphisms between partially ordered groups are the convex subgroups (see, for, example, [5, Theorem 7, p. 21]). Let f be a unital order-preserving homomorphism between partially ordered groups. Since $f(u) = u'$, f is not the zero homomorphism. Hence $\ker f$ is a proper convex subgroup. Consequently, $\ker f$ is the ρ-image of a proper convex subgroup of G. The kernel of a c-homomorphism between wound-rounds is not necessarily the ρ-image of a proper convex subgroup. For example, let $n \geq 2$, $G = G' = \mathbb{Z}$, $u = 2n$, $u' = 2$. Then $C = \mathbb{Z}/2n\mathbb{Z}$ and $C' = \mathbb{Z}/2\mathbb{Z}$. Consider the homomorphism which sends the class of 1 modulo $2n$ to the class of 1 modulo 2. Since $C' = \mathbb{Z}/2\mathbb{Z}$ is trivially cyclically ordered, this a c-homomorphism, but its kernel is not the image of a proper convex subgroup of $\mathbb{Z}$ (since $\{0\}$ is the only proper convex subgroup of $\mathbb{Z}$).

In the linearly ordered case, we can characterize the c-homomorphisms f such that the kernel of f is the ρ-image of a proper convex subgroup. For this purpose, we need to define the linear part of a co-group, its winding part, its c-convex subgroups, and to state some properties of co-groups, which we will use only in the remainder of this subsection.

- Linear part. Let (G, u) be a linearly ordered group with a strong unit, C be the cyclically ordered group $G/\mathbb{Z}u$ and ρ be the canonical epimorphism from G to C. Since G contains a strong unit, it contains a greatest proper convex subgroup $l(G)$. Then, the restriction of ρ to $l(G)$ is one-to-one. We denote by $l(C)$ its image. It is called the *linear part* of C. Indeed, one can check that the restriction of R to $l(C)$ is a linear cyclic order. An element a is in the positive cone of $l(C)$ if, and only if, it satisfies $R(0, a, 2a, \ldots, na)$, for every integer $n \geq 2$ ([11, Section 3]). Note that C is linearly cyclically ordered if, and only if, $l(C) = C$.

- Winding part. The group $C/l(C)$ is equipped with a structure of co-group which is induced by the cyclic order on C (see. [14, Proposition 2.9]). This cyclically ordered group is called the *winding part* of C, and is denoted by $U(C)$. Then, the cyclically ordered group $U(C)$ embeds in a unique way in $\mathbb{U}$ ([11, Lemma 5.1, Theorem 5.3]). If $x \in C$, we denote by $U_C(x)$ the image of its class $x + l(C)$. Then there exists a real number θ such that $U_C(x) = e^{i\theta}$. If $x \notin l(C)$, then θ is not congruent to 0 modulo 2π, we let $\theta_C(x)$ be the unique element of $]0, 2\pi[$ such that $U_C(x) = e^{i\theta_C(x)}$. If $x \in l(C)$, then we set $\theta_C(x) = 0$.

Denote by α the canonical unital order-preserving epimorphism from (G, u) onto $(G/l(G), u+l(G))$ (see diagram below). Since $l(G)$ is the greatest proper convex subgroup of G, the ordered group $G/l(G)$ is archimedean. Hence it embeds in $\mathbb{R}$, and there is a unique order preserving embedding β such that $\beta(u+l(G)) = 2\pi$. We let $\rho_{\mathbb{R}}$ be the canonical epimorphism from $(\mathbb{R}, 2\pi)$ to its wound-round $\mathbb{U} \simeq \mathbb{Z}/2\pi\mathbb{Z}$. We denote by α_1 the canonical c-epimorphism from C onto $C/l(C)$, and by β_1 the unique c-embedding of $C/l(C)$ in $\mathbb{U}$. Note that since $l(C)$ is order isomorphic to $l(G)$, the co-group $C/l(C) = U(C)$ is isomorphic to the unwound of $(G/l(G), u + l(G))$. Hence there is a group epimorphism ρ_1 from $(G/l(G), u + l(G))$ onto $C/l(C)$ such that $\rho_1 \circ \alpha = \alpha_1 \circ \rho$.

The restriction of $\rho_{\mathbb{R}}$ to $[0, 2\pi[$ is one-to-one and onto. We denote by $\tau_{\mathbb{R}}$ the one-to-one mapping from $\mathbb{U}$ onto $[0, 2\pi[\subset \mathbb{R}$ such that $\rho_{\mathbb{R}} \circ \tau_{\mathbb{R}}$ is the identity of $\mathbb{U}$. We let $I = (\beta \circ \alpha)^{-1}([0, 2\pi[)$. Then I is the interval $[l(G), u + l(G)[:= l(G) \cup \{x \in G \mid \forall (y, z) \in l(G) \times l(G) \;\; y < x < u + z\}$. The restriction of ρ to I is one-to-one and onto. We denote by τ the one-to-one mapping from C onto I such that $\rho \circ \tau$ is the identity of C. We summarize this in the following commutative diagram.

$$\begin{array}{ccccccccc}
I & \xrightarrow{\text{inclusion}} & (G,u) & \xrightarrow{\alpha} & (G/l(G), u+l(G)) & \xrightarrow{\beta} & (\mathbb{R}, 2\pi) & \xleftarrow{\text{inclusion}} & [0, 2\pi[\\
 & \nwarrow^{\tau} & \downarrow \rho & & \downarrow \rho_1 & & \downarrow \rho_{\mathbb{R}} & \nearrow^{\tau_{\mathbb{R}}} & \\
 & & C = G/\mathbb{Z}u & \xrightarrow{\alpha_1} & C/l(C) = U(C) & \xrightarrow{\beta_1} & \mathbb{U} & &
\end{array}$$

Here, α, β, α_1 and β_1 are homomorphisms of groups, but τ and $\tau_{\mathbb{R}}$ are not. However, for a, b in $\mathbb{U}$ the reals $\tau_{\mathbb{R}}(a) + \tau_{\mathbb{R}}(b)$ and $\tau_{\mathbb{R}}(a + b)$ are congruent modulo 2π (here and in the following lemma, in order to make a distinction between the elements of linearly ordered groups and those of their wound-rounds, we use the letters $a, b, \ldots$ when dealing with elements of C or $\mathbb{U}$). More exactly, we have $\tau_{\mathbb{R}}(a) + \tau_{\mathbb{R}}(b) = \tau_{\mathbb{R}}(a + b)$ if, and only if, $\tau_{\mathbb{R}}(a) + \tau_{\mathbb{R}}(b) \in [0, 2\pi[$, and $\tau_{\mathbb{R}}(a) + \tau_{\mathbb{R}}(b) = \tau_{\mathbb{R}}(a + b) + 2\pi$ otherwise. In the same way, for a, b in C, we have $\tau(a) + \tau(b) = \tau(a + b)$ if, and only if, $\tau(a) + \tau(b) \in I$, and $\tau(a) + \tau(b) = \tau(a + b) + u$ otherwise.

Let θ_C be the mapping from C to $[0, 2\pi[$ defined above. Then $\theta_C = \tau \circ \beta_1 \circ \alpha_1$. So, for a, b in C we have $\theta_C(a+b) = \tau(\beta_1 \circ \alpha_1(a) + \beta_1 \circ \alpha_1(b))$, which is equal to $\theta_C(a) + \theta_C(b)$ if, and only if, $\theta_C(a) + \theta_C(b) \in [0, 2\pi[$. Otherwise, $\theta_C(a) + \theta_C(b) = \theta_C(a + b) + 2\pi$. Consequently, for every x, y in I, we have $x + y \in I$ if, and only if, $\theta_C(\rho(x)) + \theta_C(\rho(y)) = \theta_C(\rho(x) + \rho(y))$, which in turn is equivalent to $\theta_C(\rho(x)) + \theta_C(\rho(y)) < 2\pi$. Otherwise we have $\theta_C(\rho(x)) + \theta_C(\rho(y)) = \theta_C(\rho(x) + \rho(y)) + 2\pi$.

- C-convex subgroups. Since the ordered groups $l(G)$ and $l(C)$ are isomorphic, there is a one-to-one mapping from the set convex subgroups of $l(G)$ onto the set convex subgroups of $l(C)$ (see [11, Section 4]) which preserves the inclusion order. The convex subgroups of $l(C)$ are said to be *c-convex subgroups* of C. We also assume that C is a c-convex subgroup of itself. The formal definition of c-convex subgroups is the following. A subgroup C_1 of C is a c-convex subgroup if $C_1 = C$ or it satisfies for every a, b in C,

$(b \in C_1 \Rightarrow b = \rho(0)$ or $b \neq -b)$ & $(b \in C_1$ & $R(\rho(0), b, -b)$ & $R(\rho(0), a, b)) \Rightarrow a \in C_1$. Let H be a proper convex subgroup of G. In any case $\rho(H)$ is linearly ordered. If C is not linearly cyclically ordered, then $\rho(H)$ is a proper c-convex subgroup of C.

In order to study the functor between linearly ordered groups and co-groups, we define a new class of homomorphisms. In the following definition and the following lemma, (G, u) and (G', u') are unital linearly ordered groups, and the notations are the same as in the proof of Proposition 3.9.

Definition 3.10 A c-homomorphism from C to C' is said to be a *w-homomorphism* if its kernel is a linearly ordered c-convex subgroup of C.

Note that if f is a unital order-preserving homomorphism from (G, u) to (G', u'), then $\bar{f}$ is a w-homomorphism.

We can prove that φ being a w-homomorphism from C to C' is equivalent to φ being a c-homomorphism and $\forall a \in C\backslash\{0\}$ $(2a = 0$ or $(R(0, a, 2a)$ and $\neg R(0, a, 2a, 3a))) \Rightarrow \varphi(a) \neq 0'$. Indeed, let C_1 be a c-convex subgroup of C. Then C_1 is not linearly ordered if, and only if, $C_1 = C$ and $U(C) \neq \{0\}$. So we have to prove that in any case, if $U(C) \neq \{0\}$ and $\forall a \in C\backslash\{0\}$ $(2a = 0$ or $R(0, a, 2a)$ and $\neg R(0, a, 2a, 3a)) \Rightarrow \varphi(a) \neq 0'$ holds, then $\ker\varphi \neq C$. By [6, Lemma 6.9 (1)], the formula $R(0, a, 2a)$ and $\neg R(0, a, 2a, 3a)$ is equivalent to $(\theta_C(a) = \frac{2\pi}{3}$ and $0 \leq 3a \in l(C))$, or $\frac{2\pi}{3} < \theta_C(a) < \pi$ or $(\theta_C(a) = \pi$ and $0 > 2a \in l(C))$. Assume that $U(C) \neq \{0\}$. If $C \simeq \mathbb{Z}/2\mathbb{Z}$, then it has a 2-torsion element. If $U(C) \simeq \mathbb{Z}/2\mathbb{Z}$ and $C \not\simeq \mathbb{Z}/2\mathbb{Z}$, then $l(C) \neq \{0\}$, and there is $a \in C$ such that $(\theta_C(a) = \pi$ and $0 > 2a \in l(C))$. If $U(C)$ contains at least three elements, then it contains an element in the interval $[\frac{2\pi}{3}, \pi[$, and there is $a \in C$ such that $(\theta_C(a) = \frac{2\pi}{3}$ and $0 \leq 3a \in l(C))$, or $\frac{2\pi}{3} < \theta_C(a) < \pi$.

Now, we let φ be a w-homomorphism from C to C'. Note that φ satisfies the following properties.

1) $\varphi(l(C)) \subseteq l(C')$. Indeed, let $a \in l(C)$. If $a \in \ker\varphi$, then $\varphi(a) = 0' \in l(C')$. Assume that $a > 0$ belongs to $l(C)\backslash\ker\varphi$. Since $\ker\varphi$ is a convex subgroup of $l(C)$, we have $a > \ker\varphi$, and for every positive integer k: $ka \notin \ker\varphi$. Therefore, for every positive integer n, the elements $0', \varphi(a), \ldots, n\varphi(a)$ are pairwise distinct. Since a belongs to the positive cone of $l(C)$, it satisfies $R(0, a, \ldots, na)$. Now, φ is a c-homomorphism, so this implies $R(0', \varphi(a), \ldots, n\varphi(a))$, which proves that $\varphi(a) \in l(C')$.
2) For every $a \in C$, we have $\theta_{C'}(\varphi(a)) = \theta_C(a)$. Indeed, it follows that φ induces an homomorphism from $U(C) = C/l(C)$ to $U(C') = C'/l(C')$. Since $\ker\varphi \subseteq l(C)$, this homomorphism is one-to-one. Now, one can check that it is a c-homomorphism. In particular, for every $a \in C$, we have $\theta_{C'}(\varphi(a)) = \theta_C(a)$.

Lemma 3.11 *There exists a unique unital order-preserving homomorphism f from (G, u) to (G', u') such that $\bar{f} = \varphi$.*

Proof 1) Let f_1 and f_2 be unital order-preserving homomorphisms from (G, u) to (G', u') such that $\bar{f}_1 = \bar{f}_2$. By the definition of $C = G/\mathbb{Z}u$, the restriction ρ to $[0, u[$ is one-to-one. The same holds with G' and $[0', u'[$. Therefore, for every $x \in [0, u[$ we have $f_1(x) = f_2(x)$. Now, we have $f_1(u) = u' = f_2(u)$. Let $y \in G$. We know that there is a unique $k \in \mathbb{Z}$ such that $y - ku \in [0, u[$. Then $f_1(y) = f_1(y - ku) + ku' = f_2(y - ku) + ku' = f_2(y)$. Consequently, we have $f_1 = f_2$. This proves that every w-homomorphism from C to C' is induced by at most one unital order-preserving homomorphism from (G, u) to (G', u').

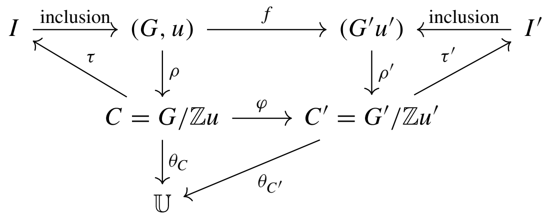
2) We denote by I the interval $[l(G), u + l(G)[:= l(G) \cup \{x \in G \mid \forall (y, z) \in l(G) \times l(G)\ y < x < u + z\}$. We define the subset I' of G' in the same way. For $x \in I$, we let $f(x)$ be the element of I' such that $\rho'(f(x)) = \varphi(\rho(x))$. For $x \in G$, there exists a unique $k \in \mathbb{Z}$ such that $x - ku \in I$. We set $f(x) = f(x - ku) + ku'$ (see diagram below). In particular, $f(0) = 0'$ and $f(u) = u'$. Furthermore, recall that ρ (resp. ρ') induces a one-to-one mapping from $l(G)$ (resp. $l(G')$) onto $l(C)$ (resp. $l(C')$). Furthermore recall that, since φ is a w-homomorphism, for every $a \in C$ we have $\theta_{C'}(\varphi(a)) = \theta_C(a)$. So $\theta_{C'}(\varphi(\rho(x))) = \theta_C(\rho(x))$. Hence:

$$\begin{aligned} f(x) \in l(G') &\Leftrightarrow ku' + f(x - ku) \in l(G') \\ &\Leftrightarrow k = 0\ \&\ f(x) \in l(G') \\ &\Leftrightarrow x \in I\ \&\ \rho'(f(x)) \in l(C') \\ &\Leftrightarrow x \in I\ \&\ \varphi(\rho(x)) \in l(C') \\ &\Leftrightarrow x \in I\ \&\ \theta_{C'}(\varphi(\rho(x))) = 0 \\ &\Leftrightarrow x \in I\ \&\ \theta_C(\rho(x)) = 0 \\ &\Leftrightarrow x \in I\ \&\ \rho(x) \in l(C) \\ &\Leftrightarrow x \in l(G). \end{aligned}$$

3) We prove that f is a group homomorphism. It suffices to consider two elements x, y in I. We already noticed that $x + y \in I \Leftrightarrow \theta_C(\rho(x)) + \theta_C(\rho(y)) < 2\pi$. In the same way, for every x', y' in I' we have $x' + y' \in I' \Leftrightarrow \theta_{C'}(\rho'(x')) + \theta_{C'}(\rho'(y')) < 2\pi$. It follows:

$$\begin{aligned} x + y \in I &\Leftrightarrow \theta_C(\rho(x)) + \theta_C(\rho(y)) < 2\pi \\ &\Leftrightarrow \theta_{C'}(\varphi(\rho(x))) + \theta_{C'}(\varphi(\rho(y))) < 2\pi \\ &\Leftrightarrow \theta_{C'}(\rho'(f(x))) + \theta_{C'}(\rho'(f(y))) < 2\pi \\ &\Leftrightarrow f(x) + f(y) \in I'. \end{aligned}$$

The restriction of ρ' to I' is a one-to-one mapping onto C'. We denote by τ' its inverse. Then the restriction of f to I is equal to the restriction of $\tau' \circ \varphi \circ \rho$ to I.



For a', b' in C' we have $\tau'(a') + \tau'(b') = \tau'(a' + b')$ if, and only if, $\tau'(a') + \tau'(b') \in I'$. Otherwise, we have $\tau'(a') + \tau'(b') = \tau'(a' + b') + u'$.

Let x, y in I such that $x + y \in I$. Since ρ and φ are homomorphisms of groups, we have

$$f(x + y) = \tau' \circ \varphi \circ \rho(x + y) = \tau'(\varphi \circ \rho(x) + \varphi \circ \rho(y)) = \tau'(\rho'(f(x)) + \rho'(f(y))).$$

Now, $\tau'(\varphi \circ \rho(x)) + \tau'(\varphi \circ \rho(y)) = \tau'(\rho'(f(x))) + \tau'(\rho'(f(y))) = f(x) + f(y)$ belongs to I', because $x + y \in I$. Therefore, $f(x + y) = \tau' \circ \varphi \circ \rho(x) + \tau' \circ \varphi \circ \rho(y) = f(x) + f(y)$.

Assume that x, y belong to I, and $x + y \notin I$ (which implies $f(x) + f(y) \notin I'$). Then $x + y - u \in I$, so

$$f(x+y) = f(x+y-u) + u' = \tau' \circ \varphi \circ \rho(x+y-u) + u' = \tau' \circ \varphi \circ \rho(x+y) + u' = \tau'(\varphi \circ \rho(x) + \varphi \circ \rho(y)) + u'.$$

Now, $\tau'(\varphi \circ \rho(x)) + \tau'(\varphi \circ \rho(y)) = f(x) + f(y) \notin I'$. Hence

$$\tau'(\varphi \circ \rho(x)) + \tau'(\varphi \circ \rho(y)) = \tau'(\varphi \circ \rho(x) + \varphi \circ (y)) + u' = \tau'(\varphi \circ \rho(x+y)) + u'.$$

Therefore, $f(x+y) = f(x) + f(y)$.

4) To prove that f is order preserving, it is sufficient to show that for every $x > 0$ in G we have $f(x) \geq 0'$. If $x > I$, then $f(x) \geq u' > 0'$. If $x \in I \backslash l(G)$, then $f(x) \in I' \backslash l(G')$, since $0 < \theta_C(\rho(x)) = \theta_{C'}(\varphi(\rho(x))) = \theta_{C'}(\rho'(f(x)))$. Hence $f(x) > 0'$. Now, assume that $x \in l(G)$. Saying that $x > 0$ is equivalent to saying that $R(\rho(0), \rho(x), 2\rho(x))$ holds. If $x \in \ker f$, then $f(x) = 0'$. Now, if $f(x) \neq 0'$, then since $f(x) \in l(G')$, we have $0' \neq f(x) \neq 2f(x) \neq 0'$. Now, ρ' is a one-to-one mapping from $l(G')$ onto $l(C')$. So $\rho'(0') \neq \rho'(f(x)) \neq 2\rho'(f(x)) \neq \rho'(0')$. Therefore $R'(\rho'(0'), \rho'(f(x)), 2\rho'(f(x))) = R'(\varphi \circ \rho(0), \varphi \circ \rho(x), 2\varphi \circ \rho(x))$ holds, since φ is a c-homomorphism. This proves that $f(x) > 0'$.
5) It follows that f is a unital order-preserving homomorphism. Furthermore, by the definition of f we have $\bar{f} = \varphi$. □

So, Rieger's Theorem 3.4 can be generalized to a functorial mapping.

Proposition 3.12 *The wound-round mapping defines a full and faithful functor from the category of unital linearly ordered groups, together with unital order-preserving group homomorphisms, to the category of co-groups, together with w-homomorphisms. The unwound mapping defines a full and faithful functor from the category of co-groups, together with w-homomorphisms, to the category of unital linearly ordered groups, together with unital order-preserving group homomorphisms. The composites of these two functors are equivalent to the identities of respective categories (where the structures are considered up to isomorphism).*

3.3.3 ℓ-groups

If C is the wound-round of an ℓ-group, one can wonder if any two elements have a minimum and a maximum with respect to some partial order. For this purpose, we restrict to the elements that we call non-isolated.

Definition 3.13 Let C be a pco-group, and denote by $A(C)$ the set whose elements are 0 and all the elements $x \in C \backslash \{0\}$ such that there exists $y \in C \backslash \{0\}$ satisfying $x <_0 y$ or $y <_0 x$. The elements of $A(C)$ are called the *non-isolated* elements.

Note that if C is the wound-round of an ℓ-group, then for every x, y in $A(C)$, we have the following:

- $x \wedge_0 y$ exists (the infimum of x and y in $(A(C), \leq_0)$),
- 0 is the smallest element,
- $x \vee_0 y$ does not exist if, and only if, $(-x) \wedge_0 (-y) = 0$, and if this holds, then for every $z \in A(C)$: $x <_0 z \Rightarrow y \not<_0 z$.

Furthermore, for every $x \in A(C)$ there exists a unique $g \in G$ such that $0 \leq g < u$ and $\rho(g) = x$, where $C \simeq G/\mathbb{Z}u$, with $u > 0$ a strong unit of G. Now, assume that $x \in C \backslash A(C)$. Then we have $0 <_0 x$, and x is not comparable with any other element. Therefore for every $y \in C$, $x \wedge_0 y = 0$, and $x \vee_0 y$ does not exist.

Let f be a c-homomorphism between wound-rounds of ℓ-groups C and C'. We say that f is an *lc-homomorphism* if for every x, y in $A(C)$ we have $f(x \wedge_0 y) = f(x) \wedge_0 f(y)$.

The wound-round mapping is not a functor from the category of ℓ-groups to the category of wound-rounds of ℓ-groups together with ℓc-homomorphisms. Indeed, let G be the ℓ-group $\mathbb{R} \times \mathbb{R}$, $u = (1, 1)$, $G' = \mathbb{R}$, $u' = 1$, and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the natural projection onto the first component. Then, f is an ℓ-homomorphism, and $f(u) = u'$. Let $x = \left(\frac{1}{2}, 2\right)$ and $y = \left(\frac{1}{4}, 4\right)$. Let ρ (resp. ρ') be the canonical epimorphism from G onto $C = G/\mathbb{Z}u$ (resp. from G' onto $C' = G'/\mathbb{Z}u'$). We sow that $\rho(x) \notin A(C)$ and $\rho(y) \notin A(C)$. Indeed, let n be an integer. Then $0 \leq \frac{1}{2} + n \leq 1 \Leftrightarrow n = 0$, and $0 \leq 2 + n \leq 1 \Leftrightarrow n \in \{-2, -1\}$. Hence there is no n such that $x + nu \in [0, u[$. This implies $\rho(x) \notin A(C)$. In the same way, $\rho(y) \notin A(C)$. Therefore $\rho(x) \wedge_0 \rho(y) = \rho(0)$. Now, $f(x) \in [0, u'[$, $f(y) \in [0, u'[$ and $f(x) \wedge f(y) = \frac{1}{4}$. It follows that $\rho'(f(x)) \wedge_0 \rho'(f(y)) = \rho'(f(y)) \neq \rho'(0)$. Consequently, the c-homomorphism $\bar{f} : C \to C'$ induced by f (see Proposition 3.9) is not an ℓc-homomorphism.

However, we have the following.

Proposition 3.14 *Let (G, u) and (G', u') be unital ℓ-groups, f be a one-to-one unital ℓ-homomorphism from (G, u) to (G', u') and $\bar{f}$ be the c-homomorphism defined in the proof of Proposition 3.9. Then, $\bar{f}$ is an ℓc-homomorphism from $C := G/\mathbb{Z}u$ to $C' := G'/\mathbb{Z}u'$.*

Proof The notations are the same as those of Proposition 3.9.

First we let $x \in G_u$, and we prove that $\rho'(f(x)) \in A(C') \Leftrightarrow \rho(x) \in A(C)$.

Recall that, since $x \in G_u$, we have that $\rho(x) \in A(C) \Leftrightarrow x \in [0, u[$. We saw in Lemmas 2.7 and 2.8 that there is a unique sequence $x_1, \ldots, x_n$ such that $x = x_1 + \cdots + x_n$, $x_1 = x \wedge u$, $x_2 = (x - x_1) \wedge u$, and so on. Since f is a unital ℓ-homomorphism, we have that $f(x_1) = f(x) \wedge u'$, $f(x_2) = (f(x) - f(x_1)) \wedge u'$, and so on. Now, f is one-to-one, so, for $1 \leq i \leq n$, $f(x_i) = 0 \Leftrightarrow x_i = 0$. Hence $f(x_1), \ldots, f(x_n)$ is the sequence associated to $f(x)$ as in Lemmas 2.7 and 2.8. It follows that

$$\rho(x) \in A(C) \Leftrightarrow x \in [0, u[\Leftrightarrow n = 1 \Leftrightarrow f(x) \in [0, u'[\Leftrightarrow \rho'(f(x)) \in A(C').$$

Let x, y in G_u. If x and y belong to $[0, u[$, then $f(x)$ and $f(y)$ belong to $[0, u'[$ and $f(x \wedge y) = f(x) \wedge f(y)$. Therefore

$$\bar{f}(\rho(x) \wedge_0 \rho(y)) = \bar{f}(\rho(x \wedge y)) = \rho'(f(x \wedge y)) = \rho'(f(x) \wedge f(y)) = \rho'(f(x)) \wedge_0 \rho'(f(y))$$
$$= \bar{f}(\rho(x)) \wedge_0 \bar{f}(\rho(y)).$$

If $x \notin [0, u[$, then $\rho(x) \wedge_0 \rho(y) = 0$. Now, we have proved that $f(x) \notin [0, u'[$, hence $\rho'(f(x)) \wedge_0 \rho'(f(y)) = 0$. The case where $y \notin [0, u[$ is similar. □

4 From MV-algebras to Wound-rounds of ℓ-groups

The correspondence between MV-algebras and pco-groups is defined as follows.

Let A be an MV-algebra and (G_A, u_A) be its Chang ℓ-group.

We saw in Section 2 that $\Xi : A \mapsto (G_A, u_A)$ is a full and faithful functor from the category of MV-algebras to the category of unital ℓ-groups, where (G_A, u_A) is the Chang

ℓ-group of A. Through misuse of language, we say that it is one-to-one, since it induces a one-to-one mapping between the classes of isomorphic structures.

Now, the wound-round functor Θ: $(G, u) \mapsto G/\mathbb{Z}u$ defined in Proposition 3.9 is a functor from the category of unital ℓ-groups to the category of wound-rounds of ℓ-groups, together with the c-homomorphisms. Trivially, $(G, u) \mapsto \Theta(G, u)$ is onto.

So, this gives rise to a functor $\Theta\Xi$ from the category of MV-algebras to the category of wound-rounds of ℓ-groups, together with the c-homomorphisms. The mapping $A \mapsto \Theta\Xi(A)$ is onto.

Now, we will prove that the mapping $A \mapsto \Theta\Xi(A)$ is one-to-one, by defining the inverse mapping. Then we will extend this inverse mapping to a larger class of pco-groups.

Note that $\Theta\Xi$ gives rise to a full and faithful functor when restricted to MV-chains and co-groups.

4.1 The Inverse Mapping of $\Theta\Xi$

Let A be an MV-algebra. We denote by $C(A)$ the wound-round of ℓ-group $G_A/\mathbb{Z}u_A$, instead of $\Theta\Xi(A)$.

First, we show that we can define a structure of MV-algebra which is isomorphic to A, on the set of non-isolated elements of $C(A)$. Then, by means of the same rules, we define an MV-algebra in the set of non-isolated elements of any wound-round of ℓ-group. It follows that if C is the wound-round of an ℓ-group, then we can construct a unital ℓ-group (G, u) such that $C \simeq G/\mathbb{Z}u$. We start setting some properties of the order $\leq_0$ on the set of non-isolated elements of a wound-round.

4.1.1 The Subset of Non-isolated Elements of a pco-group

Recall that, in a pco-group C, the set $A(C)$ of non-isolated elements is the set whose elements are 0 and all the $x \in C\backslash\{0\}$ such that there exists $y \in C\backslash\{0\}$ satisfying $x <_0 y$ or $y <_0 x$ (Definition 3.13).

Remarks 4.1 (1) Let C be a pco-group and x, y in $A(C)\backslash\{0\}$. By Proposition 3.3 since $x <_0 y \Rightarrow y - x <_0 -x$, if $x <_0 y$ in $A(C)$ and $z = y - x$, then $z \in A(C)\backslash\{0\}$. So there exists z in $A(C)$ such that $y = x + z$. We see that this is similar to condition (1) in Remark 2.6.

(2) If $-x <_0 y$, then $x + y <_0 x$.

(3) Assume that C and C' are pco-groups. It follows from Proposition 3.3 that they are c-isomorphic if, and only if, there is a group isomorphism φ from C onto C' such that for every x, y in C:

$$x \in A(C) \Leftrightarrow \varphi(x) \in A(C') \;\&\; x <_0 y \Leftrightarrow \varphi(x) <_0 \varphi(y).$$

Note that $x \in A(C) \Leftrightarrow -x \in A(C)$. We can say that $A(C)$ is *symmetric*.

Proposition 4.2 *Let (G, u) be a unital partially ordered group, $N = \{x \in]0, u[\mid \exists y \in]0, u[\ x < y$ or $y < x\}$, and C be the wound-round pco-group $G/\mathbb{Z}u$. Then, $(N, <)$ and $(A(C)\backslash\{0\}, \leq_0)$ are isomorphic ordered sets.*

Proof It follows from (1) of Proposition 3.5 that the restriction of ρ to $[0, u[$ is one-to-one. In particular, its restriction to N is one-to-one. If $x \in N$, then there exists $y \in N$ such that

$0 < x < y$ or $0 < y < x$. It follows that $\rho(x) <_0 \rho(y)$ or $\rho(y) <_0 \rho(x)$. In particular, $\rho(x) \in A(C)$, and ρ is an homomorphism of ordered sets from N to $A(C)$. Now, let $x \in G$ such that $\rho(x) \in A(C)$ and $x \notin \mathbb{Z}u$. Then, there exist $y \in G$ and integers n, n' such that $0 < x + nu < y + n'u < u$ or $0 < y + n'u < x + nu < u$. In any case $x + nu \in N$. Since $\rho(x + nu) = \rho(x)$, it follows that the restriction of ρ from N to $A(C)\backslash\{0\}$ is onto. Consequently, ρ is an isomorphism of ordered sets between $(N, \leq)$ and $(A(C)\backslash\{0\})$. □

We have already seen that an MV-algebra A is order-isomorphic to the subset $[0, u_A]$ of its Chang ℓ-group (G_A, u_A). We get a similar result in the case of wound-round of ℓ-groups.

Corollary 4.3 *Let (G, u) be a unital ℓ-group, $C = G/\mathbb{Z}u$ and ρ be the canonical mapping from G onto C. We assume that $A(C) \neq \{0\}$. We add an element 1 to $A(C)$ and we set $x <_0 1$ for every $x \in A(C)$. Then the ordered sets $([0, u], \leq)$ and $(A(C) \cup \{1\}, \leq_0)$ are isomorphic.*

Proof By the definition of $A(C)$ and of $\leq_0$ in $C = G/\mathbb{Z}u$, if $A(C) \neq \{0\}$, then there is x, y in G such that $0 < x < y < u$ or $0 < y < x < u$. Hence, by Lemma 2.4, $N =]0, u[$. Therefore the result follows from Proposition 4.2. □

4.1.2 Interpretability of A in $\Theta\Xi(A)$

We denote by ρ the canonical epimorphism from G_A onto $C(A)$, where, for $x \in G_A$, $\rho(x) \in C(A)$ is the class of x modulo $\mathbb{Z}u_A$. Without loss of generality, we assume that $A \subset G_A$ and $1 = u_A$, we denote by φ the restriction of ρ to $[0, u_A[$.

Assume that the set $A(C(A))$ of non-isolated elements of $C(A)$ is nontrivial. Since φ is an isomorphism of ordered sets between $([0, u_A[, \leq)$ and $(C(A), \leq_0)$, for every x, y in $[0, u_A[$, $\varphi(x \wedge y) = \varphi(x) \wedge_0 \varphi(y)$, and if $x \vee y < u_A$, then $\varphi(x \vee y) = \varphi(x) \vee_0 \varphi(y)$.

The set $A(C(A))$ is nontrivial if $A \neq \{0, 1\}$, $A \neq \{0, 1, x\}$ and $A \neq \{0, 1, x, \neg x\}$, for some x (Lemma 2.4). One can see that if $A = \{0, 1\}$, then $C(A) = \{0\}$. If $A = \{0, 1, x\}$, then $C(A) \simeq \mathbb{Z}/2\mathbb{Z}$. If $A = \{0, 1, x, \neg x\}$ is not an MV-chain, then $C(A) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and in any case $A(C(A)) = \{0\}$. If $A = \{0, 1, x, \neg x\}$ is an MV-chain, then $C(A) \simeq \mathbb{Z}/4\mathbb{Z}$.

In the following, we assume that $A \neq \{0, 1\}$, $A \neq \{0, 1, x\}$ and $A \neq \{0, 1, x, \neg x\}$, for some x.

Note that for every x in $]0, u_A[$, we have that $\varphi(\neg x) = -\varphi(x)$ (since $\neg x = u_A - x$). So we can set $\neg\varphi(x) = -\varphi(x)$. We add an element $\mathbf{1}$ to $A(C(A))$, and we set $\varphi(u_A) = \mathbf{1}$. We let $\neg\varphi(0) = \mathbf{1}$ and $\neg\mathbf{1} = \varphi(0)$. If $x \in [0, 1[$, we let $\varphi(x) <_0 \mathbf{1}$.

We turn to the image of $x \oplus y$.

Proposition 4.4 *Let x, y in $[0, u_A[$, we have that $\rho(x \oplus y) = \varphi(x) \wedge_0 (\neg\varphi(y)) + \varphi(y)$.*

Proof We know that $x \oplus y = (x + y) \wedge u_A$, hence $x \oplus y = x \wedge (u_A - y) + y$. Assume that $x \wedge (u_A - y) + y < 1$ and $y \neq 0$ (the case $y = 0$ being trivial). Hence

$$\begin{aligned} \rho(x \oplus y) &= \varphi(x \oplus y) \\ &= \varphi(x \wedge (u_A - y) + y) \\ &= \varphi(x \wedge (u_A - y)) + \varphi(y) \\ &= \varphi(x) \wedge_0 \varphi(u_A - y) + \varphi(y) \\ &= \varphi(x) \wedge_0 (\neg\varphi(y)) + \varphi(y). \end{aligned}$$

If $x \wedge (u_A - y) + y = u_A$ i.e. $x \wedge (u_A - y) = u_A - y$, then $u_A - y \leq x$ and $y \neq 0$. It follows that $0 <_0 \varphi(u_A - y) = -\varphi(y) \leq_0 \varphi(x)$ and

$$\begin{aligned} \varphi(x) \wedge_0 (\neg\varphi(y)) + \varphi(y) &= \varphi(x) \wedge_0 (-\varphi(y)) + \varphi(y) \\ &= -\varphi(y) + \varphi(y) \\ &= 0 \\ &= \rho(u_A) \\ &= \rho(x \oplus y). \end{aligned}$$

□

Corollary 4.5 *The MV-algebra A, in the language L_{MV}, is interpretable in the L_{lo}-structure $A(C(A))$. In particular, if A and A' are MV-algebras such that $A(C(A)) \equiv A(C(A'))$, then $A \equiv A'$. The same holds with $\prec$ instead of $\equiv$.*

Proof For every x, y in $A(C(A)) \cup \{\mathbf{1}\}$ we set $\neg x = -x$ if $0 \neq x \neq \mathbf{1}$, $\neg 0 = \mathbf{1}$, $\neg \mathbf{1} = 0$, and $x \oplus y = x \wedge_0 \neg y + y$ if $x \wedge_0 \neg y + y \neq 0$ or $x = y = 0$, and we set $x \oplus y = \mathbf{1}$ otherwise. The remainder of the proof follows from Theorem 2.13. □

4.1.3 Construction of the Inverse Mapping of $\Theta\,\Xi$

Notation 4.6 If C is a pco-group, then we add an element $\mathbf{1}$ to $A(C)$ and we set, for every $x \in A(C)$, $x <_0 \mathbf{1}$ and $\mathbf{1} + x = x + \mathbf{1} = x$.

Let $C = G/\mathbb{Z}u$ be the wound-round of an ℓ-group. Set $\neg 0 = \mathbf{1}$, $\neg \mathbf{1} = 0$ and for $x \in A(C)\backslash\{0\}$ set $\neg x = -x$. For every x, y in $A(C) \cup \{\mathbf{1}\}$ set
$x \oplus y = x \wedge_0 (\neg y) + y$ if $x \wedge_0 (\neg y) + y \neq 0$ or $x = y = 0$, and
$x \oplus y = \mathbf{1}$ otherwise.

It follows from Proposition 4.4 that $A(C) \cup \{\mathbf{1}\}$ is an MV-algebra which is isomorphic to the MV-algebra $\Gamma(G, u)$. By the uniqueness of the Chang ℓ-group, (G, u) is isomorphic to the Chang ℓ-group of $A(C) \cup \{\mathbf{1}\}$. It follows that $C \simeq \Theta\Xi(A(C) \cup \{\mathbf{1}\})$. So we defined the inverse mapping of $\Theta\Xi$, which proves that it is-one-to-one (with the same misuse of language as above).

4.1.4 The Unwound Construction

Recall that if (G, u) is a unital partially ordered group and $C \simeq G/\mathbb{Z}u$, then we say that (G, u) is the unwound of C if whenever a unital partially ordered group (G', u') satisfies $C \simeq G'/\mathbb{Z}u'$, then $(G, u) \simeq (G', u')$ (Definition 3.6). For example every co-group is an unwound, that one can get by Rieger's construction.

Turning to the wound-round C of an ℓ-group, we show that it has an unwound, which is obtained by the Chang ℓ-group construction.

Proposition 4.7 *If C is the wound-round of an ℓ-group and $A(C)$ is nontrivial, then C is generated by $A(C)$.*

Proof Let (G, u) be a unital ℓ-group such that $C \simeq G/\mathbb{Z}u$, and let $x \in G$. By properties of ℓ-groups, there exist x_+ and x_- in the positive cone of G such that $x = x_+ + x_-$ (see, for example, [1]). Now, by Lemma 2.7, this positive cone is generated by $[0, u]$. Therefore G is generated by $[0, u]$. Since $A(C)$ is nontrivial, by (2) of Lemma 2.4, for every $x \in [0, u]$ we have $\rho(x) \in A(C)$. Consequently, C is generated by $A(C)$. □

If C is the wound-round of an ℓ-group and $A(C)$ is nontrivial, then one can construct the MV-algebra $A = A(C) \cup \{\mathbf{1}\}$ associated to C. We can also construct the Chang ℓ-group G_A of A. So, it suffices to prove that $C \simeq G_A/\mathbb{Z}u_A$ and $C \simeq G'/\mathbb{Z}u' \Rightarrow (G', u') \simeq (G_A, u_A)$. This is the aim of the next proposition.

Proposition 4.8 *Let C be the wound-round of an ℓ-group. Then C has an unwound, which is the Chang ℓ-group of the MV-algebra $A(C) \cup \{\mathbf{1}\}$.*

Proof Recall that the MV-algebra $A(C) \cup \{\mathbf{1}\}$ is first-order definable in C, by the following rules.

- For $x \in A(C)\backslash\{0\}$ set $\neg x = -x$, $\neg 0 = \mathbf{1}$, $\neg \mathbf{1} = 0$.
- For every x, y in $A(C) \cup \{\mathbf{1}\}$ $x \oplus y = x \wedge_0 (\neg y) + y$ if $x \wedge_0 (\neg y) + y \neq 0$ or $x = y = 0$, and $x \oplus y = \mathbf{1}$ otherwise.

In particular, we can assume that $\oplus$ belongs to the language. Trivially, if C is isomorphic to the wound-round of the Chang ℓ-group of the MV-algebra $A(C) \cup \{\mathbf{1}\}$, then it is the wound-round of an ℓ-group.

Assume that $C = G/\mathbb{Z}u$, where (G, u) is a unital ℓ-group. Let A be the MV-algebra $A(C) \cup \{1\}$, (G_A, u_A) be the Chang ℓ-group of A and C' be the pco-group $G_A/\mathbb{Z}u_A$.

We know that $A \simeq \Gamma(G_A, u_A)$, and we already noticed that $A \simeq \Gamma(G, u)$. By uniqueness of the Chang ℓ-group, it follows that there is a unital ℓ-isomorphism between (G, u) and (G_A, u_A). Hence the pco-groups C and C' are isomorphic, and (G_A, u_A) is the unwound of C. □

Note that the wound-rounds of ℓ-groups are infinite, as show the next proposition.

Proposition 4.9 *If C is the wound-round of an ℓ-group and is not a co-group, then it is infinite.*

Proof Let (G, u) be a unital ℓ-group such that $C \simeq G/\mathbb{Z}u$. If C is not linearly ordered, then G is not linearly ordered. Hence there exist $x > 0$ and $y > 0$ in G such that $x \not\leq y$ and $y \not\leq x$. So $x \wedge y < x$ and $x \wedge y < y$. By taking $x - x \wedge y$ instead of x, and $y - x \wedge y$ instead of y, we can assume that $x > 0$ and $y > 0$ and $x \wedge y = 0$. By properties of ℓ-groups, for every positive integer n we have $nx \wedge y = 0 = x \wedge ny$ (this follows for example from 1.2.24 on p. 22 of [1]). In particular, x and y are not strong units. It follows that, for every $n \in \mathbb{N}\backslash\{0\}$, $nx \not> u$, hence $x, 2x, \ldots, nx, \ldots$ belong to different classes modulo $\mathbb{Z}u$, therefore $G/\mathbb{Z}u$ is infinite. □

4.2 MV-algebra Associated to a pco-group

We extend above construction of an MV-algebra in the set of non-isolated elements of the wound-round of an ℓ-group to a larger class of pco-groups which we define now. In [8], the lattice-cyclically-ordered groups are defined to be pco-groups such that $\leq_0$ defines a structure of distributive lattice with first element. In the present paper we look at a larger class of groups.

Definition 4.10 Let C be a pco-group. We will say that $A(C)$ *defines canonically an MV-algebra* if it satisfies the following.

(1) $(A(C) \cup \{\mathbf{1}\}, \leq_0)$ is a distributive lattice.

(2) For every x, y in $A(C)$, $x + y = x \wedge_0 y + x \vee_0 y$.
(3) For every x, y, z in $A(C)\backslash\{0\}$, we have that

$$x - y = (x \wedge_0 (-z) + z) \wedge_0 (-y) - (y \wedge_0 (-z) + z) \wedge_0 (-x).$$

We will denote by $\mathcal{AC}$ the class of pco-groups C such that $A(C)$ defines canonically an MV-algebra.

Our main purpose in this subsection is to define an MV-algebra in any element of $\mathcal{AC}$ (Theorem 4.11), and to prove that $\mathcal{AC}$ contains the wound-rounds of ℓ-groups (Theorem 4.16). We will also prove that if $C \in \mathcal{AC}$, then the subgroup generated by $A(C)$ being the wound-round of an ℓ-group depends on the first-order theory of C (Theorem 4.17).

Theorem 4.11 *Let $C \in \mathcal{AC}$. Set $\neg 0 = \mathbf{1}$, $\neg\mathbf{1} = 0$ and for $x \in A(C)\backslash\{0\}$ set $\neg x = -x$. For every x, y in $A(C) \cup \{\mathbf{1}\}$ set*
$x \oplus y = x \wedge_0 (\neg y) + y$ if $x \wedge_0 (\neg y) + y \neq 0$ or $x = y = 0$, and
$x \oplus y = \mathbf{1}$ otherwise.
Then $A(C) \cup \{\mathbf{1}\}$ is an MV algebra with natural partial order $\leq_0$.

Corollary 4.12 *Let C be a pco-group. If $A(C)$ defines canonically an MV-algebra, then the MV-algebra $A(C) \cup \{\mathbf{1}\}$ defined in Theorem 4.11 is interpretable in $C \cup \{\mathbf{1}\}$, where $\mathbf{1}$ is a new element.*

Note that if A is an MV-algebra such that there exist $x < y$ in $]0, 1[$, then the MV-algebra $A(C(A)) \cup \{\mathbf{1}\}$ (together with the operations defined in Theorem 4.11) is isomorphic to A. Indeed, since $]0, 1[$ contains $x < y$, we deduce from Corollary 2.5 that $A(C(A))$ is nonempty. By Corollary 4.3, the canonical epimorphism ρ from the Chang ℓ-group (G_A, u_A) of A induces an isomorphism φ between the lattices $[0, u_A]$ and $A(C(A)) \cup \{\mathbf{1}\}$. Now, for g, h in $]0, u_A[$, $\varphi(g)\wedge(-\varphi(h))+\varphi(h) = \varphi(g)\wedge\varphi(u_A-h)+\varphi(h) = \varphi((g+h)\wedge u_A)$. Hence $\varphi(g) \wedge (\neg\varphi(h)) + \varphi(h) = 0$ if, and only if, either $g + h \geq u_A$ or $g + h = 0$. Consequently, by Proposition 4.4, φ is an isomorphism of MV-algebras.

The proof of Theorem 4.11 is based on the following three lemmas.

Lemma 4.13 *Let C be a pco-group such that, for every x and y in $A(C)$, $x \wedge_0 y$ exists. Let x, y in $A(C)\backslash\{0\}$. If $-y \neq x \wedge_0 (-y)$, then $y <_0 x \wedge_0 (-y) + y$. In particular, $x \wedge_0 (-y) + y$ belongs to $A(C)$.*

Proof Since $-y \neq x \wedge_0 (-y)$, we have that $x \wedge_0 (-y) <_0 -y$. By hypothesis, this is equivalent to $y <_0 -(x \wedge_0 (-y))$. By Proposition 3.3, this is also equivalent to $-x \wedge_0 (-y) - y <_0 -y$. By hypothesis, this in turn is equivalent to $y <_0 x \wedge_0 (-y) + y$. The last assertion follows easily. □

Lemma 4.14 *Let C be a pco-group. Let x, y in $A(C)$ such that the infimum $z = x \wedge_0 y$ of x and y in $(A(C), \leq_0)$ exists. Then $x-z$ and $y-z$ belong to $A(C)$, the infimum $(x-z)\wedge_0(y-z)$ exists and is equal to 0.*

Proof If $x \leq_0 y$, then $z = x$, $y - z = y - x <_0 -x$ (Proposition 3.3). Hence $y - z \in A(C)$, and $(y - z) \wedge_0 (x - z) = x - z = 0$. The same holds if $y \leq_0 x$. Now, assume that nor $x \leq_0 y$ nor $y \leq_0 x$. We have that $z <_0 x$, hence $x - z <_0 -z$. In particular, $x - z \in A(C)$. Let $t \in C$ such that $0 <_0 t <_0 x - z$. Then we have: $R(0, t, x - z, -z)$. Hence $R(z, t + z, x, 0)$ holds.

Therefore $R(0, z, t+z, x)$ holds, i.e. $z <_0 t+z <_0 x$. In the same way, $0 <_0 t <_0 y-z \Rightarrow z <_0 t+z <_0 y$. Hence, since $z = x \wedge_0 y$, this yields a contradiction. Consequently, there is no $t \in A(C)\backslash\{0\}$ such that $t <_0 x - z$ & $t <_0 y - z$. □

Let C be a pco-group such that, for every x and y in $A(C)$, $x \wedge_0 y$ exists. Then, the supremum $x \vee_0 y$ exists if, and only if, $(-x) \wedge_0 (-y) \neq 0$. If this holds, then $x \vee_0 y = -((-x) \wedge_0 (-y))$. Otherwise, there is no $z \in A(C)$ such that $x \leq_0 z$ and $y \leq_0 z$. If the supremum of x and y does not exist, then we will set $x \vee_0 y = \mathbf{1}$. So $(A(C) \cup \{\mathbf{1}\}, \leq_0)$ is a lattice with smallest element 0 and greatest element $\mathbf{1}$.

Lemma 4.15 *Let $C \in \mathcal{AC}$. Then, for every x, y in $A(C) \cup \{\mathbf{1}\}$: $x \oplus y = \mathbf{1} \Leftrightarrow (-y \leq_0 x$ & $(x, y) \neq (0, 0))$. In particular: $\neg x \oplus x = \mathbf{1}$.*

Proof We have that $x \wedge_0 \neg y + y = 0$ if, and only if, $-y = x \wedge_0 (-y)$. So $x \wedge_0 \neg y + y = 0 \Leftrightarrow -y \leq_0 x$. In particular, $x \oplus y = \mathbf{1} \Leftrightarrow (-y \leq_0 x$ and $(x, y) \neq (0, 0))$. □

Proof of Theorem 4.11 Note that if $x \vee_0 y$ does not exist in $A(C)$, then $x \vee_0 y = \mathbf{1}$. By Lemma 4.13, if x and y belong to $A(C)\backslash\{0\}$ and $x \oplus y \neq \mathbf{1}$, then $x \oplus y = x \wedge_0 \neg y + y \in A(C)$. Let x, y in $A(C) \cup \{\mathbf{1}\}$. If $y = 0$, then $x \oplus y = x \wedge_0 \mathbf{1} + 0 = x$. If $x = 0$, then $x \oplus y = 0 + y = y$. If $y = 0$, then $x \oplus y = x + 0 = x$. If $y = \mathbf{1}$, then $x \wedge_0 0 + \mathbf{1} = 0$, hence $x \oplus y = \mathbf{1}$. If $x = \mathbf{1}$, then $1 \wedge \neg y + y = \neg y + y = 0$. Hence $x \oplus y = \mathbf{1}$. It follows that in any case $x \oplus y \in A(C) \cup \{\mathbf{1}\}$.

We have to prove that $\oplus$ and $\neg$ satisfy the axioms of Definition 2.1.

MV4) Trivially, for every $x \in A(C) \cup \{1\}$: $\neg\neg x = x$.

MV3) and MV5) have already been proved (i.e. $x \oplus 0 = x$, $x \oplus \neg 0 = \neg 0$).

MV2) ($x \oplus y = y \oplus x$) The case where $x \in \{0, \mathbf{1}\}$ or $y \in \{0, \mathbf{1}\}$ follows from above calculations. Assume that x and y belong to $A(C)\backslash\{0\}$. By Lemma 4.15, $x \oplus y = \mathbf{1} \Leftrightarrow -y \leq_0 x \Leftrightarrow -x \leq_0 y \Leftrightarrow y \oplus x = \mathbf{1}$. Otherwise, $x \oplus y - y \oplus x = x \wedge_0 (-y) + y - (y \wedge_0 (-x) + x) = x \wedge_0 (-y) - (y \wedge_0 (-x)) - (x - y) = x \wedge_0 (-y) + (-y) \vee_0 x - (x - y) = 0$, by 2) of Definition 4.10.

MV6) ($\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$) Trivially, we can assume that $x \neq y$. Since $x \vee_0 y = y \vee_0 x$, it is sufficient to prove that for all x, y in $A(C) \cup \{\mathbf{1}\}$ we have that $\neg(\neg x \oplus y) \oplus y = x \vee_0 y$.

If $y = 0$, then $\neg(\neg x \oplus y) \oplus y = \neg(\neg x \oplus y) = \neg(\neg x) = x = x \vee_0 y$. If $x = 0$, then $\neg(\neg x \oplus y) \oplus y = \neg(\mathbf{1} \oplus y) \oplus y = \neg\mathbf{1} \oplus y = 0 \oplus y = y = x \vee_0 y$.

If $y = \mathbf{1}$, then $\neg(\neg x \oplus y) \oplus y = \neg(\neg x \oplus y) \oplus \mathbf{1} = \mathbf{1} = x \vee_0 y$. If $x = \mathbf{1}$ and $y \in A(C)\backslash\{0\}$, then $\neg(\neg x \oplus y) \oplus y = \neg y \oplus y = \mathbf{1} = x \vee_0 y$.

If $x <_0 y$, then, by Lemma 4.15, $\neg x \oplus y = \mathbf{1}$, and $\neg(\neg x \oplus y) \oplus y = 0 \oplus y = y = x \vee_0 y$.

Otherwise, we have that $\neg x \oplus y = (-x) \wedge_0 (-y) + y \neq 0$, and

$$\neg(\neg x \oplus y) \oplus y = (-((-x) \wedge_0 (-y) + y)) \oplus y = (x \vee_0 y - y) \oplus y = (x \vee_0 y - y) \wedge_0 (-y) + y.$$

Since $y <_0 x \vee_0 y$, we have that $x \vee_0 y - y <_0 -y$ (by Proposition 3.3). Hence $(x \vee_0 y - y) \wedge_0 (-y) + y = x \vee_0 y - y + y = x \vee_0 y$.

MV1) ($x \oplus (y \oplus z) = (x \oplus y) \oplus z$). This is trivial if x, y or z belongs to $\{0, \mathbf{1}\}$. We assume that x, y, z belong to $A(C)\backslash\{0\}$. Assume that $x \oplus y = \mathbf{1}$. Then, $(x \oplus y) \oplus z = \mathbf{1}$. By Lemma 4.15, we have that $-y \leq_0 x$. If $y \oplus z = \mathbf{1}$, then $(x \oplus y) \oplus z = \mathbf{1} = x \oplus (y \oplus z)$. We assume that $y \oplus z \neq \mathbf{1}$. By Lemma 4.13, $y \leq_0 y \oplus z$. Hence $-(y \oplus z) \leq_0 -y \leq_0 x$.

Therefore, $x \wedge_0 (-(y \oplus z)) + (y \oplus z) = -(y \oplus z) + (y \oplus z) = 0$. It follows that $(x \oplus y) \oplus z = \mathbf{1} = x \oplus (y \oplus z)$.

Assume that $x \oplus y \neq \mathbf{1} \neq y \oplus z$, so we have that $-y \not\leq_0 x$ and $-y \not\leq_0 z$. Therefore:

$$\begin{aligned}(x \oplus y) \oplus z - x \oplus (y \oplus z) &= (x \oplus y) \oplus z - (z \oplus y) \oplus x \\ &= (x \wedge_0 (-y) + y) \wedge_0 (-z) + z - (z \wedge_0 (-y) + y) \wedge_0 (-x) - x \\ &= (x \wedge_0 (-y) + y) \wedge_0 (-z) - (z \wedge_0 (-y) + y) \wedge_0 (-x) - (x - z).\end{aligned}$$

Now, it follows from 3) of Definition 4.10 that $(x \oplus y) \oplus z - x \oplus (y \oplus z) = 0$. □

A pco-group needs not define canonically an MV-algebra, as shows the following example. Let n_1 and n_2 be integers, greater than 4, C_1 be the co-group $\mathbb{Z}/n_1\mathbb{Z}$ and C_2 be the co-group $\mathbb{Z}/n_2\mathbb{Z}$. C_1 and C_2 define MV-algebras. We can define a pco-group $C_1 \times C_2$ by setting $R((x_1, x_2), (y_1, y_2), (z_1, z_2)) \Leftrightarrow R(x_1, y_1, z_1) \,\&\, R(x_2, y_2, z_2)$. Then

$$A(C_1 \times C_2) = C_1 \times C_2 \backslash [((\mathbb{Z}/n_1\mathbb{Z}) \times \{0\}) \cup (\{0\} \times (\mathbb{Z}/n_2\mathbb{Z})) \cup \{(n_1 - 1, 1), (1, n_2 - 1)\}].$$

Now, $-(3, 1) = (n_1 - 3, n_1 - 1)$ and $(1, 3)$ belong to $A(C_1 \times C_2)$, $(3, 1) \not\leq_0 (1, 3)$, $(1, 3) \not\leq_0 (3, 1)$, but $(1, 3) - (3, 1) = (n_1 - 2, 2) \in A(C_1 \times C_2)$. Hence the rule $x \in A(C), y \in A(C) \Rightarrow (x + y \in A(C) \Leftrightarrow x \leq_0 -y$ or $-y \leq_0 x)$ does not hold. Consequently $C_1 \times C_2$ does not define canonically an MV-algebra.

We can define another partial cyclic order on $C_1 \times C_2$ by setting $(x_1, x_2) \leq_0 (y_1, y_2) \Leftrightarrow (x_1 \leq_0 y_1 \,\&\, x_2 \leq_0 y_2)$. In this case $A(C_1 \times C_2) = C_1 \times C_2$, and we conclude in the same way that $C_1 \times C_2$ does not define canonically an MV-algebra.

Now, Theorem 4.16 proves that the pco-group $(\mathbb{Z} \times \mathbb{Z})/\mathbb{Z}(n_1, n_2)$ defines canonically an MV-algebra.

Theorem 4.16 *Let C be the wound-round of an ℓ-group. Then $C \in \mathcal{AC}$.*

Proof We have to prove that C satisfies conditions (1), (2), (3) of Definition 4.10.

Recall that we saw after Definition 3.13 that if C is the wound-round of an ℓ-group, then for every x, y in $A(C)$, $x \wedge_0 y$ exists. Furthermore, $x \vee_0 y$ does not exist if, and only if, $(-x) \wedge_0 (-y) = 0$, and if this holds, then for every $z \in A(C)$: $x <_0 z \Rightarrow y \not<_0 z$.

(1) Let (G, u) be a unital ℓ-group such that $C \simeq G/\mathbb{Z}u$ and ρ be the natural mapping from G onto C. By Lemma 3.7 the restriction of ρ is a one-to-one mapping from $G_u = \{g \in G \mid 0 \leq g \,\&\, g \not\geq 0\}$ onto C. We saw in Proposition 4.2 that $A(C)$ can be identified with a subset of $[0, u[$. Let g, h in $[0, u[$ such that $\rho(g) \in A(C)$ and $\rho(h) \in A(C)$. We have that $g < h \Leftrightarrow \rho(g) <_0 \rho(h)$. It follows that $\rho(g \wedge h) \in A(C)$, $\rho(g \wedge h) = \rho(g) \wedge_0 \rho(h)$, and if $g \vee h \neq u$, then $\rho(g \vee h) \in A(C)$, $\rho(g \vee h) = \rho(g) \vee_0 \rho(h)$. By setting $\rho(u) = \mathbf{1}$, we have that $g \vee h = u \Leftrightarrow \rho(g) \vee_0 \rho(h) = \mathbf{1}$, hence $A(C) \cup \{\mathbf{1}\}$ embeds into a sublattice of $[0, u]$, so it is a distributive lattice, with smallest element 0 and greatest element $\mathbf{1}$. Note that by Corollary 4.3, if $A(C) \neq \{0\}$, then $A(C) \cup \{\mathbf{1}\}$ is isomorphic to the lattice $[0, u]$.

(2) Let x, y in $A(C)$, and g, h be the elements of $[0, u[$ such that $\rho(g) = x$ and $\rho(h) = y$. Since G is an ℓ-group, we have that $g + h = g \wedge h + g \vee h$, with $0 \leq g \wedge h < u$ and $0 \leq g \vee h \leq u$. We saw in Corollary 4.3 that ρ induces an isomorphism of ordered sets between $[0, u[$ and $A(C)$. Hence $\rho(g \wedge h) = x \wedge_0 y$, and if $g \vee h < u$, then $\rho(g \vee h) = x \vee_0 y$. If $g + h \in G_u$, then $g \wedge h + g \vee h \in G_u$. Hence $g \vee h < u$ and $x + y = \rho(g + h) = \rho(g \wedge h + g \vee h) = \rho(g \wedge h) + \rho(g \vee h) = x \wedge_0 y + x \vee_0 y$.

Assume that $g + h \notin G_u$, then $g + h - u \in G_u$. If $g \vee h < u$, then we have that

$$\begin{aligned} x + y &= \rho(g) + \rho(h) \\ &= \rho(g + h - u) \\ &= \rho(g \wedge h + g \vee h - u) \\ &= \rho(g \wedge h) + \rho(g \vee h) \\ &= x \wedge_0 y + x \vee_0 y. \end{aligned}$$

If $g \vee h = u$, then $x \vee_0 y = \mathbf{1}$. Hence $x \wedge_0 y + x \vee_0 y = x \wedge_0 y$ (see Notation 4.6). Then:

$$\begin{aligned} x + y &= \rho(g) + \rho(h) \\ &= \rho(g + h - u) \\ &= \rho(g \wedge h + g \vee h - u) \\ &= \rho(g \wedge h) \\ &= x \wedge_0 y. \end{aligned}$$

(3) Let x, y, z in $A(C)\backslash\{0\}$ and g, h, k in $]0, u[$ such that $\rho(g) = x$, $\rho(h) = y$ and $\rho(k) = z$.

$$(x \wedge_0 (-z) + z) \wedge_0 (-y) - (y \wedge_0 (-z) + z) \wedge_0 (-x) =$$

$$(\rho(g) \wedge_0 \rho(u - k) + \rho(k)) \wedge_0 \rho(u - h) - (\rho(h) \wedge_0 \rho(u - k) + \rho(k)) \wedge_0 \rho(u - g) =$$

$$(\rho(g \wedge (u - k)) + \rho(k)) \wedge_0 \rho(u - h) - (\rho((h) \wedge (u - k)) + \rho(k)) \wedge_0 \rho(u - g) =$$

$$\rho(g \wedge (u - k) + k) \wedge_0 \rho(u - h) - \rho(h \wedge (u - k) + k) \wedge_0 \rho(u - g).$$

$g \wedge (u-k) < u-k$, hence $g \wedge (u-k)+k < u-k+k = u$, hence $\rho(g \wedge (u-k)+k) \in A(C)$, and in the same way $\rho(h \wedge (u - k) + k) \in A(C)$, so:

$$(x \wedge_0 (-z) + z) \wedge_0 (-y) - (y \wedge_0 (-z) + z) \wedge_0 (-x) =$$

$$\rho((g \wedge (u - k) + k) \wedge (u - h)) - \rho((h \wedge (u - k) + k) \wedge (u - g)) =$$

$$\rho((g + k) \wedge u \wedge (u - h)) - \rho((h + k) \wedge u \wedge (u - g)) =$$

$$\rho((g+k)\wedge(u-h)-(h+k)\wedge(u-g)) = \rho((g+k+h)\wedge u-h-(h+k+g)\wedge u+g) =$$

$$\rho(g - h) = \rho(g) - \rho(h) = x - y.$$ □

One can wonder if being the wound-round of an ℓ-group can be characterized by first-order sentences. We will see that this holds if the pco-group C is generated by $A(C)$. This characterization relies on good sequences.

If C is a pco-group, then we denote by $\langle A(C)\rangle$ the subgroup of C generated by $A(C)$.

Theorem 4.17 *Let $C \in \mathcal{AC}$. The group $\langle A(C)\rangle$ being the wound-round of an ℓ-group is expressible by countably many first-order formulas of the language L_c.*

Proof Assume that $\langle A(C)\rangle = G/\mathbb{Z}u$, where (G, u) is a unital ℓ-group. Let A be the MV-algebra $A(C) \cup \{1\}$, (G_A, u_A) be the Chang ℓ-group of A. By Proposition 2.11, every element x of the positive cone of G_A is a sum of elements $x_1, \dots, x_n$ of A satisfying the conditions of Lemmas 2.7 and 2.8, where $x \leq nu$. Furthermore, if $x \not\geq u_A$, then the x_i's are different from u_A. Let call the sequence $(x_1, \dots, x_n, 0, \dots)$ the *good sequence* associated to x. By Lemma 3.7, the canonical epimorphism $\rho : G_A \to G_A/\mathbb{Z}u_A$ induces a one-to-one mapping between $G_{u_A} = \{x \in G_A \mid x \geq 0 \;\&\; x \not\geq u_A\}$ and $\langle A(C)\rangle$. It follows that every element x of $\langle A(C)\rangle$ can be represented by a unique good sequence of elements of $A(C)$. Furthermore, by Lemma 2.7, if x is a sum of n elements of $A(C)$, then the good sequence associated to x contains at most n elements different from 0. So C satisfies the following family of first-order formulas. For every $n \in \mathbb{N}\backslash\{0\}$,

$$\forall(x_1,\ldots,x_n) \in A(C)^n\ \exists(y_1,\ldots,y_n) \in A(C)^n \bigwedge_{1\le i<n} (y_{i+1} \wedge_0 -y_i = 0\ \&\ y_i = 0 \Rightarrow y_{i+1} = 0)$$

$$\&\ x_1+\cdots+x_n = y_1+\cdots+y_n\ \&\ (\forall(z_1,\ldots,z_n) \in A(C)^n \bigwedge_{1\le i<n} (z_{i+1}\wedge_0 -z_i = 0\ \&\ z_i = 0 \Rightarrow z_{i+1} = 0)$$

$$\&\ x_1 + \cdots + x_n = z_1 + \cdots + z_n) \Rightarrow z_1 = y_1,\ldots,z_n = y_n$$

Every element x of the positive cone of G_A is equivalent modulo $\mathbb{Z}u_A$ to an element x' of G_{u_A}, and the good sequence associated to x' is obtained by dropping the u_A's from the good sequence associated to x. Hence so is the good sequence associated to $\rho(x)$. Now, if x and y belong to G_{u_A}, then the good sequence associated to $z = x + y$ is obtained by the rules $z_i = x_i \oplus (x_{i-1} \odot y_1) \oplus \cdots \oplus (x_1 \odot y_{i-1}) \oplus y_i$. The good sequence associated to $\rho(x)$ is obtained by dropping the u_A's from the good sequence associated to z. Therefore, C satisfies the following family of first-order formulas. For every $n \in \mathbb{N}\backslash\{0\}$,

$$\forall(x_1,\ldots,x_n) \in A(C)^n\ \forall(y_1,\ldots,y_n) \in A(C)^n\ \forall(z_1,\ldots,z_n) \in A(C)^n \bigwedge_{1\le i<n} (x_{i+1} \wedge_0 -x_i = 0$$

$$\&\ x_i = 0 \Rightarrow x_{i+1} = 0)\ \&\ \bigwedge_{1\le i<n} (y_{i+1}\wedge_0 -y_i = 0\ \&\ y_i = 0 \Rightarrow y_{i+1} = 0)\ \&\ \bigwedge_{1\le i<n} (z_{i+1}\wedge_0 -z_i = 0$$

$$\&\ z_i = 0 \Rightarrow z_{i+1} = 0)\ \&\ x_1 + \cdots + x_n + y_1 + \cdots + y_n = z_1 + \cdots + z_n \Rightarrow \bigcup_{1\le i_0<n}$$

$$\bigcup_{1\le i<i_0} (x_i \oplus (x_{i-1} \odot y_1) \oplus\cdots\oplus (x_1 \odot y_{i-1}) \oplus y_i = 1)\ \&\ x_{i_0} \oplus (x_{i_0-1} \odot y_1) \oplus\cdots\oplus (x_1 \odot y_{i_0-1}) \oplus y_{i_0} \neq 1$$

$$\&\ \bigcup_{i_0\le i\le n} z_i = x_i \oplus (x_{i-1} \odot y_1) \oplus \cdots \oplus (x_1 \odot y_{i-1}) \oplus y_i$$

Conversely, assume that $C \in \mathcal{AC}$ and that C satisfies above families of formulas (recall that $C \in \mathcal{AC}$ is expressible by countably many first-order formulas). We prove that $\langle A(C)\rangle$ is isomorphic to the wound-round of $G_A/\mathbb{Z}u_A$, where A is the MV-algebra $A(C) \cup \{\mathbf{1}\}$. The group operation on $\langle A(C)\rangle$ is determined by $A(C)$ and by above formulas, which are also satisfied by the group $C' = G_A/\mathbb{Z}u_A$. It follows that the groups $\langle A(C)\rangle$ and C' are isomorphic. Furthermore, the ordered sets $[0, u]$ and $(A(C) \cup \{\mathbf{1}\}, \le_0)$ are isomorphic. By Corollary 4.3 they are isomorphic to $(A(C') \cup \{\mathbf{1}\}, \le_0)$. By Remarks 4.1, the pco-groups C' and $\langle A(C)\rangle$ are isomorphic. This proves that $\langle A(C)\rangle$ being the wound-round of an ℓ-group is expressible by countably many first-order formulas. □

5 Case of MV-chains

Here we focus on the correspondence between MV-chains and co-groups.

We first give a direct description of the restriction of the functor $\Theta\Xi$ to the class of MV-chains. Then, we show that this correspondence preserves elementary equivalence. Thanks to this transfer principle, we deduce, in the case of MV-chains, similar results to those of co-groups proved in [12].

5.1 Direct Description of the Functor $\Theta\,\Xi$

We proved in Proposition 3.12 that Θ induces a functorial one-to-one correspondence between co-groups and unital linearly ordered groups.

We saw the description of the wound-round $G/\mathbb{Z}u$ of a unital linearly ordered group (G, u) before Rieger's Theorem 3.4. Now, given a co-group C, the linearly ordered group $uw(C)$ is the set $\mathbb{Z}\times C$ together with the following order and operation. The linear order $\leq$ is the lexicographic order of $(\mathbb{Z}, \leq)\times(C, \leq_0)$, $u_C = (1, 0)$ and $(m, x)+(n, y) = (m+n, x+y)$ if $x = y = 0$ or $\min_0(x, y) <_0 x + y$, and $(m, x) + (n, y) = (m + n + 1, x + y)$ otherwise (see [5]).

Now, we turn to Ξ. The one-to-one correspondence between unital linearly ordered groups and MV-chains is the following. A unital linearly ordered group (G, u) is associated to the MV-chain $\Gamma(G, u) = [0, u]$ (see [2, Lemma 6]). Conversely, an MV-chain A is associated to its Chang ℓ-group G_A.

The construction of the Chang ℓ-group of $(A(C)\cup\{\mathbf{1}\}, \mathbf{1})$, in the linearly ordered case, is similar to the construction of the unwound of a co-group. Indeed, let A be an MV-chain. By [2, Lemmas 5 and 6], G_A is isomorphic to $\mathbb{Z} \times (A\backslash\{1\})$ lexicographically ordered and with the rules: $(m, x)+(n, y) = (m+n, x\oplus y)$ if $x\oplus y < 1$ and $(m, x)+(n, y) = (m+n+1, x\odot y)$ otherwise.

Furthermore, this correspondence is a functorial one (see Section 2).

The functor $\Theta\,\Xi$ induces a functorial one-to-one correspondence between MV-chains and co-groups. If A is an MV-chain, then $C(A) = G_A/\mathbb{Z}u_A$ is a co-group. Note that if C is a co-group with at least three elements, then $A(C) = C$. It follows that if C contains at least three elements, then the unital linearly ordered group $(uw(C), u_C)$ is isomorphic to the Chang ℓ-group of $(A(C) \cup \{\mathbf{1}\}, \mathbf{1})$.

By the comments above, in the linearly ordered case, we can give a direct description of the functor $\Theta\,\Xi$ and the description of its inverse is simpler than in Theorem 4.11.

Before going into details, let us note the following fact.

If ρ is the natural mapping from $uw(C)$ onto $C \simeq uw(C)/\mathbb{Z}u_C$, then for g, h in $[0, u_C[$ we have that $\rho(g) <_0 \rho(h) \Leftrightarrow g < h$, and if $g \leq h \neq 0$, then $g < g + h < g + u_C$. So, $\rho(g + h) = \rho(g) + \rho(h)$ if, and only if, $g + h < u_C$, which in turn is equivalent to: $\rho(g) <_0 \rho(g) + \rho(h)$. Otherwise, we have that $\rho(g) + \rho(h) = \rho(g + h - u_C)$.

Proposition 5.1 *Let A be an MV-chain.*

(1) *In the MV-chain A, the set $C(A)$ is interpreted by $A\backslash\{1\}$, the cyclic order is given by $R(x, y, z) \Leftrightarrow x < y < z$ or $y < z < x$ or $z < x < y$, the addition is given by $x + y = x \oplus y$ if $x \odot y = 0$ and $x + y = x \odot y$ otherwise.*
(2) *In $C(A) \cup \{\mathbf{1}\}$, the set A is interpreted by $C(A) \cup \{\mathbf{1}\}$, $\neg x$ is interpreted by $-x$ if $x \notin \{0, \mathbf{1}\}$, $\neg 0 = \mathbf{1}$ and $\neg\mathbf{1} = 0$. $\oplus$ is interpreted by $\mathbf{1} \oplus x = \mathbf{1}$ and for x, y in $C(A)$ $x \oplus y = x + y$ if $x + y \neq 0$ and $\min_0(x, y) <_0 x + y$, $x \oplus y = \mathbf{1}$ if $x \neq 0 \neq y$ and $x + y \leq_0 \min_0(x, y)$, and $0 \oplus 0 = 0$.*

Proof (1) We saw in Section 2 that in G_A $x\oplus y = (x+y)\wedge 1$, and $x\odot y = (x+y-1)\vee 0$. Since G_A is linearly ordered, $x \oplus y = \min(x + y, 1)$, and $x \odot y = \max(x + y, 1) - 1$.

Let $z \in [0, 1[$ such that $x+y-z \in \mathbb{Z}\cdot 1$, then $z = x+y$ if $x+y < 1$ and $z = x+y-1$ otherwise.

(2) We saw that if g, h are the elements of $[0, u_C[\subset uw(C)$ such that $\rho(g) = x$ and $\rho(h) = y$, then $g + h < u_C \Leftrightarrow \min_0(x, y) <_0 x + y$.

□

5.2 Transfer Principles of Elementary Equivalence

We prove that the correspondence between MV-chains and co-groups in Proposition 5.1 also preserves elementary equivalence.

Proposition 5.2 *Let A be an MV-chain.*

- *The co-group $C(A)$, in the language L_c, is interpretable in the L_{MV}-structure A.*
- *The L_{MV}-structure A is interpretable in the L_c structure $C(A) \cup \{\mathbf{1}\}$.*
- *If A and A' are MV-chains, then:*
  $A \equiv A' \Leftrightarrow C(A) \cup \{\mathbf{1}\} \equiv C(A') \cup \{\mathbf{1}\} \Leftrightarrow C(A) \equiv C(A')$, *and*
  $A \prec A' \Leftrightarrow C(A) \cup \{\mathbf{1}\} \prec C(A') \cup \{\mathbf{1}\} \Leftrightarrow C(A) \prec C(A')$.

Proof The first two items have been proved in Proposition 5.1.

It follows from Theorem 2.13 that $A \equiv A' \Rightarrow C(A) \equiv C(A')$ and $C(A) \cup \{\mathbf{1}\} \equiv C(A') \cup \{\mathbf{1}\} \Rightarrow A \equiv A'$. Now we see that $C(A) \equiv C(A') \Rightarrow C(A) \cup \{\mathbf{1}\} \equiv C(A') \cup \{\mathbf{1}\}$. The last proposition can be proved in the same way. □

Proposition 5.3 *Let A and $(A_i)_{i\in\mathbb{N}\backslash\{0\}}$ be MV-chains, U be an ultrafilter on $\mathbb{N}\backslash\{0\}$ and ΠA_i be the ultraproduct of $(A_i)_{i\in\mathbb{N}\backslash\{0\}}$. Then $A \equiv \Pi A_i \Leftrightarrow C(A) \equiv \Pi C(A_i)$.*

Proof Let Φ be an L_{MV}-sentence and Φ_c be the corresponding L_c-sentence. Then: $A \models \Phi \Leftrightarrow C(A) \models \Phi_c$, and for every i in $\mathbb{N}\backslash\{0\}$, $A_i \models \Phi \Leftrightarrow C(A_i) \models \Phi_c$. Hence $\{i \in \mathbb{N}\backslash\{0\} \mid A_i \models \Phi\} \in U \Leftrightarrow \{i \in \mathbb{N}\backslash\{0\} \mid C(A_i) \models \Phi_c\} \in U$. The equivalence follows. □

We saw in Section 3.2 that if (G, u) and (G', u') are unital linearly ordered groups, then we have $(G, u) \equiv (G', u') \Leftrightarrow G/\mathbb{Z}u \equiv G'/\mathbb{Z}u'$ and $(G, u) \prec (G', u') \Leftrightarrow G/\mathbb{Z}u \prec G'/\mathbb{Z}u'$. In the same way we have a transfer principle of elementary equivalence between the MV-chains and their Chang ℓ-groups.

We consider the language $L_{oMV} = (0, +, -, \leq, \oplus, \neg)$. The L_o-structure $\mathbb{Z}$ will be seen as an L_{oMV}-structure where $x \oplus y = z \Leftrightarrow x = y = z = 0$ and $\neg x = y \Leftrightarrow x = y = 0$. If A is an MV-chain, then it will be seen as an L_{oMV}-structure, where $x + y = z \Leftrightarrow x = y = z = 0$ and $x - y = z \Leftrightarrow x = y = z = 0$.

Proposition 5.4 *Let A be an MV-chain.*

- *The L_{MV}-structure A is interpretable in the L_{loZu}-structure $(G_A, \mathbb{Z}u_A)$ (resp. in the L_{lou}-structure (G_A, u_A)).*
- *The L_{loZu}-structure $(G_A, \mathbb{Z}u_A)$ (resp. the L_{lou}-structure (G_A, u_A)) is interpretable in the L_{oMV}-structure $\mathbb{Z} \times A$.*
- *If A and A' are MV-chains, then:*
  $(G_A, \mathbb{Z}u_A) \equiv (G_{A'}, \mathbb{Z}u_{A'}) \Leftrightarrow \mathbb{Z} \times A \equiv \mathbb{Z} \times A' \Leftrightarrow A \equiv A'$, *and*
  $(G_A, u_A) \equiv (G_{A'}, u_{A'}) \Leftrightarrow \mathbb{Z} \times A \equiv \mathbb{Z} \times A' \Leftrightarrow A \equiv A'$.
  The same holds with $\prec$ instead of $\equiv$.

Proof In $(G_A, \mathbb{Z}u_A)$ (resp. in (G_A, u_A)), 1 is the smallest positive element of $\mathbb{Z}u$ (resp. $1 = u$), the set A is interpreted by $\{x \in G_A \mid 0 \leq x \leq u_A\}$, $x \oplus y = \min(x + y, u_A)$, $\neg x = u_A - x$.

In $\mathbb{Z} \times A$, G_A is interpreted by $\mathbb{Z} \times (A \backslash \{u\})$, $\mathbb{Z}u$ is interpreted by $\mathbb{Z} \times \{0\}$ (resp. $u = (1, 0)$). The order relation is the lexicographic order: $(m, x) \leq (n, y) \Leftrightarrow m < n$ or ($m = n$ and $x \leq y$). The sum is defined by $(m, x) + (n, y) = (m + n, x \oplus y)$ if $x \oplus y < 1$, and $(m, x) + (n, y) = (m + n + 1, x \odot y)$ if $x \oplus y = 1$.

It follows from Theorem 2.13) that $(G_A, \mathbb{Z}u_A) \equiv (G_{A'}, \mathbb{Z}u_{A'}) \Rightarrow A \equiv A'$, $(G_A, u_A) \equiv (G_{A'}, u_{A'}) \Rightarrow A \equiv A'$, $\mathbb{Z} \times A \equiv \mathbb{Z} \times A' \Rightarrow (G_A, \mathbb{Z}u_A) \equiv (G_{A'}, \mathbb{Z}u_{A'})$ and $\mathbb{Z} \times A \equiv \mathbb{Z} \times A' \Rightarrow (G_A, u_A) \equiv (G_{A'}, u_{A'})$ (the same holds with $\prec$). Now, we deduce from Theorem 2.12 that in the language L_{oMV}: $A \equiv A' \Rightarrow \mathbb{Z} \times A \equiv \mathbb{Z} \times A'$. Now, clearly, if $A \equiv A'$ in L_{MV}, then $A \equiv A'$ in L_{oMV} (the same holds with $\prec$). □

5.3 MV-chains Elementarily Equivalent to Archimedean Ones

We deduce from [12] similar results in the case of MV-chains. In particular we characterize pseudofinite and pseudo-hyperarchimedean MV-chains.

First we list some results about co-groups, starting with definitions. Next, turning to MV-chains, we also review some definitions. Then we will deduce characterizations of those MV-algebras which are elementarily equivalent to archimedean ones.

Let C be a co-group.

(1) C is said to be *c-archimedean* if for every x and y in $C \backslash \{0\}$ there exists an integer $n > 0$ such that $R(0, nx, y)$ does not hold. Note that $R(0, nx, y)$ is equivalent to $y \leq_0 nx$, since $(C, \leq_0)$ is linearly ordered. So one can say that C is c-archimedean if the ordered set $(C, \leq_0)$ is archimedean.
(2) C is said to be *discrete* if there is $\varepsilon_C \in C \backslash \{0\}$ such that for every $x \in C \backslash \{0, \varepsilon_C\}$ we have $R(0, \varepsilon_C, x)$. In the same way as above C being discrete is equivalent to the ordered set $(C, \leq_0)$ being discretely ordered.
(3) C is said to be *dense* if it is not discrete.
(4) C is said to be *c-regular* if for every integer $n \geq 2$ and every $0 <_0 x_1 <_0 \cdots <_0 x_n$ in C there exists $x \in C$ such that $x_1 \leq_0 nx \leq_0 x_n$ and $x <_0 2x <_0 \cdots <_0 (n-1)x <_0 nx$. This is equivalent to saying that its unwound is a *regular* linearly ordered group, that is, for every $n \geq 2$ and every $0 < x_1 < \cdots < x_n$ in $uw(C)$ there exists $x \in uw(C)$ such that $x_1 \leq nx \leq x_n$.
(5) C is said to be *pseudo-c-archimedean* if C belongs to the elementary class generated by the c-archimedean co-groups.
(6) C is said to be *pseudofinite* if C belongs to the elementary class generated by the finite co-groups.

One can prove that C is c-archimedean if, and only if, its unwound is archimedean, and C is discrete if, and only if, its unwound is a discrete linearly-ordered group (that is, it contains a smallest positive element). Furthermore, C is dense if, and only if, its unwound is densely ordered, that is, for every $x < y$ in $uw(C)$ there exists $z \in uw(C)$ such that $x < z < y$.

Now we state some main results of [12].

In the next theorem, the prime invariants of Zakon of an abelian group B are define as follows. If p is a prime, then we define the p-th *prime invariant of Zakon* of B, denoted by $[p]B$, to be the maximum number of p-incongruent elements in B. In the infinite case, we set $[p]B = \infty$, without distinguishing between infinities of different cardinalities (see [15]).

Theorem 5.5 ([12] Theorem 1.9 and Lemma 4.16) (1) *A dense co-group is pseudo-c-archimedean if, and only if, it is c-regular. If this holds, then it is elementarily equivalent to some c-archimedean dense co-group.*

(2) *Any two dense c-regular co-groups are elementarily equivalent if, and only if, their torsion subgroups are isomorphic and they have the same family of prime invariants of Zakon. This in turn is equivalent to: their torsion subgroups are isomorphic and their unwounds have the same family of prime invariants of Zakon.*

In the theorem below, if C is a discrete co-group, then for a prime p, integers $n \in \mathbb{N}\backslash\{0\}$ and $k \in \{0, \ldots, p^n - 1\}$, we denote by $D_{p^n,k}$ the formula: $\exists x,\ \ R(0, x, 2x, \ldots, (p^n - 1)x)$ & $p^n x = k\varepsilon_C$. Here, ε_C is the smallest positive element of C. Since it is definable, we can assume that it lies in the language.

Theorem 5.6 ([12] Theorems 1.11, 4.33 and 4.42) (1) *Any two non-c-archimedean c-regular discrete co-groups are elementarily equivalent if, and only if, they satisfy the same formulas $D_{p^n,k}$.*

(2) *A co-group is pseudofinite if, and only if, it is discrete and c-regular.*

(3) *Let U be a nonprincipal ultrafilter on $\mathbb{N}\backslash\{0\}$, C be the ultraproduct of the co-groups $\mathbb{Z}/n\mathbb{Z}$, p be a prime, $n \in \mathbb{N}\backslash\{0\}$ and $k \in \{0, \ldots, p^n - 1\}$. Then C satisfies the formula $D_{p^n,k}$ if, and only if, the set $p^n\mathbb{N}\backslash\{0\} - k := \{p^n j - k \mid j \in \mathbb{N}\backslash\{0\}\}$ belongs to U.*

We turn to the MV-chains. We start with definitions (see [3, Chapter 6]).

(1) In an ordered set, by an *atom* we mean an element x such that $x > 0$ and whenever $y \leq x$ then either $y = 0$ or $y = x$ ([3, Definitions 6.4.2 and 6.7.1]).

(2) An ℓ-group is *hyperarchimedean* if for every positive x and y there exists $n \in \mathbb{N}\backslash\{0\}$ such that $nx \wedge y = (n + 1)x \wedge y$ (see [1, Theorem 14.1.2]).

(3) An MV-algebra is *atomic* if for each $x \neq 0$ there is an atom y with $y \leq x$. It is *atomless* if no element is an atom ([3, Definition 6.7.1]).

(4) An element x of an MV-algebra is *archimedean* if there exists $n \in \mathbb{N}\backslash\{0\}$ such that $\neg x \vee n.x = 1$. This is equivalent to saying that there exists $n \in \mathbb{N}$ such that $n.x = (n + 1).x$ ([3, Corollary 6.2.4]).

(5) An MV-algebra is *hyperarchimedean* if all its elements are archimedean ([3, Definition 6.3.1]).

(6) An MV-algebra is *simple* if it embeds in the interval [0, 1] of $\mathbb{R}$ ([3, Theorem 3.5.1]).

Note that if an MV-chain A is atomic, then it contains only one atom, and the underlying ordered set is discretely ordered. If it is atomless, then the underlying ordered set is densely ordered.

Saying that an MV-chain is hyperarchimedean is equivalent to saying that it is simple.

Recall the notations, for x in an MV-algebra, $2.x = x \oplus x$, $x^2 = x \odot x$ and so on.

Definition 5.7 Let A be an MV-chain.

(1) We say that A is *regular* if for every integer $n \geq 2$ and every $0 < x_1 < \cdots < x_n$ in A there exists $x \in A$ such that $x_1 \leq n.x \leq x_n$, and $0 < x < 2.x < \cdots < (n-1).x < n.x$.

(2) We say that A is *pseudo-simple* if A belongs to the elementary class generated by the simple MV-chains.

(3) We say that A is *pseudofinite* if A belongs to the elementary class generated by the finite MV-chains.

Let A be an MV-chain, it is easy to see that A is regular if, and only if, $C(A)$ is c-regular. Since the unwound of $C(A)$ is isomorphic to the Chang ℓ-group G_A of A, this is equivalent to saying that G_A is regular. One can also see that A is atomic if, and only if, $C(A)$ is discrete. Moreover, A is simple if, and only if, G_A is archimedean. Note that a linearly ordered group is hyperarchimedean if, and only if, it is archimedean.

In the MV-chain $A(C)$, the formula $R(0, x, 2x, \dots, (p^n - 1)x)$ can be reformulated as $0 < x < 2.x < \dots < (p^n - 1).x$, (which is equivalent to $x \neq 0$ and $0 = x^2 = \dots = x^{p^n-1}$, since $x^2 = 0 \Leftrightarrow 2.x \leq 1$) hence we can define formulas $D_{p^n,k}$ in MV-chains in the following way. If A is an atomic and not simple MV-chain, then the atom ε_A of A is definable (it is the smallest positive element). We can assume that it lies in the language. For a prime p, for $n \in \mathbb{N}\backslash\{0\}$ and $k \in \{0, \dots, p^n - 1\}$, we denote by $D_{p^n,k}$ the formula: $\exists x,\ 0 < x < 2.x < \dots < (p^n - 1).x) \wedge p^n.x = k.\varepsilon_A$.

In the same way, the torsion subgroup has an analogue in MV-chains. Let x be an element of an MV-chain. We will say that x is a *torsion element* if there exists $n \in \mathbb{N}\backslash\{0\}$, such that $n.x = 1$ and $x^n = 0$.

One can see that $x \in A\backslash\{1\}$ is a torsion element in the MV-chain A if, and only if, it is a torsion element in the group $C(A)$.

Now, thanks to Proposition 5.3, Theorems 5.5 and 5.6 can be expressed in terms of MV-chains.

Theorem 5.8 (1) *Any atomless regular MV-chain is elementarily equivalent to some simple atomless MV-chain.*

(2) *Any two atomless regular MV-chains are elementarily equivalent if, and only if, their subchain of torsion elements are isomorphic and their Chang ℓ-groups have the same family of prime invariants of Zakon.*

In particular, an atomless MV-chain is pseudo-simple if, and only if, it is regular.

Theorem 5.9 (1) *Any two infinite atomic regular MV-chains are elementarily equivalent if, and only if, they satisfy the same formulas $D_{p^n,k}$.*

(2) *An infinite MV-chain is pseudofinite if, and only if, it is atomic and regular.*

(3) *Let U be a nonprincipal ultrafilter on $\mathbb{N}\backslash\{0\}$, A be the ultraproduct of the MV-chains $[0, n]$, p be a prime, $n \in \mathbb{N}\backslash\{0\}$ and $k \in \{0, \dots, p^n - 1\}$. Then A satisfies the formula $D_{p^n,k}$ if, and only if, $p^n\mathbb{N}\backslash\{0\} - k \in U$.*

6 Non-linearly Ordered Case

Let A be a finite MV-algebra. By [3, Proposition 3.6.5], A is isomorphic to a product of finite MV-chains and its Chang ℓ-group G_A is isomorphic to some $\mathbb{Z} \times \dots \times \mathbb{Z}$. Now, an ultraproduct of MV-chains need not be a product of a finite number of MV-chains. Hence we will focus on the MV-algebras which are isomorphic to a product of n MV-chains, where $n \in \mathbb{N}\backslash\{0\}$.

We will say that an MV-algebra is *n-pseudofinite* if it is elementarily equivalent to some ultraproduct of a family of finite MV-algebras which are isomorphic to products of n MV-chains.

In the case of hyperarchimedean MV-algebras, we will make a similar restriction. If an MV-algebras A is isomorphic to a product of n MV-chains $[0, 1_1], \dots, [0, 1_n]$, then saying that A is hyperarchimedean is equivalent to saying that each of $[0, 1_1], \dots, [0, 1_n]$ is simple.

Recall that simple and hyperarchimedean MV-algebras have been defined and characterized before Definitions 5.7. We will say that an MV-algebra is *n-pseudo-hyperarchimedean* if it is elementarily equivalent to some ultraproduct of a family of hyperarchimedean MV-algebras which are isomorphic to products of n simple MV-algebras.

By [3, Corollary 6.5.6], being hyperarchimedean is equivalent to being a boolean product of simple MV-algebras. One can prove that if an hyperarchimedean MV-algebra is isomorphic to a finite product of simple MV-algebras, then every sub-MV-algebra is projectable, hyperarchimedean and is isomorphic to a finite product of simple MV-algebras.

Our goal is to get characterizations of n-pseudofinite and n-pseudo-hyperarchimedean MV-algebras, and to give necessary and sufficient conditions for two such MV algebras being elementarily equivalent.

Let us list some properties of abelian ℓ-groups (see [1]).

We let G be an ℓ-group. We know that, for every $x \in G$, there exists a unique pair x_+, x_- of elements of the positive cone of G such that $x = x_+ + x_-$ and $x_+ \wedge x_- = 0$. We let $|x| := x_+ + x_-$.

- Two elements x, y of G are said to be *orthogonal* if $|x| \wedge |y| = 0$. This is equivalent to: $x_+ \wedge y_+ = x_+ \wedge y_- = x_- \wedge y_+ = x_- \wedge y_- = 0$. A subset A of G is said to be *orthogonal* if its elements are pairwise orthogonal. Every orthogonal subset is contained in a maximal orthogonal subset.
- If $A \subset G$, then the *polar* of A is the set $A^\perp := \{y \in G \mid \forall x \in A,\ |x| \wedge |y| = 0\}$; if $A = \{x\}$, then we let $x^\perp := \{x\}^\perp$. The set $A^{\perp\perp}$ is called a *bipolar*. Every polar of G is a convex ℓ-subgroup of G. A polar $A^\perp$ is said to be *principal* if $A^\perp = x^{\perp\perp}$ for some $x \in G$ (see [1, Chapter 3]).
- An element x of G_+ is said to be *basic* if $x^{\perp\perp}$ is a linearly ordered group, which is equivalent to saying that the set $[0, x]$ is linearly ordered. If x and y are basic elements, then either $x \leq y$ or $y < x$ or $x \wedge y = 0$. If $x \wedge y > 0$, then $x^\perp = y^\perp$, hence $x^{\perp\perp} = y^{\perp\perp}$ (see [1, pp. 133-135]).
- The group G is said to be *projectable* if, for every $x \in G$, G is the direct sum of $x^\perp$ and $x^{\perp\perp}$. Note that being projectable is a first-order property. Indeed, let x, y in G_+. Then $y \in x^\perp \Leftrightarrow x \wedge y = 0$, and $y \in x^{\perp\perp} \Leftrightarrow \forall z\ (x \wedge z = 0 \Rightarrow y \wedge z = 0)$. Hence $y \in x^\perp$ and $y \in x^{\perp\perp}$ are first-order properties.

We will not characterize pseudofinite MV-algebras. However we can review basic properties that they satisfy. Since an MV-algebra embeds in the positive cone of its Chang ℓ-group, we have for every a: $|a| = a$. Hence we can define orthogonal elements and polars in the following way. Let A be an MV-algebra. Two elements a, b are *orthogonal* if $a \wedge b = 0$. The *polar* of a subset B of A is $B^\perp = \{a \in A \mid \forall b \in B\ a \wedge b = 0\}$. The MV-algebra A is said to be *projectable* if, for every a, b in A, b can be written in a unique way as $b = b_1 \oplus b_2$, with $b_1 \in a^\perp$ and $b_2 \in a^{\perp\perp}$.

One can also prove that an MV-algebra is projectable if, and only if, its Chang ℓ-group is projectable.

Now, every finite MV-algebra is projectable, and every minimal principal polar is discrete. Consequently, every pseudofinite MV-algebra is projectable, and its minimal principal polars are discrete and regular.

Before characterizing n-pseudofinite and n-pseudo-hyperarchimedean MV-algebras, we characterize those unital ℓ-groups which are isomorphic to products of n linearly ordered ℓ-groups, and the MV-algebras which are isomorphic to products of n MV-chains.

6.1 Products of *n* Linearly Ordered Groups

Lemma 6.1 *If $\{x_1, \dots, x_n\}$ is a maximal orthogonal set of an ℓ-group G whose elements are basic elements, then:*

- *for every $x \in G$ there is some $i \in \{1, \dots, n\}$ such that $x_i^{\perp\perp} \subset x^{\perp\perp}$,*
- *$x > 0$ is basic if, and only if, there exists $i \in \{1, \dots, n\}$ such that $x_i^{\perp\perp} = x^{\perp\perp}$,*
- *the minimal principal polars of G are $x_1^{\perp\perp}, \dots, x_n^{\perp\perp}$.*

Proof Let $0 < x \in G$. Since $\{x_1, \dots, x_n\}$ is maximal orthogonal, there is some $i \in \{1, \dots, n\}$ such that $x \wedge x_i > 0$. Now, $[0, x_i]$ is linearly ordered, hence for every $y \in G_+$, we have that either $x_i \wedge x \leq x_i \wedge y$ or $x_i \wedge y \leq x_i \wedge x$. If $x_i \wedge y \leq x_i \wedge x$, then $x_i \wedge y \leq x_i \wedge (x \wedge y)$. Hence $y \in x^\perp \Rightarrow y \in x_i^\perp$. If $x_i \wedge x < x_i \wedge y$, then $x_i \wedge x \leq x_i \wedge (x \wedge y)$. Hence $y \wedge x > 0$. It follows that $x^\perp \subset x_i^\perp$. Therefore $x_i^{\perp\perp} \subset x^{\perp\perp}$.

Let $x > 0$ and $i \in \{1, \dots, n\}$ such that $x^\perp = x_i^\perp$. If x is basic, then we have that $x_i \leq x$ or $x \leq x_i$. Assume that $x_i \leq x$. For every $y > 0$ we have that $y \wedge x_i = y \wedge x_i \wedge x = y \wedge x \wedge x_i = \min(y \wedge x, x_i)$, since $[0, x]$ is linearly ordered. Therefore: $y \wedge x_i = 0 \Leftrightarrow y \wedge x = 0$, hence $x^\perp = x_i^\perp$. The case $x \leq x_i$ is similar.

The last assertion follows trivially. □

By [10, Théorème 6, Chapitre II], we know that an ℓ-group G is a direct sum of linearly ordered groups if, and only if, the following holds:

- for every $x^{\perp\perp}$ which is minimal, G is the direct sum of $x^\perp$ and $x^{\perp\perp}$,
- every $x^{\perp\perp}$ contains some $y^{\perp\perp}$ which is minimal,
- there is at most a finite number of minimal $y^{\perp\perp}$.

Therefore, G is the direct sum of n linearly ordered groups if, and only if, it contains a maximal orthogonal set $\{x_1, \dots, x_n\}$ whose elements are basic elements and, for every $i \in \{1, \dots, n\}$, G is the direct sum of $x_i^\perp$ and $x_i^{\perp\perp}$. This is equivalent to saying that G is projectable and contains a maximal orthogonal set of n element which are basic.

Now, we also have the following result.

Proposition 6.2 *Let (G, u) be a unital ℓ-group. Then G is the product of n linearly ordered groups if, and only if, G_+ contains a maximal orthogonal set $\{u_1, \dots, u_n\}$, whose elements are basic, such that $u = u_1 + \cdots + u_n$.*

Proof $\Rightarrow$ is straightforward. Assume that G_+ contains a maximal orthogonal set $\{u_1, \dots, u_n\}$, whose elements are basic, such that $u = u_1 + \cdots + u_n$. We know that if x, y, z in G_+ satisfy $x \wedge y = 0$, then $x + y = x \vee y$ and $(x + z) \wedge y = z \wedge y$ (see, for example, [7, Lemma 2.3.4]). Let $x \in G_+$, and $p \in \mathbb{N}\backslash\{0\}$ such that $x \leq pu$. We have that $pu = pu_1 + \cdots + pu_n$, where the pu_i's are pairwise orthogonal. Hence $x = x \wedge pu = x \wedge pu_1 + \cdots + x \wedge pu_n \in u_1^{\perp\perp} + \cdots + u_n^{\perp\perp}$. Note that, since $\{u_1, \dots, u_n\}$ is a orthogonal, we have that $x = (x \wedge pu_1) \vee \cdots \vee (x \wedge pu_n)$. Assume that $x = x_1 + \cdots + x_n = x_1 \vee \cdots \vee x_n$ with $x_i \in u_i^{\perp\perp}$ $(1 \leq i \leq n)$. Then $x_i = x \wedge pu_i$, which proves the uniqueness of the decomposition. It follows that G is the direct sum of $u_1^{\perp\perp}, \dots, u_n^{\perp\perp}$. □

6.2 Products of *n* MV-chains

From [7, Lemma 2.3.4], which we recalled in the proof of Proposition 6.2, one deduces by induction that for every orthogonal family $\{x_1, \dots, x_n\}$ in the positive cone of an ℓ-group we have $x_1 + \cdots + x_n = x_1 \vee \cdots \vee x_n$. Now, in an MV-algebra, if $\{x_1, \dots, x_n\}$ is an orthogonal family, then $x_1 \oplus \cdots \oplus x_n = x_1 \vee \cdots \vee x_n$.

The following proposition is similar to Lemma 6.4.5 in [3].

Proposition 6.3 *Let A be an MV-algebra. Then, the Chang ℓ-group of A is isomorphic to a product of n linearly ordered groups if, and only if, there exist nonzero elements $u_1, \dots, u_n$ of A such that:*

- $1 = u_1 \oplus \cdots \oplus u_n$,
- *for all i, j in $\{1, \dots, n\}$: $i \neq j \Rightarrow u_i \wedge u_j = 0$,*
- *for all x, y in A, if $x \leq u_i$ and $y \leq u_i$, then $x \leq y$ or $y \leq x$.*

If this holds, then the Chang ℓ-group of A is interpretable in $(\mathbb{Z} \times [0, u_1[) \times \cdots \times (\mathbb{Z} \times [0, u_n[)$, where $(p_1, x_1, \dots, p_n, x_n) \leq (q_1, y_1, \dots, q_n, y_n)$ if, and only if, for every $i \in \{1, \dots, n\}$, $p_i < q_i$ or ($p_i = q_i$ and $x_i \leq y_i$). The addition is defined componentwise, $(p_i, x_i) + (q_i, y_i) = (p_i + q_i, x_i \oplus y_i)$ if $x_i \oplus y_i < u_i$, and $(p_i, x_i) + (q_i, y_i) = (p_i + q_i + 1, x_i \odot y_i)$ if $x_i \oplus y_i = u_i$.

Proof The equivalence follows from Proposition 6.2. Now let $x \in G_{A+}$. We know that there exists a good sequence $(x_1, \dots, x_p)$ of elements of $[0, u]$ such that $x = x_1 + \cdots + x_p$, where, for $1 \leq k \leq p - 1$, $(u - x_k) \wedge x_{k+1} = 0$ (see Section 2). For $j \in \{1, \dots, n\}$ let $x_j = x_{1,j} + \cdots + x_{n,j}$, with $x_{i,j} \in u_i^{\perp\perp}$ $(1 \leq i \leq n)$. We have that $u - x_j = (u_1 - x_{1,j}) + \cdots + (u_n - x_{n,j})$, hence $u_i - x_{i,j} > 0 \Rightarrow x_{i,j+1} = 0$ i.e. $x_{i,j} \neq u_i \Rightarrow x_{i,j+1} = 0$. Therefore we can write x as $x = k_1 u_1 + x_1 + \cdots + k_n u_n + x_n$, with $0 \leq k_i \leq p$ and $x_i \in [0, u_i[$ $(1 \leq i \leq n)$. So, every element of G_A can be written in a unique way as $x = k_1 u_1 + x_1 + \cdots + k_n u_n + x_n$, with $k_i \in \mathbb{Z}$ and $x_i \in [0, u_i[$ $(1 \leq i \leq n)$. Let $x = k_1 u_1 + x_1 + \cdots + k_n u_n + x_n$, and $y = l_1 u_1 + y_1 + \cdots + l_n u_n + y_n$ in G_A.

Trivially, $x \leq y$ if, and only if, for every $i \in \{1, \dots, n\}$, $k_i < l_i$ or ($k_i = l_i$ and $x_i \leq y_i$).

Set $x + y = z = m_1 u_1 + z_1 + \cdots + m_n u_n + z_n$. Since $k_i u_i + x_i + l_i u_i + y_i \in u_i^{\perp\perp}$, we have that $m_i u_i + z_i = (k_i + l_i) u_i + x_i + y_i$, i.e. $x_i + y_i - z_i = (m_i - k_i - l_i) u_i$. If $x_i + y_i < u_i$, then $-u_i < x_i + y_i - z_i < u_i$, hence $x_i + y_i - z_i = 0$ and $z_i = x_i + y_i = x_i \oplus y_i$ and $m_i = k_i + l_i$. Otherwise, in the same way we prove that $z_i = x_i + y_i - u_i$ and $m_i = k_i + l_i + 1$. Now, $x_i \oplus y_i = (x_i + y_i) \wedge u = (x_i + y_i) \wedge u_i = u_i$ and $x_i \odot y_i = u - [(2u_i - x_i - y_i) \wedge u] = (x_i + y_i - u) \vee 0 = x_i + y_i - u_i$. □

Since in G_A we have that $A = [0, u_A]$, saying that G_A is isomorphic to a product of n linearly ordered groups is equivalent to saying that A is isomorphic to a product of n MV-chains.

It follows from Proposition 6.3 that being isomorphic to a product of n MV-chains is a first-order property.

6.3 *n*-pseudofinite MV-algebras

We define the language $L_{MVn} = (0, \oplus, \neg, \underline{1_1}, \dots, \underline{1_n})$, with n new constant symbols. Let $(A_1, 1_1)$, $\dots$, $(A_n, 1_n)$, $(A'_1, 1'_1)$, $\dots$, $(A'_n, 1'_n)$ be MV-chains. For $i \in \{1, \dots, n\}$, we assume that $(A, 1_i)$ is an L_{MVn}-structure, by setting, for $j \in \{1, \dots, n\}$, $x = \underline{1_j}$ if either $i = j$ and $x = 1_i$, or $i \neq j$ and $x = 0$. We define in the same way the L_{MVn}-structures $(A'_1, 1'_1), \ldots, (A'_n, 1'_n)$. Let A be the L_{MVn}-structure $A_1 \times \cdots \times A_n$ and A' be the L_{MVn}-structure $A'_1 \times \cdots \times A'_n$.

Now, we consider families of MV-chains, $(A_{1,\alpha_1}, 1_{1,\alpha_1})_{\alpha_1 \in I_1}, \ldots, (A_{n,\alpha_n}, 1_{n,\alpha_n})_{\alpha_n \in I_n}$ (we can do the same thing with families of linearly ordered groups $(T_{1,\alpha_1})_{\alpha_1 \in I_1}, \ldots, (T_{n,\alpha_n})_{\alpha_n \in I_n}$). For every $(\alpha_1, \ldots, \alpha_n)$ in $I_1 \times \cdots \times I_n$ we set $(A_{(\alpha_1,\ldots,\alpha_n)}, 1_{1,\alpha_1}, \ldots, 1_{n,\alpha_n}) = (A_{1,\alpha_1} \times \cdots \times A_{,n\alpha_n}, 1_{1,\alpha_1}, \ldots, 1_{n,\alpha_n})$. We let U be an ultrafilter on $I_1 \times \cdots \times I_n$ and $(A, 1_1, \ldots, 1_n)$ be the ultraproduct of the family $(A_{(\alpha_1,\ldots,\alpha_n)}, 1_{1,\alpha_1}, \ldots, 1_{n,\alpha_n})$. We know that for $i \in \{1, \ldots, n\}$, the canonical projection $p_i(U)$ on I_i is an ultrafilter on I_i. Denote by A_i the ultraproduct of the family (A_{i,α_i}). Then one can prove that $(A, 1_1, \ldots, 1_n) \simeq (A_1 \times \cdots \times A_n, 1_1, \ldots, 1_n)$, where 1_i is the greatest element of A_i. Note that the maximal element of A is $1 = 1_1 + \cdots + 1_n$.

Theorem 6.4 *Let $(A, 1)$ and $(A', 1')$ be MV-algebras.*

(1) *$(A, 1)$ is n-pseudofinite if, and only if, A is projectable, it is isomorphic to a product of n MV-chains $[0, 1_1] \times \cdots \times [0, 1_n]$ and, for every $i \in \{1, \ldots, n\}$, the MV-chain $[0, 1_i]$ is either finite or infinite discrete regular.*
(2) *If $(A, 1)$ and $(A', 1')$ are n-pseudofinite, then $(A, 1_1, \ldots, 1_n) \equiv (A', 1'_1, \ldots, 1'_n)$ if, and only if, for every $i \in \{1, \ldots, n\}$ either the MV-chains $[0, 1_i]$, $[0, 1'_i]$ are finite and isomorphic, or they are infinite regular and satisfy the same formulas $D_{p^m,k}$.*

Proof (1) $\Rightarrow$. Let $(A_{1,\alpha_1}, 1_{1,\alpha_1})_{\alpha_1 \in I_1}, \ldots, (A_{n,\alpha_n}, 1_{n,\alpha_n})_{\alpha_n \in I_n}$ be families of MV-chains. For every $(\alpha_1, \ldots, \alpha_n)$ in $I_1 \times \cdots \times I_n$ we set $(A_{(\alpha_1,\ldots,\alpha_n)}, 1_{1,\alpha_1}, \ldots, 1_{n,\alpha_n}) = (A_{1,\alpha_1} \times \cdots \times A_{,n\alpha_n}, 1_{1,\alpha_1}, \ldots, 1_{n,\alpha_n})$. We let U be an ultrafilter on $I_1 \times \cdots \times I_n$ and for every i let A_i be the ultraproduct of the family (A_{i,α_i}) (associated to $p_i(U)$). If A is the ultraproduct of the family $(A_{(\alpha_1,\ldots,\alpha_n)}, 1_{1,\alpha_1}, \ldots, 1_{n,\alpha_n})$, then $(A, 1_1, \ldots, 1_n) \simeq (A_1 \times \cdots \times A_n, 1_1, \ldots, 1_n)$. Now, by Theorem 5.9 every A_i is an MV-chain which is either finite or infinite discrete regular.

$\Leftarrow$ If this holds, then in the language L_{MVn} A is isomorphic to $[0, 1_1] \times \cdots \times [0, 1_n]$. By Theorem 5.9, every $[0, 1_i]$ is isomorphic to an ultraproduct of a family of finite MV-chains $(A_{i,\alpha_i}, 1_{i,\alpha_i})_{\alpha_i \in I_i}$. Hence, A is isomorphic to the ultraproduct of the family $(A_{1,\alpha_1} \times \cdots \times A_{,n\alpha_n}, 1_{1,\alpha_1}, \ldots, 1_{n,\alpha_n})$.

(2) We deduce from Theorem 2.12 that in the language L_{MVn} we have: $(A, 1_1, \ldots, 1_n) \equiv (A', 1'_1, \ldots, 1'_n) \Leftrightarrow (A_1, 1_1) \equiv (A'_1, 1'_1), \ldots, (A_n, 1_n) \equiv (A'_n, 1'_n)$. Hence, the result follows from Theorem 5.9.

□

6.4 *n*-hyperarchimedean MV-algebras

In the same way as Theorem 6.4, and by using Theorem 5.8, one can prove the following.

Theorem 6.5 *Let $(A, 1)$ and $(A', 1')$ be MV-algebras.*

(1) *$(A, 1)$ is n-pseudo-hyperarchimedean if, and only if, A is projectable, it is isomorphic to a product of n MV-chains $[0, 1_1] \times \cdots \times [0, 1_n]$ and, for every $i \in \{1, \ldots, n\}$, the MV-chain $[0, 1_i]$ is either finite or infinite and regular.*
(2) *If $(A, 1)$ and $(A', 1')$ are n-pseudo-hyperarchimedean, then $(A, 1_1, \ldots, 1_n) \equiv (A', 1'_1, \ldots, 1'_n)$ if, and only if, for every $i \in \{1, \ldots, n\}$ either the MV-chains $[0, 1_i]$, $[0, 1'_i]$ are finite and isomorphic, or they are both discrete infinite regular and satisfy*

the same formulas $D_{p^m,k}$, or they are infinite dense regular and their Chang ℓ-groups have the same prime invariants of Zakon.

Data Availability Data sharing not applicable to this article as no data-sets were generated or analyzed during the current study.

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